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OSCILLATION OF SECOND ORDER NONLINEAR NEUTRAL ADVANCED FUNCTIONAL DIFFERENCE EQUATIONS

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ABSTRACT

In this paper we establish some sufficient conditions for the oscillations of all solutions of the second order nonlinear neutral advanced functional difference Equation

$$\Delta\left[r(n)\Delta(x(n)+p(n)x(n+\tau))\right]+q(n)f(n+\sigma))=0: \quad n\geq n_0, \tag{*}$$

where $\sum_{n=n_0}^{\infty}\frac{1}{r(n)}\leq\infty$ *and* $0\leq p(n)\leq p_0<\infty$, T *is an integer, and* σ *is a positive integer. The result proved here improve some known results in the literature.*

Keywords: *Neutral functional difference equation, oscillation and nonoscillation solutions.*

1. INTRODUCTION

In this paper we consider the following second order nonlinear neutral advanced functional difference equations of the form:

$$\Delta\left[r(n)\Delta(x(n)+p(n)x(n+\tau))\right]+q(n)f(x(n+\sigma))=0: \quad n\geq n_0, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, n is a positive integer, n_0 is a fixed positive integer, x is a delay function.

The following conditions are assumed to hold through the paper;

(a) $\{r(n)\}_{n=n_0}^{\infty}$ is a sequence of positive real numbers;

(b) $\{p(n)\}_{n=n_0}^{\infty}$ is a sequence of nonnegative real numbers with the property that $0<p(n)\leq p_0<\infty$;

(c) $\{q(n)\}_{n=n_0}^{\infty}$ is a sequence of nonnegative real numbers and $q(n)$ is not identically zero for large values of n ;

(d) $\frac{f(u)}{u}\geq k>0$ for $u\neq 0$, k is constant;

\and

(e) τ is an integer and σ is a positive integer.

We shall consider the following two cases, separately

$$\sum_{n=n_0}^{\infty}\frac{1}{r(n)}=\infty \tag{2}$$

and

$$\sum_{n=n_0}^{\infty}\frac{1}{r(n)}<\infty. \tag{3}$$

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We note that the second order neutral functional difference equations have applications in problems dealing with vibrating masses attached to elastic ball and in some variational problems. In recent years there has been an increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of different classes of difference equations. We refer [1, 2, 6, 7, 13]. Also the oscillatory behaviour of neutral functional difference equations has been the subject of intensive study, see for example, [3-5, 8-12, 14].

In [12], Zhang *et al.* established that every solution of the equation

$$\Delta[a(n)\Delta(x(n)-px(x-\tau))]+f(n,x(\sigma(n)))=0;\ n\geq n_0 \tag{4}$$

is either oscillatory or eventually
where $p,q,\sigma(n)$ are non-negative constants, f is a bounded set, $a(n)$ is a positive constant.

$$|x(n)|\leq p|x(n-\tau)|$$

if

$$\sum_{s=n}^{\infty}q(s)\sum_{i=0}^{m}p^{i}=\infty,$$

where

$$\frac{f(n,u)}{u}\geq q(n)>0\quad\text{for}\quad u\neq 0.$$

In [3], Budincevic established that every solution of equation

$$\Delta[a(n)\Delta(x(n)+px(n-n_0))]+q(n)f(x(n-m_0))=0 \tag{5}$$

is either oscillatory or else $x(n)\to 0$ as $n\to\infty$.

In [7], Murugesan *et al.* established that every solution of equation (1) is oscillatory if $\tau>0,\ 0\leq p(n)\leq p_0<\infty$ and at least one of the first order advanced difference inequalities

where $Q_1(n),\ Q_2(n)$ are non-negative functions

$$\Delta w(n)-\frac{1}{1+p_0}Q_1(n)w(n+\sigma)\geq 0 \tag{6}$$

$$\Delta w(n)-\frac{1}{1+p_0}Q_2(n)w(n+\sigma)\geq 0 \tag{7}$$

has no positive solution .

In this paper our aim is to obtain sufficient conditions for oscillation of all solution of equation (1) under the condition

$$\sum_{n=n_0}^{\infty}\frac{1}{r(n)}=\infty$$

$$\sum_{n=n_0}^{\infty}\frac{1}{r(n)}<\infty.$$

or

Let n_0 be a fixed nonnegative integer. By a solution of (1) we mean a nontrival real sequence $\{x(n)\}$ which is defined for $n\geq n_0$ and satisfies equation (1) for $n\geq n_0$. A solution $\{x(n)\}$ of (1) is said to be oscillatory if for every positive integer integer $N>0,$ there exists an $n\geq N$ such that $\mathrm{x}(\mathrm{n})x(n+1)\leq 0$ otherwise $\{x(n)\}$ is said to be nonoscillatory. Equation (1) is said to be oscillatory if all it solutions are oscillatory. In the sequel for the sake of convenience, where we write a functional inequality without specifying its domain of validity we assume that it hold for all sufficiently large n

2. MAIN RESULTS

In this section, we establish some new oscillation criteria for oscillatory of equation (1). For the sake of convenience, we define the following notations

$$Q(n):=\min\{q(n),q(n+\tau)\},$$
$$(\Delta\rho(n))_{+}=\max\{0,\Delta\rho(n)\},$$

$$R(n)=\sum_{s=n_0}^{n-1}\frac{1}{r(s)},$$

and $$\delta(n)=\sum_{s=n}^{\infty}\frac{1}{r(s)}.$$

Theorem 2.1: Assume that (2) holds and $\tau > 0$. Moreover suppose that there exist a positive real valued sequence $\{p(n)\}_{n=n_0}^{\infty}$ such that

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[k\rho(s)Q(s)-\frac{((1+p_0)r(s)(\Delta\rho(s))_+)^2}{4\rho(s)}\right]=\infty.$$

Then every solution of equation (1) is oscillatory.

Proof: Assume the contrary. without loss of generality we assume that $\{x(n)\}$ is an eventually positive solution of (1) .Then there exists integer $n_1 \geq n_0$ such that $x(n)>0$ for all $n \geq n_1$. Define

$$z(n)=x(n)+p(n)x(n+\tau). \tag{9}$$

Then $z(n)>0$ for all $n \geq n_1$. From (1), we have

$$\Delta(r(n)\Delta z(n))\leq -kq(n)x(n+\sigma)\leq 0;\quad n\geq n_1 \tag{10}$$

Therefore the sequence $\{r(n)\Delta z(n)\}$ is non-increasing. We claim that

$$\Delta z(n)>0 \text{ for } n\geq n_1. \tag{11}$$

If not, there exists an integer $n_2 \geq n_1$ such that $\Delta z(n_2)<0$. Then from (10), we obtain

$$r(n)\Delta z(n)\leq r(n_2)\Delta z(n_2),\quad n\geq n_2.$$

Here

$$z(n)\leq z(n_2)+r(n_2)\Delta z(n_2)\sum_{s=n_2}^{n-1}\frac{1}{r(s)}.$$

Letting $n\to\infty$ we get $z(n)\to\infty$. This contradiction proves that $\Delta z(n)>0$ for $n \geq n_1$. On the other hand, using the definition of $z(n)$ and applying (1) for all sufficiently large values of n,

$$\Delta[r(n)\Delta z(n)]+kq(n)x(n+\sigma)+p_0\Delta[r(n+\tau)\Delta z(n+\tau)]+p_0kq(n+\tau)x(n+\sigma+\tau)\leq 0$$
$$\Delta[r(n)\Delta z(n)]+p_0\Delta[r(n+\tau)\Delta z(n+\tau)]+kQ(n)z(n+\sigma)\leq 0.$$

Since $\Delta z(n)>0$ we have $z(n+\sigma)\geq z(n)$ and hence

$$\Delta[r(n)\Delta z(n)]+p_0\Delta[r(n+\tau)\Delta z(n+\tau)]+kQ(n)z(n)\leq 0. \tag{12}$$

Define

$$w(n)=\rho(n)\frac{r(n)\Delta z(n)}{z(n)},\quad n\geq n_1 \tag{13}$$

Clearly $w(n)>0$. From (9) we have

$$r(n+1)\Delta z(n+1)\leq r(n)\Delta z(n).$$

From (13), we have

$$\begin{aligned}\Delta w(n)&=\rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n)}-\frac{\rho(n)r(n+1)\Delta z(n+1)}{z(n)z(n+1)}\Delta z(n)+\frac{r(n+1)\Delta z(n+1)}{z(n+1)}\Delta\rho(n)\\ &\leq \rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n)}-\frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n)}+\frac{w(n+1)}{\rho(n+1)}\Delta\rho(n)\\ &\leq \rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n)}-\frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n)}+\frac{(\Delta\rho(n))_+}{\rho(n+1)}w(n+1).\end{aligned} \tag{14}$$

Similarly, we define

$$v(n)=\rho(n)\frac{r(n+\tau)\Delta z(n+\tau)}{z(n)}, \quad n\ge n_1. \tag{15}$$

From (9) we have $r(n+\tau+1)\Delta z(n+\tau+1)\le r(n)\Delta z(n)$.

Using this in (15), we have

$$\Delta v(n)\le \rho(n)\frac{\Delta(r(n+\tau)\Delta z(n+\tau))}{z(n)}-\frac{\rho(n)v^2(n+1)}{\rho^2(n+1)r(n)}+\frac{\Delta\rho(n)}{\rho(n+1)}v(n+1). \tag{16}$$

It follows from (14) and (16), that

$$\begin{aligned}\Delta w(n)+p_0\Delta v(n)&\le \rho(n)\frac{\Delta[r(n)\Delta z(n)]}{z(n)}+p_0\rho(n)\frac{\Delta[r(n+\tau)\Delta z(n+\tau)]}{z(n)}\\&-\frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n)}+\frac{(\Delta\rho(n))_+}{\rho(n+1)}w(n+1)-p_0\frac{\rho(n)v^2(n+1)}{\rho^2(n+1)r(n)}\\&+p_0\frac{(\Delta\rho(n))_+}{\rho(n+1)}v(n+1).\end{aligned}$$

In view of (12) and the above inequality, we obtain

$$\begin{aligned}\Delta w(n)+p_0\Delta v(n)&\le -k\rho(n)Q(n)-\frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n)}+\frac{(\Delta\rho(n))_+}{\rho(n+1)}w(n+1)\\&-p_0\frac{\rho(n)v^2(n+1)}{\rho^2(n+1)r(n)}+p_0\frac{(\Delta\rho(n))_+}{\rho(n+1)}+v(n+1)\\&\le -k\rho(n)Q(n)+\frac{(1+p_0)r(n)((\Delta\rho(s))_+)^2}{4\rho(n)}.\end{aligned}$$

Summing the above inequality from n to $n-1$, we have

$$w(n)+p_0v(n)\le w(n_1)+p_0v(n_1)-\sum_{s=n_1}^{n-1}\left[kp(s)Q(s)-\frac{(1+p_0)r(s)((\Delta\rho(s))_+)^2}{4\rho(s)}\right].$$

It follows that

$$\sum_{s=n_1}^{n-1}\left[k\rho(s)Q(s)-\frac{(1+p_0)r(s)((\Delta\rho(s))_+)^2}{4\rho(s)}\right]\le w(n_1)+p_0v(n_1).$$

which contradicts (8). This completes the proof. Choosing $\rho(n)=R(n+\sigma)$. By Theorem 2.1 we have the following results.

Corollary 2.2: Assume that (2) holds and $\tau>0$. If

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[kR(s+\sigma)Q(s)-\frac{(1+p_0)}{4r(s)R(s+\sigma)}\right]=\infty, \tag{17}$$

then every solution of equation (1) is oscillatory.

Corollary 2.3: Assume that (2) hold and $\tau>0$. If

$$\liminf_{n\to\infty}\frac{1}{\ln R(n+\sigma)}\sum_{s=n_0}^{n}R(s+\sigma)Q(s)>\frac{1+p_0}{4k}, \tag{18}$$

then every solution of equation (1) is oscillatory.

Proof: From (18), there exists an $\varepsilon > 0$ such that for all large n,

$$\frac{1}{\ln R(n+\sigma)}\sum_{s=n_0}^{n-1} R(s+\sigma)Q(s) > \frac{1+p_0}{4k} + \varepsilon .$$

It follows that

$$\sum_{s=n_0}^{n} R(s+\sigma)Q(s) - \left(\frac{1+p_0}{4k}\right)\ln R(s+\sigma) \geq \varepsilon \ln R(n+\sigma).$$

That is $$\sum_{s=n_0}^{n-1}\left[R(s+\sigma)Q(s) - \frac{1+p_0}{4kr(s)R(s+\sigma)}\right] \geq \varepsilon \ln R(n+\sigma) + \left(\frac{1+p_0}{4k}\right)\ln R(n_0+\sigma) \quad (19)$$

Now, it is obvious that (19) implies (17) and the assertion of corollary 2.3 follows from corollary 2.2

Corollary 2.4: Assume that (2) holds and $\tau > 0.$ If

$$\liminf_{n\to\infty}\left[Q(n)R^2(n+\sigma)r(n)\right] > \frac{1+p_0}{4k}, \quad (20)$$

then every solution of (1) is oscillatory.

Proof: It is easy to verify that (20) yields the existence $\varepsilon > 0$ such that for all large n,

$$Q(n)R^2(n+\sigma)r(n) \geq \frac{1+p_0}{4k} + \varepsilon$$

Multiplying by $\dfrac{1}{R(n+\sigma)r(n)}$ on both sides of the above inequality, we have

$$R(s+\sigma)Q(s) - \frac{1+p_0}{4kr(s)R(s+\sigma)} \geq \frac{\varepsilon}{R(s+\sigma)r(s)}$$

which implies that (17) holds. Therefore by Corollary 2.2. every so solution of equation (1) is oscillatory.

Next, choosing $\rho(n) = n.$ By Theorem 2.1 we have the following result.

Corollary 2.5: Assume that (2) holds and $\tau > 0.$ if

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[ksQ(s) - \frac{(1+p_0)r(s)}{4(s)}\right] = \infty, \quad (21)$$

then every solution of equation (1) is oscillatory.

Theorem 2.6: Assume that (3) holds, $\tau > 0$ and $\sigma < \tau$. Suppose also that there exists a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that (8) holds, and

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[k\delta(s+1)Q(s-\tau) - \frac{(1+p_0\delta(s))}{4r(s)\delta^2(s+1)}\right] = \infty \quad (22)$$

Then every solution of equation (1) is oscillatory.

Proof: Assume the contrary. without loss of generality; we may suppose that $\{x(n)\}$ is an eventually positive solution of equation (1) Then there exists an integer $n_1 \geq n_0$ such that $x(n+\sigma-\tau) > 0$ for all $n \geq n_1$. From (1) it follows that $\{r(n)\Delta z(n)\}$ is non-increasing eventually, where $z(n)$ is defined by (9). Consequently, it is easy to conclude that there exist two possible cases of the sign of $\Delta z(n)$, that is, $\Delta z(n) > 0$ or $\Delta z(n) < 0$ eventually. If $\Delta z(n) > 0$ eventually, then we are back to the case of Theorem 2.1 and we can get a contradiction to (8). If $\Delta z(n) < 0$, $n \geq n_2 \geq n_1$, then we define the sequence $\{v(n)\}$ by

$$v(n) = \frac{r(n)\Delta z(n)}{z(n)}, \quad n \geq n_3 = n_2 + \tau. \quad (23)$$

Clearly $v(n)<0.$ Noting that $\{r(n)\Delta z(n)\}$ is n non increasing, we get

$$r(s)\Delta z(s)\le r(n)\Delta z(n),\quad s\ge n\ge n_3.$$

Dividing the above by $r(s)$ and summing it from n to $l-1$, we obtain

$$z(l)\le z(n)+r(n)\Delta z(n)\sum_{s=n}^{l-1}\frac{1}{r(s)}.,\quad l\ge n\ge n_3$$

