International Journal of Mathematical Archive-9(2), 2018, 257-260
Available online through www.ijma.info ISSN 2229 – 5046

CHARACTERIZATION OF A FOUR-DIMENSIONAL LORENTZIAN MANIFOLDS USING JACOBI OPERATOR

Prof. Dr. Sci. VESELIN TOTEV VIDEV*

Dept. Mathematics and Informatics,
Trakia University, 6000 Stara Zagora, Bulgaria, Europe Union.

(Received On: 27-11-17; Revised & Accepted On: 18-01-18)

ABSTRACT

In the present note we characterize a four-dimensional Lorentzian manifolds using characteristic coefficients of the Jacobi operator.

Mathematical subject classification: *53 B 20.*

Keywords: *Jacobi operator, Lorentzian manifolds, reducible space, space of maximal mobility.*

An n-dimensional Riemannian manifold M with metric g is called a Lorentzian manifold if at any point $p\in M$, the tangent space M_p to the manifold is an n-dimensional vector space with signature (-,+,...,+) or (+,+,...,+,-). An unit tangent vector X is called spacelike tangent vector if $g(X, X) = 1$ and X is called timelike tangent vector if $g(X, X) = -1$. The set of all spacelike unit tangent vectors in the tangent space M_p we denote by S^+_pM, and the set of all unit timelike tangent vectors in M_p, we denote by S^-_pM. If ∇ is the Levi-Civita connection induced by g, then the curvature tensor R of type (1, 3), on the manifold M, is defined by the equality

$$R(x, y, z) = \nabla_x z + \nabla_y z - \nabla_{[x,y]} z\,,$$

where $x, y, z\in M_p$, $p\in M$, and [.,.] are the Lee brackets. Using this tensor we define the curvature tensor of type (0, 4) in the following way:

$$R(x, y, z, u) = g\big(R(x, y, z), u\big)\,.$$

The curvature tensor R has the following properties:

$$R(x, y, z, u) = - R(y, x, z, u) = - R(x, y, u, z),$$
$$R(x, y, z, u) + R(y, z, x, u) + R(z, x, y, u) = 0,$$
$$R(x, y, z, u) = R(z, u, x, y),$$
$$\sigma_{xyz}(\nabla_x R)(y, z, u) = 0,$$

where $x, y, z, u\in S^{\pm}_pM$, and σ is a cyclic sum over x, y, z. The Ricci tensor ρ on the manifold M is a bilinear symmetric function defined by the equality:

$$\rho(x, y) = \operatorname{trace}\big(z\to R(z, x, y)\big),$$

where $x, y, z\in S^{\pm}_pM$, $p\in M$. Any Lorentzian manifold M with the property

$$\rho(x, y) = \lambda g(x, y),$$

λ=const., $x, y\in S^{\pm}_pM$ is called Einstein Lorentzian manifold [1].

Let M be a four-dimensional Lorentzian manifold and let e_1, e_2, e_3, e_4 ($e_4\in S^-_pM$) be an arbitrary Lorentzian basis in the tangent space M_p, at a point $p\in M$. A bivector space $\wedge^2 M_p$ is a 6-dimensional vector space of signature (+,+,+,-,-,-), in which $e_1\wedge e_2$, $e_1\wedge e_3$, $e_1\wedge e_4$, $e_3\wedge e_4$, $e_4\wedge e_2$, $e_2\wedge e_3$ is an orthonormal basis, where $\wedge$ is the second exterior product in M_p, $p\in M$[1].

Corresponding Author: Prof. Dr. