International Journal of Mathematical Archive-9(2), 2018, 284-294
Available online through www.ijma.info ISSN 2229 – 5046

CLASSIFICATIONS OF SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS OF MIXED TYPE

N. SIKENDER*[1], P. RAMI REDDY[2], M. VENKATA KRISHNA[3] AND M. CHENNA KRISHNA REDDY[4]

[1,3,4]Department of Mathematics,
Osmania University, Hyderabad, Telangana, India.

[2]Department of Mathematics,
Khairathabad Govt. Degree College, Hyderabad, Telangana, India.

(Received On: 22-12-17; Revised & Accepted On: 06-02-18)

ABSTRACT

In this paper, the authors classified all solutions of the second-order nonlinear neutral delay dynamic equations of mixed type into four classes and obtained conditions for the existence / non-existence of solutions in these classes. Examples are included to illustrate the validiation of the main results.

Keywords: *second-order, neutral dynamic equations, oscillatory solution, mixed type, weakly oscillatory solution, asymptotic behavior, time scales.*

INTRODUCTION

In this paper, we are concerned to obtain the conditions for the existence / non-existence solutions of a class M^+, M^-, OS, WOS and asymptotic behavior of solutions of a class M^+ and M^- of second order nonlinear neutral delay dynamic equations of mixed type

$$(a(t)(x(t)+b(t)x(\alpha(t))+c(t)x(\beta(t)))^{\Delta})^{\Delta} + p(t)f(x(\gamma(t)))+q(t)g(x(\delta(t))) = 0, \tag{1}$$

for $t \in [t_0, \infty)_T$, where T is a time scale with $\sup T = \infty$. In what follows, it is always assume that

1. $a \in C^1_{rd}([t_0,\infty)_T,(0,\infty))$, $b, c \in C^2_{rd}([t_0,\infty)_T, R)$ and $p, q \in C_{rd}([t_0,\infty)_T, R)$;
2. $f, g \in C(R,R)$ such that $uf(u) > 0$ an $ug(u) > 0$ for $u \neq 0$;
3. $\alpha, \beta, \gamma, \delta \in C_{rd}(R,R)$ are strictly increasin functions such that
$$\alpha(t) \le t,\ \beta(t) \ge t,\ \gamma(t) \le t,\ \delta(t) \ge t \text{ and } \lim_{t\to\infty}\alpha(t) = \lim_{t\to\infty}\beta(t) = \lim_{t\to\infty}\gamma(t) = \lim_{t\to\infty}\delta(t) = \infty$$

Let $t_x \in [t_0,\infty)_T$ such that $\alpha(t) \ge t_0$ and $\gamma(t) \ge t_0$ for all $t \in [t_x,\infty)_T$. By a solution of equation (1), we shall mean a function $x \in C_{rd}([t_x,\infty)_T, R)$ which has the properties $x(t)+b(t)x(\alpha(t))+c(t)x(\beta(t)) \in C^1_{rd}([t_x,\infty)_T, R)$ and $a(t)(x(t)+b(t)x(\alpha(t))+c(t)x(\beta(t))) \in C^1_{rd}([t_x,\infty)_T, R)$ and satisfies equation (1) on $[t_x,\infty)_T$. As is customary, a solution of equation (1) is oscillatory solution (**OS**) if it is neither eventually positive nor eventually negative, otherwise it is called non-oscillatory. A non-oscillatory solution $x(t)$ of equation (1) is said to be weakly oscillatory solution (**WOS**) if $x(t)$ is non-oscillatory and $x^{\Delta}(t)$ is oscillatory for large value of $t \in [t_0,\infty)_T$.

In recent years, there has been much research activity concerning the oscillation, nonoscillation and asymptotic behavior of solutions of various differential equations, difference equations and dynamic equations. For instance in [7, 8, 9] etc., authors have been studied by classifying all solutions into four classes such as M^+, M^-, OS, WOS and obtained criteria for the existence/non-existence of solutions. In order to extend and generalize the

Corresponding Author: N. Sikender*[1],
[1]Department of Mathematics, Osmania University, Hyderabad, Telangana, India.

papers [7, 8, 9], Rami Reddy *et al.* [10] were concerned the solutions of existence/non-existence of a class M^+, M^-, OS, WOS of second order nonlinear neutral delay dynamic equation of the form

$$(r(t)(x(t)+p(t)x(\alpha(t)))^{\Delta})^{\Delta}+q(t)f(x(\beta(t)))=0 \tag{2}$$

for $t \in [t_0,\infty)_T$, subject to the conditions:

4. $r \in C^1_{rd}([t_0,\infty)_T,(0,\infty))$, $p \in C^2_{rd}([t_0,\infty)_T, R$;
5. $q \in C_{rd}([t_0,\infty)_T, R)$ and q does not vanish eventually;
6. $f \in C^1(R,R)$ such that f satisfies $uf(u)>0$ for $u \neq 0$ and $f'(u) \geq 0$ for $u \in R$;
7. $\alpha, \beta \in C_{rd}([t_0,\infty)_T, R)$ are strictly increasing functions such that

$$\alpha(t) \leq t,\ \beta(t) \leq t \ \text{ and } \ \lim_{t\to\infty}\alpha(t)=\infty=\lim_{t\to\infty}\beta(t).$$

The study of the oscillation and other asymptotic properties of solutions of neutral delay difference / differential / dynamic equations with positive and negative coefficients attracted a good bit of attention in the last several years. For instance in [16, 17, 18, 20] etc., and the references cited therein, authors have been studied certain criteria of second order nonlinear dynamic equations with positive and negative coefficients. In order to extend and generalize the papers [16, 17, 18, 19], sikender *et al.* [11] were concerned the solutions of existence/non-existence of a class M^+, M^-, OS, WOS of second order nonlinear neutral delay dynamic equation of the form

$$(r(t)(x(t)+p(t)x(\alpha(t)))^{\Delta})^{\Delta}+q(t)f(x(\beta(t)))-h(t)g(x(\gamma(t)))=0 \tag{3}$$

for $t \in [t_0,\infty)_T$, subject to the conditions:

8. $r \in C^1_{rd}([t_0,\infty)_T,(0,\infty))$, $p \in C^2_{rd}([t_0,\infty)_T, R)$ and $q, h \in C_{rd}([t_0,\infty)_T, R)$;
9. $f, g \in C(R,R)$ such that $uf(u)>0$ and $ug(u)>0$ for $u \neq 0$;
10. $\alpha, \beta, \gamma \in C_{rd}([t_0,\infty)_T, T)$ are strictly increasing functions such that

$$\alpha(t) \leq t,\quad \beta(t) \leq t,\quad \gamma(t) \leq t \ \text{ and } \ \lim_{t\to\infty}\alpha(t)=\infty=\lim_{t\to\infty}\beta(t).$$

Interest in Functional differential and difference equations is growing due to the development in science and technology and the varied applications in many areas. For examples, equations involving delay and those involving advance and a combination of both arise in nerve conduction (Life Science), organizational behaviour (Social sciences), Signal processing pantograph equations (Mechanical Engineering), to mention a few (see for e.g [12, 13, 14, 15]). Study of such equations has been an active area of research for many researchers and recently an importance is given to the unification of continuous and discrete aspects of analysis on time scales. In [10], the authors considered equation (1) with $c(t)=0$, $q(t)=0$ for all $t \in [t_0,\infty)_T$ and classified all solutions of (1) into four classes and obtained criteria for the existence / non-existence of solutions in these classes. In [11], the authors considered equation (1) with $c(t)=0$, $p(t)\geq 0$, $q(t) \leq 0$ and $\delta(t) \leq t$ for all $t \in [t_0,\infty)_T$ and classified all solutions of (1) into four classes and obtained criteria for the existence / non-existence of solutions in these classes. But, now the authors are so much interested on mixed delays. In [6], the authors considered the neutral differential equations of mixed type of the form

$$(a(t)(x(t)+bx(t-\tau_1)+cx(t+\tau_2))')'+p(t)x^{\alpha}(t-\sigma_1)+q(t)x^{\beta}(t+\sigma_2)=0,\quad t \geq t_0 > 0 \tag{4}$$

subject to the following conditions:

11. $a \in C^1([t_0,\infty), R)$ and is positive for all $t \geq t_0$;
12. b and c are constants, τ_1, τ_2, σ_1 and σ_2 are nonnegative constants;
13. α and β are the ratio of odd positive integers;
14. $p, q \in C_{rd}([t_0,\infty)_T, R)$.

Motivated and inspired by the papers mentioned above, in this paper the authors are interested to study the solutions of existence / non-existence of a class M^+, M^-, OS, WOS and asymptotic behavior of M^+ and M^- of equation (1). Here we consider two cases, namely $p \leq 0$, $q \geq 0$ and $p \geq 0$, $q \leq 0$ for large $t \in [t_0,\infty)_T$, to give sufficient conditions in order that every solution of equation (1) oscillates and to study the asymptotic nature of non-oscillatory solutions of equation (1) with respect to their asymptotic behavior all the solutions of equation (1) may be *priori* divided in to the following classes:

1. $M^+ = \{x \in S :$ there exists some $t_{x_1} \in [t_0,\infty)_T$ such that $x(t)x^{\Delta}(t) \geq 0$ for $t \in [t_{x_1},\infty)_T\}$;
2. $M^- = \{x \in S :$ there exists some $t \in [t_{x_1},\infty)_T\}$ such that $x(t)x^{\Delta}(t) \leq 0$ for $t \in [t_{x_1},\infty)_T\}$;

3. $OS = \{x \in S:$ there exists a sequence $t_n \in [t_0, \infty)_T$, $t_n \to \infty$ such that $x(t_n)x(t_{n+1}) \le 0\}$;
4. $WOS = \{x \in S:$ $x(t)$ is non-oscillatory for large $t \in [t_0, \infty)_T$, but $x^{\Delta}(t)$ oscillates$\}$.

With a very simple argument we can prove that M^+, M^-, OS, WOS are mutually disjoint. By the above definitions, it turns out that solutions in the class M^+ are eventually either positive nondecreasing or negative nonincreasing, solutions in the class M^- are eventually either positive nonincreasing or negative nondecreasing, solutions in the class OS are oscillatory, and finally solutions in the class WOS are weakly oscillatory.

In next Section, we mentioned important lemma's which are existing in the literature. In Section 2, we obtain sufficient conditions for the existence/non-existence in the above said classes. In Section 3, we discuss the asymptotic behavior of solutions in the class of M^+ and M^-.

1. IMPORTANT LEMMA'S

Theorem 1.1 ***(Chain rule)*** **([1, Theorem 1.90]):** Let $f: R \to R$ is continuously differentiable and suppose $g: T \to R$ is delta differentiable. Then $f \circ g: T \to R$ is delta differentiable and the formula

$$(f \circ g)^{\Delta}(t) = \{\int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh\}g^{\Delta}(t)$$

Theorem 1.2 ([1, Theorem 1.117]): Let $a \in T^k$, $b \in T$ and assume $f: T \times T^k \to R$ is continuous at (t,t), where $t \in T^k$ with $t > a$. Also assume that $f^{\Delta}(t,.)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighbourhood U of t, independent of $\tau \in [a, \sigma(t)]$, such that

$$| f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t) - s) | \le \varepsilon | \sigma(t) - s | \text{ for all } s \in U$$

where f^{Δ} denotes the derivative of f with respect to the first variable. Then

1. $g(t) := \int_a^t f(t,\tau)\Delta\tau$ implies $g^{\Delta}(t) = \int_a^t f^{\Delta}(t,\tau)\Delta\tau + f(\sigma(t),t)$
2. $h(t) := \int_t^b f(t,\tau)\Delta\tau$ implies $h^{\Delta}(t) = \int_t^b f^{\Delta}(t,\tau)\Delta\tau - f(\sigma(t),t)$.

For more basic concepts in the time scale theory the readers are referred to the books (see [1, 2]).

2. EXISTENCE AND NON-EXISTENCE OF SOLUTIONS IN M^+, M^-, OS AND WOS

First, we examine the existence of solutions of equation (1) in the class M^+.

Theorem 2.1: Assume that

(H_1) $b(t)$, $c(t)$ are non negative and non-decreasing;

(H_2) $p(t)$ is non positive and $q(t)$ is non negative for $t \in [t_0, \infty)_T$;

(H_3) there exists $M_1 > 0$ such that $\dfrac{f(u)}{g(u)} \le M_1$ for $u \ne 0$;

(H_4) g is non-decreasing and

(H_5) $\limsup_{t\to\infty} \int_{t_0}^t (M_1 p(s) + q(s))\Delta s = \infty$ hold. Then for equation (1), we have $M^+ = \varphi$.

Proof: Suppose that the equation (1) has a solution $x \in M^+$. Without loss of generality, we may assume that $x(t) > 0$ and $x^{\Delta}(t) \ge 0$ for large $t \in [t_0, \infty)_T$ (the proof is similar if $x(t) < 0$ and $x^{\Delta}(t) \le 0$ for large $t \in [t_0, \infty)_T$). Then there exists $t_1 \in [t_0, \infty)_T$ such that $x(t)$, $x(\alpha(t))$, $x(\gamma(t))$ all are positive and $x^{\Delta}(t)$, $x^{\Delta}(\alpha(t))$, $x^{\Delta}(\gamma(t))$ all are non-negative for all $t \in [t_1, \infty)_T$.

Define,

$$z(t) = x(t) + b(t)x(\alpha(t)) + c(t)x(\beta(t)) \tag{2.1}$$

for $t \in [t_1, \infty)_T$. Then in view of (H_1), we have $z(t) > 0$ and $z^{\Delta}(t) \ge 0$ for all $t \in [t_1, \infty)_T$.

Using (2.1), equation (1) becomes

$$(a(t)z^{\Delta}(t))^{\Delta}+p(t)f(x(\gamma(t)))+q(t)g(x(\delta(t)))=0$$

or

$$\frac{(a(t)z^{\Delta}(t))^{\Delta}}{g(x(\delta(t)))}=-p(t)\frac{f(x(\gamma(t)))}{g(x(\delta(t)))}-q(t)\text{ for } t\in[t_1,\infty)_T. \tag{2.2}$$

Now for $t\in[t_1,\infty)_T$,

$$\left(\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))}\right)^{\Delta}=\frac{(a(t)z^{\Delta}(t))^{\Delta}}{g(x(\delta(t)))}+(a(\sigma(t))z^{\Delta}(\sigma(t)))\left(\frac{1}{g(x(\delta(t)))}\right)^{\Delta}$$

$$=\frac{(a(t)z^{\Delta}(t))^{\Delta}}{g(x(\delta(t)))}-(a(\sigma(t))z^{\Delta}(\sigma(t)))\left(\frac{(g(x(\delta(t))))^{\Delta}}{g^{\sigma}(x(\delta(t)))g(x(\delta(t)))}\right),$$

implies that,

$$\left(\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))}\right)^{\Delta}=\frac{(a(t)z^{\Delta}(t))^{\Delta}}{g(x(\delta(t)))}-(a(\sigma(t))z^{\Delta}(\sigma(t)))\left(\frac{\{\int_0^1 g'(x(\delta(t))+h\mu(t)(x(\delta(t)))^{\Delta})dh\}(x(\delta(t)))^{\Delta}}{g^{\sigma}(x(\delta(t)))g(x(\delta(t)))}\right)$$

or

$$\left(\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))}\right)^{\Delta}=\frac{(a(t)z^{\Delta}(t))^{\Delta}}{g(x(\delta(t)))}-(a(\sigma(t))z^{\Delta}(\sigma(t)))\left(\frac{\{\int_0^1 g'(x(\delta(t))+h\mu(t)(x(\delta(t)))^{\Delta})dh\}x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{g^{\sigma}(x(\delta(t)))g(x(\delta(t)))}\right) \tag{2.3}$$

Therefore,

$$\left(\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))}\right)^{\Delta}\le\frac{(a(t)z^{\Delta}(t))^{\Delta}}{g(x(\delta(t)))} \tag{2.4}$$

for all $t\in[t_1,\infty)_T$, due to (H_4), $z^{\Delta}(t)\ge 0$ and $x^{\Delta}(\delta(t))\ge 0$ for all $t\in[t_1,\infty)_T$. From (2.2) and (2.4), we have

$$\left(\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))}\right)^{\Delta}\le -p(t)\frac{f(x(\gamma(t)))}{g(x(\delta(t)))}-q(t)$$

$$\le -p(t)\frac{f(x(\gamma(t)))}{g(x(\gamma(t)))}-q(t)$$

$$\le -(M_1p(t)+q(t))$$

for $t\in[t_1,\infty)_T$, due to (H_2)-(H_4). Integrating the last inequality from t_1 to t, we obtain

$$\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))}-\frac{a(t_1)z^{\Delta}(t_1)}{g(x(\delta(t_1)))}\le-\int_{t_1}^{t}(M_1p(s)+q(s))\Delta s.$$

From (H_5), we obtain

$$\liminf_{t\to\infty}\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))}=-\infty,$$

which contradicts the assumption $z^{\Delta}(t)\ge 0$ for large t. Thus, the theorem is proved.

Example 2.2: In Theorem 2.1, the assumption (H_5) cannot be dropped. For this, suppose $T = R$, and consider the following differential equation

$$\left(\frac{1}{t^2}(x(t)+(3t+1)x(t-1))+2(2t-5)x(t+\frac{1}{2}))'\right)'+\frac{(-18)}{t^3(t-2)^3}x^3(t-2)$$

$$+\frac{14}{t^2(t+5)(t^2+10t+28)}x(t+5)(x^2(t+5)+3)=0, \tag{2.5}$$

$t\ge 3$. For this differential equation, all assumptions of Theorem 2.1 hold, but (H_5) is violated. The equation (2.5) has a solution $x(t)=t\in M^+$.

Theorem 2.3: Assume that condition (H_1) and

(H_6) $p(t)$ is non negative and $q(t)$ is non positive for $t \in [t_0, \infty)_T$;

(H_7) there exists $M_2 > 0$ such that $|\frac{g(u)}{f(u)}| \leq M_2$ for $u \neq 0;$

(H_8) f is non-decreasing;

(H_9) $\limsup_{t\to\infty} \int_{t_0}^{t} (p(s) + M_2 q(s))\Delta s = \infty$, hold. Then for equation (1), we have $M^+ = \varphi$.

Proof: Suppose that the equation (1) has a solution $x \in M^+$. Without loss of generality, we may assume that $x(t) > 0$ and $x^\Delta(t) \geq 0$ for large $t \in [t_0, \infty)_T$ (the proof is similar if $x(t) < 0$ and $x^\Delta(t) \leq 0$ for large $t \in [t_0, \infty)_T$). Then there exists $t_1 \in [t_0, \infty)_T$ such that $x(t)$, $x(\alpha(t))$, $x(\gamma(t))$ all are positive and $x^\Delta(t)$, $x^\Delta(\alpha(t))$, $x^\Delta(\gamma(t))$ all are non-negative for all $t \in [t_1, \infty)_T$.

Define $z(t)$ as in Theorem 2.1, and from (H_1), we get $z(t) > 0$ and $z^\Delta(t) \geq 0$ for all $t \in [t_1, \infty)_T$. Using (2.1), equation (1) becomes

$$(a(t)z^\Delta(t))^\Delta + p(t) f(x(\gamma(t))) + q(t) g(x(\delta(t))) = 0$$

or

$$\frac{(a(t)z^\Delta(t))^\Delta}{f(x(\gamma(t)))} = -p(t) - \frac{g(x(\delta(t)))}{f(x(\gamma(t)))} q(t) \quad \text{for } t \in [t_1, \infty)_T \tag{2.6}$$

Now for $t \in [t_1, \infty)_T$,

$$\begin{aligned}\left(\frac{a(t)z^\Delta(t)}{f(x(\gamma(t)))}\right)^\Delta &= \frac{(a(t)z^\Delta(t))^\Delta}{f(x(\gamma(t)))} + (a(\sigma(t))z^\Delta(\sigma(t)))\left(\frac{1}{f(x(\gamma(t)))}\right)^\Delta \\ &= \frac{(a(t)z^\Delta(t))^\Delta}{f(x(\gamma(t)))} - (a(\sigma(t))z^\Delta(\sigma(t)))\left(\frac{(f(x(\gamma(t))))^\Delta}{f^\sigma(x(\gamma(t)))f(x(\gamma(t)))}\right),\end{aligned}$$

implies that,

$$\left(\frac{a(t)z^\Delta(t)}{f(x(\gamma(t)))}\right)^\Delta = \frac{(a(t)z^\Delta(t))^\Delta}{f(x(\gamma(t)))} - (a(\sigma(t))z^\Delta(\sigma(t)))\left(\frac{\{\int_0^1 f'(x(\gamma(t)) + h\mu(t)(x(\gamma(t)))^\Delta)dh\}(x(\gamma(t)))^\Delta}{f^\sigma(x(\gamma(t)))f(x(\gamma(t)))}\right)$$

or

$$\left(\frac{a(t)z^\Delta(t)}{f(x(\gamma(t)))}\right)^\Delta = \frac{(a(t)z^\Delta(t))^\Delta}{f(x(\gamma(t)))} - (a(\sigma(t))z^\Delta(\sigma(t)))\left(\frac{\{\int_0^1 f'(x(\gamma(t)) + h\mu(t)(x(\gamma(t)))^\Delta)dh\}x^\Delta(\gamma(t))\gamma^\Delta(t)}{f^\sigma(x(\gamma(t)))f(x(\gamma(t)))}\right) \tag{2.7}$$

Therefore,

$$\left(\frac{a(t)z^\Delta(t)}{f(x(\gamma(t)))}\right)^\Delta \leq \frac{(a(t)z^\Delta(t))^\Delta}{f(x(\gamma(t)))} \tag{2.8}$$

for all $t \in [t_1, \infty)_T$, due to (H_8), $z^\Delta(t) \geq 0$ and $x^\Delta(\gamma(t)) \geq 0$ for all $t \in [t_1, \infty)_T$.

From (2.6) and (2.8), we have

$$\begin{aligned}\left(\frac{a(t)z^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta &\leq -p(t) - q(t)\frac{g(x(\delta(t)))}{f(x(\gamma(t)))} \\ &\leq -p(t) - q(t)\frac{g(x(\delta(t)))}{f(x(\delta(t)))} \\ &\leq -(p(t) + M_2 q(t))\end{aligned}$$

for $t \in [t_1, \infty)_T$, due to (H_6)-(H_7). Integrating the last inequality from t_1 to t, we obtain

$$\frac{a(t)z^{\Delta}(t)}{f(x(\gamma(t)))}-\frac{a(t_1)z^{\Delta}(t_1)}{f(x(\gamma(t_1)))}\leq -\int_{t_1}^{t}(p(s)+M_2q(s))\Delta s.$$

From (H_9), we obtain

$$\liminf_{t\to\infty}\frac{a(t)z^{\Delta}(t)}{f(x(\gamma(t)))}=-\infty,$$

which contradicts the assumption $z^{\Delta}(t)\geq 0$ for large t. Thus, the theorem is proved.

Example 2.4: Let $\mathrm{T}=\mathrm{R}$, and consider the following differential equation

$$\left(\frac{1}{3t^2-8t}(x(t)+(t^2+2t)x(t-2)+(2t-1)x(t+9))'\right)'$$
$$+\frac{12}{\mathrm{t(3t-8)^2(t-1/2)^3(t^2-t+5/4)}}\mathrm{x^3(t-1/2)(x^2(t-1/2)+1)}-\frac{8}{(3t^2-8t)^2(5t+6)^5}x^5(5t+6)=0, \tag{2.9}$$

$t\geq 1$. For this differential equation, all assumptions of Theorem 2.3 hold, but (H_9) is violated. The equation (2.9) has a solution $x(t)=t\in M^+$.

Theorem 2.5: Assume that conditions (H_2), (H_3), (H_4), (H_5) and

(H_{10}) $-1<b(t)+c(t)\leq 0$ for $\mathrm{t}\in[\mathrm{t_0},\infty)_{\mathrm{T}}$ and $c(t)$ is non negative;

(H_{11}) $M_1p(t)+q(t)\geq 0$ for $\mathrm{t}\in[\mathrm{t_0},\infty)_{\mathrm{T}}$;

(H_{12}) $\int_{t_0}^{\infty}\frac{1}{a(t)}\Delta t=\infty$ hold. Then for equation (1), we have $M^+=\varphi$.

Proof: Suppose that the equation (1) has a solution $x\in M^+$. Without loss of generality, we may assume that $x(t)>0$ and $x^{\Delta}(t)\geq 0$ for large $\mathrm{t}\in[\mathrm{t_0},\infty)_{\mathrm{T}}$ (the proof is similar if $x(t)<0$ and $x^{\Delta}(t)\leq 0$ for large $\mathrm{t}\in[\mathrm{t_0},\infty)_{\mathrm{T}}$). Then there exists $\mathrm{t_1}\in[\mathrm{t_0},\infty)_{\mathrm{T}}$ such that $x(t)$, $x(\alpha(t))$, $x(\gamma(t))$ all are positive and $x^{\Delta}(t)$, $x^{\Delta}(\alpha(t))$, $x^{\Delta}(\gamma(t))$ all are non-negative for all $\mathrm{t}\in[\mathrm{t_1},\infty)_{\mathrm{T}}$.

Define $z(t)$ as in Theorem 2.1, we have

$$\begin{aligned} z(t) &= x(t)+b(t)x(\alpha(t))+c(t)x(\beta(t)) \\ &\geq (1+b(t)+c(t))x(\alpha(t)) \\ &> 0 \end{aligned}$$

for $\mathrm{t}\in[\mathrm{t_1},\infty)_{\mathrm{T}}$. From (1), we have

$$\begin{aligned} (a(t)z^{\Delta}(t))^{\Delta} &= -(p(t)\frac{f(x(\gamma(t)))}{g(x(\delta(t)))}+q(t))g(x(\delta(t))) \\ &\leq -(p(t)\frac{f(x(\gamma(t)))}{g(x(\gamma(t)))}+q(t))g(x(\delta(t))) \\ &\leq -(M_1p(t)+q(t))g(x(\delta(t))) \\ &\leq 0, \quad \text{for } \mathrm{t}\in[\mathrm{t_1},\infty)_{\mathrm{T}}. \end{aligned}$$

Hence, $a(t)z^{\Delta}(t)$ is non-increasing for $\mathrm{t}\in[\mathrm{t_1},\infty)_{\mathrm{T}}$.

We claim that, $a(t)z^{\Delta}(t)\geq 0$ for $\mathrm{t}\in[\mathrm{t_1},\infty)_{\mathrm{T}}$.

If $a(t)z^{\Delta}(t)<0$ for $\mathrm{t}\in[\mathrm{t_1},\infty)_{\mathrm{T}}$, then $a(t)z^{\Delta}(t)\leq a(t_1)z^{\Delta}(t_1)$
or

$$z^{\Delta}(t)\leq a(t_1)z^{\Delta}(t_1)\frac{1}{a(t)}. \tag{2.10}$$

Integrating the above inequality from t_1 to t, we obtain

$$z(t)-z(t_1)\le a(t_1)z^{\Delta}(t_1)\int_{t_1}^{t}\frac{1}{a(s)}\Delta s.$$

This gives that $z(t)\to-\infty$ as $t\to\infty$ due to (H_{12}), which is a contradiction to $z(t)>0$. Thus $a(t)z^{\Delta}(t)\ge 0$ for $t\in[t_1,\infty)_T$. Now proceeding as in the proof of Theorem 2.1, and using (H_{11}),
we get

$$\lim_{t\to\infty}\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))}=-\infty.$$

This contradicts the fact that $z^{\Delta}(t)\ge 0$ for large t. Hence, the theorem is proved.

Example 2.6: Let $T = hZ$, h is a ratio of odd positive integers, and consider the following h-difference equation

$$\Delta_h(e^t\Delta_h(x(t)-e^{2h}x(t-2h)+\frac{1}{2}e^{-2h}x(t+2h)))$$
$$-\frac{e^{3h}+1}{2h^2}e^{-2t+12h}x(4t-12h)+\frac{e^h+1}{2h^2}(e^h+1)e^t x(t+h)=0, \tag{2.11}$$

$t>12h$. For this difference equation, all assumptions of Theorem 2.5 hold, but (H_{12}) is violated. The equation (2.11) has a solution $x(t)=e^t\in M^+$.

Theorem 2.7: Assume that conditions (H_6), (H_7), (H_8), (H_9), (H_{10}), (H_{12}) and (H_{13}). $p(t)+M_2q(t)\ge 0$ for $t\in[t_0,\infty)_T$ hold. Then for equation (1), we have $M^+=\phi$.

Proof: The proof is similar to that of Theorem 2.5.

Next, we examine the problem of the existence of solutions of equation (1) in the class M^-:

Theorem 2.8: Assume that (H_6), (H_7), (H_8) and

(H_{14}) $b(t)$ and $c(t)$ are non-negative and non-increasing;

(H_{15}) $f(uv)=f(u)f(v)$;

(H_{16}) $\gamma(t)\le\alpha(t)$;

(H_{17}) the function $\dfrac{1}{f(u)}$ is locally integrable on $(0,c)$ and $(-c,0)$ for some $c>0$,

i.e., $\int_0^c\frac{du}{f(u)}<\infty$, $\int_{-c}^0\frac{du}{f(u)}>-\infty$;

(H_{18}) $\limsup_{t\to\infty}\int_T^t\frac{1}{a(s)f(1+b(s)+c(s))}\left(\int_T^s(p(v)+M_2q(v))\Delta v\right)\Delta s=\infty$ for $t\ge T$.

Then for equation (1), we have $M^-=\phi$.

Proof: Suppose that equation (1) has a solution $x\in M^-$. Without loss of generality, we may assume that $x(t)>0$ and $x^{\Delta}(t)\le 0$ for large $t\in[t_0,\infty)_T$ (the proof is similar if $x(t)<0$ and $x^{\Delta}(t)\ge 0$ for large $t\in[t_0,\infty)_T$). Then there exists $t_1\in[t_0,\infty)_T$ such that $x(t)$, $x(\alpha(t))$, $x(\gamma(t))$ all are positive and $x^{\Delta}(t)$, $x^{\Delta}(\alpha(t))$, $x^{\Delta}(\gamma(t))$ all are non-positive for all $t\in[t_1,\infty)_T$. Define $z(t)$ as in Theorem 2.1 and using (H_{14}), we see that $z(t)>0$ and $z^{\Delta}(t)\le 0$ for all $t\in[t_1,\infty)_T$.

As in the proof of Theorem 2.3, we obtain

$$\frac{a(t)z^{\Delta}(t)}{f(x(\gamma(t)))}-\frac{a(t_1)z^{\Delta}(t_1)}{f(x(\gamma(t_1)))}=-\int_{t_1}^{t}(p(s)+M_2q(s))\Delta s$$

or

$$\frac{a(t)z^{\Delta}(t)}{f(x(\gamma(t)))}\le-\int_{t_1}^{t}(p(s)+M_2q(s))\Delta s$$

or

$$\frac{z^{\Delta}(t)}{f(x(\gamma(t)))}\le-\frac{1}{a(t)}\int_{t_1}^{t}(p(s)+M_2q(s))\Delta s \quad \text{for } t\in[t_1,\infty)_T . \tag{2.12}$$

We see that $z(t)\le(1+b(t)+c(t))x(\gamma(t))$. Since x is non-increasing and from (H_{15}), we see that

$$f(z(t))\le f(1+b(t)+c(t))\,f(x(\gamma(t))) \tag{2.13}$$

Combining (2.12) and (2.13), we obtain

$$\begin{aligned}\frac{z^{\Delta}(t)}{f(z(t))}&\le\frac{z^{\Delta}(t)}{f(1+b(t)+c(t))\,f(x(\gamma(t)))}\\&\le-\frac{1}{a(t)\,f(1+b(t)+c(t))}\int_{t_1}^{t}(p(s)+M_2q(s))\Delta s.\end{aligned}$$

Integrating from t_1 to t, we obtain

$$-\int_{z(t_1)}^{z(t)}\frac{\Delta s}{f(s)}\ \le-\int_{t_1}^{t}\frac{1}{a(s)f(1+b(s)+c(s))}\left(\int_{t_1}^{s}(p(v)+M_2q(v))\Delta v\right)\Delta s$$

or

$$\int_{z(t)}^{z(t_1)}\frac{\Delta s}{f(s)}\ge\ \int_{t_1}^{t}\frac{1}{a(s)f(1+b(s)+c(s))}\left(\int_{t_1}^{s}(p(v)+M_2q(v))\Delta v\right)\Delta s$$

Using (H_{18}), we obtain

$$\limsup_{t\to\infty}\int_{z(t)}^{z(t_1)}\frac{\Delta s}{f(s)}=\infty,$$

which is a contradiction to (H_{17}). Hence, the theorem is proved.

Theorem 2.9: Assume that (H_2), (H_3), (H_4), (H_5), (H_{11}), (H_{12}) and

(H_{19}) b, c are non negative constants,

(H_{20}) f is nonincreasing hold.

Then every solution of equation (1) is either oscillatory or weakly oscillatory.

Proof: From Theorem 2.1, it follows that for equation (1), we have $M^{+}=\varphi$. To complete the proof, it suffices to show that for equation (1), $M^{-}=\phi$. Let $x\in M^{-}$, then $x(t)>0$ and $x^{\Delta}(t)\le 0$ for $t\in[t_1,\infty)_T$. Define $z(t)$ as in Theorem 2.1, we have $z(t)>0$ and $z^{\Delta}(t)\le 0$ for $t\in[t_1,\infty)_T$ due to (H_{19}). Using (2.1), equation (1) becomes

$$\begin{aligned}(a(t)z^{\Delta}(t))^{\Delta}&=-p(t)f(x(\gamma(t)))-q(t)g(x(\delta(t)))\\&=\left(-p(t)\frac{f(x(\gamma(t)))}{g(x(\delta(t)))}-q(t))g(x(\delta(t))\right)\\&\le(-p(t)\frac{f(x(\delta(t)))}{g(x(\gamma(t)))}-q(t))g(x(\delta(t)))\\&\le-(M_1p(t)+q(t)) \quad \text{for } t\in[t_1,\infty)_T .\end{aligned}$$

Hence,

$$(a(t)z^{\Delta}(t))^{\Delta} \leq 0 \text{ for } t \in [t_1, \infty)_T, \text{ due to } (H_{11}).$$

This implies that, $a(t)z^{\Delta}(t)$ is nonincreasing for $t \in [t_1, \infty)_T$. Then there exists $k(<0) \in R$ and $t_2 \in [t_1, \infty)_T$ such that

$$a(t)z^{\Delta}(t) \leq k$$

or

$$z^{\Delta}(t) \leq k \frac{1}{a(t)}$$

Integrating the above inequality from t_2 to t, we obtain

$$z(t) - z(t_2) \leq k \int_{t_2}^{t} \frac{1}{a(s)} \Delta s.$$

This implies that $z(t) \to -\infty$ as $t \to \infty$. Which gives a contradiction to $z(t) > 0$.

Hence, the theorem is proved.

Theorem 2.10: Assume that (H_6), (H_7), (H_8), (H_{12}), (H_{13}), (H_{19}) and

(H_{21}) g is nonincreasing hold.

Then every solution of equation (1) is either oscillatory or weakly oscillatory.

Proof: The proof is similar to that of Theorem 2.9 and hence complete the proof.

Remark 2.11: In Theorem 2.9 and Theorem 2.10, we used f is nonincreasing and g is nondecreasing respectively. In this cases f and g are defined on $R - \{0\}$ only.

3. BEHAVIOR OF SOLUTIONS IN M^+ AND M^-

Theorem 3.1: Assume that (H_6), (H_7), (H_8), (H_{14}), (H_{15}), (H_{16}) and (H_{18}) hold. Then every solution $x(t) \in M^-$, we have $\lim_{t\to\infty} x(t) = 0.$

Proof: The argument used in the proof of Theorem 2.8, again leads that

$$\limsup_{t\to\infty} \int_{z(t)}^{z(t_1)} \frac{1}{f(s)} \Delta s = \infty.$$

This implies that, $\lim_{t\to\infty} z(t) = 0$. But $z(t) > x(t)$ for all $t \in [t_1, \infty)_T$.

This shows that $\lim_{t\to\infty} x(t) = 0.$ Hence complete the proof of the theorem.

Theorem 3.2: Assume that (H_1), (H_2), (H_3), (H_4), (H_{11}) and

(H_{22}) $1 + b(t) + c(t)$ is bounded,

(H_{23}) $\limsup_{t\to\infty} \int_T^t (M_1 p(s) + q(s)) (\int_T^s (\frac{1}{a(v)} \Delta v) \Delta s = \infty$ for all $T \geq t_0$ hold.

Then every solution of (1) in the class M^+ is unbounded.

Proof: Let x be a solution of (1) such that $x \in M^+$. Proceeding as in the proof of Theorem 2.1, by defining $z(t)$ as in (2.1), we have $z(t) > 0$, $z^{\Delta}(t) \geq 0$ for all $t \in [t_1, \infty)$ due to (H_1). Then from (1), we obtain (2.2). Consider the function

$$w(t) = -\frac{a(t)z^{\Delta}(t)}{g(x(\delta(t)))} \int_{t_1}^{t} \frac{1}{a(s)} \Delta s,$$

we have,

$$w^{\Delta}(t) = -(a(t)z^{\Delta}(t))^{\Delta}\left(\frac{1}{g(x(\delta(t)))}\int_{t_1}^{t}\frac{1}{a(s)}\Delta s\right) - \left(a(\sigma(t))z^{\Delta}(\sigma(t))\right)\left(\frac{1}{g(x(\delta(t)))}\int_{t_1}^{t}\frac{1}{a(s)}\Delta s\right)^{\Delta}$$

$$= \left(p(t)\frac{f(x(\gamma(t)))}{g(x(\delta(t)))} + q(t)\right)\int_{t_1}^{t}\frac{1}{a(s)}\Delta s - (a(\sigma(t))z^{\Delta}(\sigma(t))\left(\frac{1}{g(x(\delta(t)))}\left(\int_{t_1}^{t}\frac{1}{a(s)}\Delta s\right)^{\Delta}\right)$$

$$-(a(\sigma(t))z^{\Delta}(\sigma(t))\left(\int_{t_1}^{\sigma(t)}\frac{1}{a(s)}\Delta s\left(\frac{1}{g(x(\delta(t)))}\right)^{\Delta}\right),$$

implies that

$$w^{\Delta}(t) \geq (M_1 p(t) + q(t))\int_{t_1}^{t}\frac{1}{a(s)}\Delta s - \left(a(t)z^{\Delta}(t)\right)\left(\frac{1}{a(t)g(x(\delta(t)))}\right) - \left(a(\sigma(t))z^{\Delta}(\sigma(t))\right)\left(\int_{t_1}^{\sigma(t)}\frac{1}{a(s)}\Delta s\left(\frac{1}{g(x(\delta(t)))}\right)^{\Delta}\right)$$

$$= (M_1 p(t) + q(t))\int_{t_1}^{t}\frac{1}{a(s)}\Delta s - \frac{z^{\Delta}(t)}{g(x(\delta(t)))} + \left(a(\sigma(t))z^{\Delta}(\sigma(t))\left(\int_{t_1}^{\sigma(t)}\frac{1}{a(s)}\Delta s\right)\left(\frac{1}{g(x(\delta(t)))}\right)\right)^{\Delta}$$

$$\geq (M_1 p(t) + q(t))\int_{t_1}^{t}\frac{1}{a(s)}\Delta s - \frac{z^{\Delta}(t)}{g(x(\delta(t)))}.$$

Integrating the above inequality from t_1 to t, we get

$$w(t) \geq \int_{t_1}^{t}(Mp(s) + q(s))\int_{t_1}^{s}\frac{1}{a(\tau)}\Delta\tau\Delta s - \int_{t_1}^{t}\frac{z^{\Delta}(s)}{g(x(\delta(s)))}\Delta s. \tag{3.14}$$

As the function $\dfrac{z^{\Delta}(t)}{g(x(\delta(t)))}$ is positive for $t \in [t_1, \infty)_T$, then

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{z^{\Delta}(s)}{g(x(\delta(s)))}\Delta s \quad \text{exists.}$$

Assume that $\lim_{t\to\infty}\int_{t_1}^{t}\dfrac{z^{\Delta}(s)}{g(x(\delta(s)))}\Delta s = k < \infty.$ Taking into account (H_{23}), and from (3.14) we get

$$\limsup_{t\to\infty} w(t) = \infty,$$

which gives a contradiction, because w is negative for all $t \in [t_1, \infty)_T$. Thus,

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{z^{\Delta}(s)}{g(x(\delta(s)))}\Delta s = \infty. \tag{3.15}$$

Consequently,

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{z^{\Delta}(s)}{g(x(\delta(s)))}\Delta s \leq \frac{1}{g(x(\delta(t_1)))}\int_{t_1}^{t}z^{\Delta}(s)\Delta s$$

$$= \frac{1}{g(x(\delta(t_1)))}(z(t) - z(t_1)).$$

From (3.15), we get

$$\lim_{t\to\infty} z(t) = \infty. \tag{3.16}$$

Since $z(t) = x(t) + b(t)x(\alpha(t)) + c(t)x(\beta(t))$ and x is non-negative, we have

$$z(t) \leq (1 + p(t))x(t).$$

Thus from (3.16), we have

$$\lim_{t\to\infty} x(t) = \infty.$$

This completes the proof of the theorem.

We conclude this paper with the following remark.

Remark 3.3: In this paper we obtained conditions for the non-existence of solutions in the classes M^+, M^- and the existence of solutions in the classes OS and WOS. It would be interesting to extend the results of this paper to the following equation

$$(a(t)(x(t)+b(t)x(\alpha(t))+c(t)x(\beta(t)))^{\Delta})^{\Delta}+p(t)f(x(\gamma(t)))+q(t)g(x(\delta(t)))=r(t) \text{ for } t\in[t_0,\infty)_T$$

where $r(t)$ is a rd-continuous function.

REFERENCES

1. M. Bohner, and A. Peterson, *Dynamic Equations on Time Scales: An Introduction With Applications*, Birkhäuser, Boston, 2001.
2. M. Bohner, and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
3. S. Hilger, *Analysis on measure chains: a unified approach to continuousand discrete calculus*, Results. Math. 18 (1990), 18–56.
4. S. Panigrahi, J. R. Graef and P. R. Reddy, *Oscillation Results for fourth order nonlinear neutral dynamic equations*, Commu. Math. Anal., 15 (2013), 11–28.
5. J. R. Graef, S. Panigrahi, and P. Rami Reddy, *On oscillatory and asymptotic behavior of fourth order nonlinear neutral delay dynamic equations with positive and negative coefficients*, Math. Slovaca, 64 (2014), No. 2, 347–366.
6. E. Thandapani, S. Padmavathi, and S. Pinelas, *Classifications of solutions of second-order nonlinear neutral differential equations of mixed type*, Advances in Difference Equations, 226 (2012), (1-13).
7. M. Cecchi, and M. Marini, *Oscillatory and non-oscillatory behavior of a second order functional differential equation*, Rocky Mountain Journal of Mathematics 22 (1992) No. 4, 1259–1276.
8. R. P. Agarwal, M. M. S. Manuel, and E. Thandapani, *Oscillatory and non-oscillatory behavior of second order neutral delay difference equations*, Math. Comput. Modelling 24 (1996) No. 1. 5–11.
9. Sabah Hafez Abdallah, *Oscillatory and non-oscillatory behavior of second order neutral delay differential equations*, Appl. Math. Comput. 135 (2003) 333–344.
10. P. Rami Reddy, N. Sikender, and M. V. Krishna, *Classification of solutions of second order neutral delay dynamic equations* IOSR Jornal of Mathematics (IOSR - JM), e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 5 Ver. I (Sep. - Oct.2016), 72-81.
11. N. Sikender, M. V. Krishna, and P. Rami Reddy *Classification of solutions of second order neutral delay dynamic equations with positive and negative coefficients* International Jornal of Mathematics Trends and Technology (IJMTT), e-ISSN: 2278-5728, Volume 37, Number I- September 2016, 20-28.
12. E. Boe, H. C. Chang, *Dynamics of delayed systems under feedback control*, Chem. Eng. Sci. 44 (1989), 1281-1294.
13. R. D. Driver, *A mixed neutral systems, Nonlinear Analysis*; Theory Methods and Applications (1984), 155-158.
14. J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York,1977
15. E. P. Popov, *Automatic regulation and Control*, Nauka Mascow (in Russian), 1966.
16. Da-Xue Chen, Pei-Xin Qu, Yong-Hong Lan, *Oscillation of second-order nonlinear dynamic equations with positive and negative coefficients*, Advances in Difference Equations 2013, 2013:168.
17. S.Padhi, *Oscillation and asymptotic behavior of solutions of second order neutral differential equation with positive and negative coefficients*, Fasc. Math., 38 (2007), 105-111.
18. S. Padhi, *Oscillation and asymptotic behavior of solutions of second order homogeneous neutral differential equations with positive and negative coefficients*, Funct. Differ. Equ. 14 (2007), 363-371.
19. E. Thandapani, V. Muthulakshmi, John R. Graef, *Classification of solutions of second order nonlinear neutral delay differential equations with positive and negative coefficients*, International J. Pure Appl. Math. 64 No. (2010), 159-170.

Source of support: Nil, Conflict of interest: None Declared.