Letting $l\to\infty$ in the above inequality, we see that

$$0\le z(n)+r(n)\Delta z(n)\delta(n),\quad n\ge n_{3.}.$$

Therefore,

$$\frac{r(n)\Delta z(n)}{z(n)}\delta(n)\ge -1,\quad n\ge n_{3.}.$$

From (23), we have

$$-1\le v(n)\delta(n)\le 0,\quad n\ge n_3 \tag{24}$$

Similarly, we introduce the sequence $\{w(n)\}$ where

$$w(n)=\frac{r(n-\tau)\Delta z(n-\tau)}{z(n)},\quad n\ge n_3 \tag{25}$$

Obviously $w(\mathrm{n})<0.$ Noting that $\{r(n)\Delta z(n)\}$ is non increasing, we have $r(n-\tau)\Delta z(n-\tau)\ge r(n)\Delta z(n)$. Then $w(n)\ge v(n)$. From (24), we have

$$-1\le w(n)\delta(n)\le 0,\quad n\ge n_{3.}. \tag{26}$$

From (24) and (25), we have

$$\Delta v(n)=\frac{\Delta\big(r(n)\Delta z(n)\big)}{z(n)}-\frac{v^2(n)}{r(n)} \tag{27}$$

and

$$\Delta w(n)=\frac{\Delta\big(r(n-\tau)\Delta z(n-\tau)\big)}{z(n)}-\frac{w^2(n)}{r(n)}. \tag{28}$$

In view of (27) and (28) we obtain,

$$\Delta w(\mathrm{n})+\mathrm{p}_0\Delta v(\mathrm{n})\le\frac{\Delta\big(r(n-\tau)\Delta z(n-\tau)\big)}{\Delta z(n)}+p_0\frac{\Delta\big(r(n)\Delta z(n)\big)}{\Delta z(n)}-\frac{w^2(n)}{r(n)}-\frac{v^2(n)}{r(n)}. \tag{29}$$

On the otherhand, proceeding as in the proof of theorem 2.1, we have

$$\Delta\big(r(n-\tau)\Delta z(n-\tau)\big)+p_0\Delta\big(r(n)\Delta z(n)\big)\le -kQ(n-\tau)z(n).$$

Using the above inequality in (29), we have

$$\Delta w(n)+p_0\Delta v(n)\le -kQ(n-\tau)-\frac{w^2(n)}{r(n)}-\frac{v^2(n)}{r(n)}. \tag{30}$$

Multiplying (30) by $\delta(n+1)$ and summing from n_3 to $n-1$, we have

$$\begin{aligned}&\delta(n)w(n)-\delta(n_3+1)w(n_3)+\sum_{s=n_3}^{n-1}\frac{w(s)}{r(s)}+\sum_{s=n_3}^{n-1}\frac{w^2(s)}{r(s)}\delta(s+1)\\&+p_0\delta(n)v(n)-p_0\delta(n_3+1)v(n_3)+p_0\sum_{s=n_3}^{n-1}\frac{v(s)}{r(s)}\\&+p_0\sum_{s=n_3}^{n-1}\frac{v^2(s)}{r(s)}\delta(s+1)+k\sum_{s=n_3}^{n-1}Q(s-\tau)\delta(s+1)\le 0.\end{aligned}$$

From the above inequality, we have

$$\delta(n)w(n)-\delta(n_3+1)w(n_3)+p_0\delta(n)v(n)-p_0\delta(n_3+1)v(n_3)$$
$$+k\sum_{s=n_3}^{n-1}\delta(s+1)Q(s-\tau)-\frac{(1+p_0)}{4}\sum_{s=n_3}^{n-1}\frac{\delta(s)}{r(s)\delta^2(s+1)}\le 0.$$

Thus, it follows from the above inequality that

$$\delta(n)w(n)+p_0\delta(n)v(n)+\sum_{s=n_3}^{n-1}\left[k\delta(s+1)Q(s-\tau)-\frac{(1+p_0)\delta(s)}{4r(s)\delta^2(s+1)}\right]$$
$$\le \delta(n_3+1)w(n_3)+p_0\delta(n_3+1)v(n_3).$$

By (24) and (26), we get a contradiction with (22) and this completes the proof.

Corollary 2.7: Assume that (3) holds, $\tau>0$ and $\sigma\le\tau.$ Further assume that (22) one of the conditions (17), (18), (20) and (21) holds. Then every solution of equation (1) is oscillatory**.**

Theorem 2.8: Assume that (3) holds, $\tau>0$ and $\sigma\le\tau.$ Furthermore suppose that there exists a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that (8) holds, and

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\delta^2(s+1)Q(s-\tau)=\infty \tag{31}$$

Then every solution of equation (1) is oscillatory.

Proof: Assume the contrary. Without loss of generality, we assume that $\{x(n)\}$ is an eventually positive solution of equation (1). Then there exists an integer $n_1\ge n_0$ such that $x(n-\tau)>0$ for all $n_0\ge n_1$. By (1), $\{r(n)\Delta z(n)\}$ is non-increasing, eventually where $z(n)$ is defined by (9) . Consequently, it is easy to conclude that there exist two possible cases of the sign of $\Delta z(n)$ that is, $\Delta z(n)>0$ or $\Delta z(n)<0$ eventually. If $\Delta z(n)<0$, $n\ge n_2\ge n_1$, then we define $w(n)$ and $v(n)$ as in theorem 2.6, Then proceed as in the proof of theorem 2.6, we obtain (24), (26) and (30) Multiplying (30) by $\delta^2(n+1)$, and summing from n_3 to $n-1$, where $n_3\ge n+\tau,$ we get

$$\delta^2(n)w(n)-\delta^2(n_3+1)w(n_3+1)+2\sum_{s=n_3}^{n-1}\frac{w(s)\delta(s+1)}{r(s+1)}+p_0\delta^2(n)v(n)$$
$$-p_0\delta^2(n_3+1)v(n_3)+2\sum_{s=n_3}^{n-1}\frac{v(s)\delta(s+1)}{r(s+1)}+\sum_{s=n_3}^{n-1}\frac{w^2(s)\delta^2(s+1)}{r(s)}$$
$$+p_0\sum_{s=n_3}^{n-1}\frac{v^2(s)}{r(s)}\delta^2(s+1)+k\sum_{s=n_3}^{n-1}Q(s-\tau)\delta^2(s+1)\le 0. \tag{32}$$

It follows from (3) and (24) that

$$\left|\sum_{s=n_3}^{\infty}\frac{w(s)\delta(s+1)}{r(s+1)}\right|\le\sum_{s=n_3}^{\infty}\frac{|w(s)\delta(s+1)|}{r(s+1)}\le\sum_{s=n_3}^{\infty}\frac{1}{r(s+1)}<\infty,$$

and $$\sum_{s=n_3}^{n-1}\frac{w^2(s)\delta^2(s+1)}{r(s)}\le\sum_{s=n_3}^{\infty}\frac{1}{r(s)}<\infty.$$

In view of (26), we have

$$\left|\sum_{s=n_3}^{\infty}\frac{v(s)\delta(s+1)}{r(s+1)}\right|\le\sum_{s=n_3}^{\infty}\frac{|v(s)\delta(s+1)|}{r(s+1)}\le\sum_{s=n_3}^{\infty}\frac{1}{r(s+1)}<\infty,$$
$$\sum_{s=n_3}^{\infty}\frac{v^2(s)\delta^2(s+1)}{r(s)}\le\sum_{s=n_3}^{\infty}\frac{1}{r(s)}<\infty.$$

From (32), we get

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\delta^2(s+1)Q(s-\tau)<\infty,$$

which is a contradiction with (31). This completes the proof.

Corollary 2.9: Assume that (3) holds, $\tau>0$ and $\sigma\le\tau$ Suppose se also that one of conditions (17), (18) and (20) and (21) holds, also condition (31) holds The every solution of equation (1) is oscillatory.

Theorem 2.10: Assume that (2) holds and $\tau<0$. Also, suppose that there exist a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[k\rho(s)Q(s)-\frac{(1+p_0)r(s+\tau)\left(\Delta\rho(s)_+\right)^2}{4\rho(s)}\right]=\infty. \tag{33}$$

Then every solution of equation (1) is oscillatory.

Proof: Assume the contrary. Without loss of generality we may assume that $\{x(n)\}$ is an eventually positive solution of equation (1). Then there exists an integer $n_1\ge n_0$ such that $x(n-\tau)>0$ for all $n\ \ge n.$ Define $z(n)$ by (9). Similar to the proof of Theorem 2.1, there exists an integer $n_2\ge n_1$ such that $\Delta z(n+\tau)>0$ for $n\ \ge n_2$ and

$$\Delta\left[r(n)\Delta z(n)\right]+p_0\left[r(n+\tau)\Delta z(n+\tau)\right]+kQ(n)z(n+\tau)\le 0\quad fo\ \ r\ n\ge n_2. \tag{34}$$

Define the sequence $\{w(n)\}$ by

$$w(n)=\rho(n)\frac{r(n)\Delta z(n)}{z(n+\tau)},\ n\ge n_2. \tag{35}$$

Then $w(n)>0$. By (10) we get

$$r(n+1)\Delta z(n+1)\le r(n+\tau)\Delta z(n+\tau),$$

and

$$\begin{aligned}\Delta w(n)&\le\rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n+\tau)}-\frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n+\tau)}+\frac{w(n+1)}{\rho(n+1)}\Delta\rho(n)\\&\le\rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n+\tau)}-\frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n+\tau)}+\frac{w(n+1)}{\rho(n+1)}(\Delta p(n))_+\end{aligned} \tag{36}$$

Again define the sequence $\{v(n)\}$ by

$$v(n)=\rho(n)\ \frac{r(n+\tau)\Delta z(n+\tau)}{z(n+\tau)},\quad n\ge n_2. \tag{37}$$

The rest of the proof is similar to that of Theorem 2.1 and so is omitted. This completes the proof.

Choosing $\rho(n)=R(n+\tau)$. By Theorem 2.10 we have the following oscillation criteria.

Corollary 2.11: Assume that (2) holds and $\tau<0$. If

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[kR(s+\tau)Q(s)-\frac{1+p_0}{4r(s+\tau)R(s+\tau)}\right]=\infty. \tag{38}$$

Then every solution of equation (1) is oscillatory.

Corollary 2.12: Assume that (2) holds and $\tau<0$. If

$$\liminf_{n\to\infty}\frac{1}{\ln R(n+\tau)}\sum_{s=n_0}^{n-1}R(s+\tau)Q(s)>\frac{1+p_0}{4k}, \tag{39}$$

Then every solution of equation (1) is oscillatory.

Corollary 2.13: Assume that (2) holds and $\tau < 0$. If

$$\liminf_{n\to\infty}\left[Q(n)R^2(n+\tau)r(n+\tau)\right] > \frac{1+p_0}{4k}, \tag{40}$$

then every solution of equation (1) is oscillatory.

Corollary 2.14: Assume that (2) holds and $\tau < 0$. If

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[ksQ(s) - \frac{(1+p_0)r(s+\tau)}{4s}\right] = \infty, \tag{41}$$

then every solution of equation (1) is oscillatory.

Next, we will give the following results under the case when (3) holds and $\sigma < \tau$.

Theorem 2.15: Assume that (3) holds, $\tau < 0$ and $\sigma \le -\tau$. Further, suppose that there exists a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that (33) holds. Suppose also that one of the following holds:

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[k\delta(s+1)Q(s+\tau) - \frac{(1+p_0)\delta(s)}{4r(s)\delta^2(s+1)}\right] = \infty, \tag{42}$$

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\delta^2(s+1)Q(s+\tau) = \infty. \tag{43}$$

Then every solution of equation (1) is oscillatory.

Proof: Assume the contrary. Without loss of generality. we may assume that $\{x(n)\}$. is an eventually positive solution of equation (1). Then there exists an integer $n_1 \ge n_0$ such that $x(n) > 0$ for all $n \ge n_1$. From (1) $\{r(n)\Delta z(n)\}$ is non-increasing eventually where $z(n)$ is defined by (9). Consequently, it is easy to conclude that there exist two possible cases of sign of $\Delta z(n)$. That is, $\Delta z(n) > 0$ or $\Delta z(n) < 0$ eventually. If $\Delta z(n) > 0$ eventually, then we are back of the case of theorem 2.10 and we can get a contradiction to (33). If $\Delta z(n) < 0$ eventually, then there exists an integer $n_2 \ge n_1$ such that $\Delta z(n) < 0$ for all $n \ge n_2$.

Define the sequences $\{w(n)\}$ and $\{v(n)\}$ as follows:

$$w(n) = \frac{r(n+\tau)\Delta z(n+\tau)}{z(n)} \quad \text{for} \quad n \ge n_3 = n_2 + 2\tau$$

and

$$v(n) = \frac{r(n+2\tau)\Delta z(n+2\tau)}{z(n)} \quad \text{for} \quad n \ge n_3 = n_2 + 2\tau$$

The rest of the proof can be proceed as in theorem 2.6 or theorem 2.8 we can obtain a contradiction to (42) or (43) respectively. This completes the proof.

3. EXAMPLES

Example 3.1: Consider the second order neutral advanced difference equation

$$\Delta\left[(n+\tau)\Delta x(n) + p(n)x(n+\tau)\right] + \tfrac{\lambda}{n} f\left(x(n+\sigma)\right) = 0, \; n = 1,2,3,..., \tag{44}$$

where $r(n) = n+\tau$, τ and σ are positive integers, $\rho(n) = n+\tau$, $f(x) = x(1+x^2)$.

$0 \le p(n) \le p_0 < \infty$, $q(n) = \dfrac{\lambda}{n}$ and $\lambda > 0$. Let $k = 1$. Then

$$\limsup_{n\to\infty}\sum_{s=1}^{n} kp(s)Q(s) - \frac{(1+p_0)r(s)\left((\Delta\rho(s))_+\right)^2}{4\rho(s)} = \limsup_{n\to\infty}\sum_{s=1}^{n}\left[\lambda - \frac{(1+p_0)}{4}\right] = \infty, \text{ for } \lambda > \frac{1+p_0}{4}.$$

Hence by theorem 2.1, every solution of equation (44) is oscillatory for $\lambda > \dfrac{1+p_0}{4}$.

Example 3.2: Consider the following second order neutral advanced difference equation

$$\Delta\left[\left(n-\tau\right)\Delta x\left(n\right)+p\left(n\right)x\left(n+\tau\right)\right]+q\left(n\right)\,f\left(x\left(n+\sigma\right)\right)=0, \qquad n\geq 1, \tag{45}$$

where $r(n)=n-\tau$, $q(n)=\dfrac{\lambda}{n}$, $0\leq p(n)\leq p_0<\infty$, τ is a negative integer, σ is a positive integer, and $f(x)=x(1+x^2)$ and $\lambda>0.$ It is easy to see that
Hence, by Corollary 2.14 every solution of equation (45) is oscillatory.

Example 3.3: Consider the following second order neutral advanced difference equation

$$\Delta\left[e^{n}\left(x\left(n\right)+p\left(n\right)x\left(n+2\right)\right)\right]+e^{2n}f\left(x\left(n+1\right)\right)=0, \qquad n\geq 1, \tag{46}$$

where $0\leq p(n)\leq p_0<\infty$, $f(x)=x(1+x^2)$, $r(n)=e^{n}$, $q(n)=e^{2n}$. It is easy to see that

$$\sum_{n=1}^{\infty}\frac{1}{r(n)}<\infty .$$

Choose $k=1$

$$\limsup_{n\to\infty}\sum_{s=1}^{n}\delta^{2}\left(s+1\right)Q\left(s-2\right)=\limsup_{n\to\infty}\sum_{s=1}^{n}\frac{1}{e^{2n}\left(e-1\right)^{2}}e^{2(n-2)}=\infty$$

Also

$$\liminf_{n\to\infty}\left[Q\left(n\right)R^{2}\left(n+1\right)r\left(n\right)\right]=\liminf_{n\to\infty}\left[e^{2n}\left(\frac{e^{n}-1}{e^{n}\left(e-1\right)}\right)^{2}e^{n}\right]>\frac{1+p_0}{4}$$

Then by Corollary 2.9 every solution of equation (46) is oscillatory.

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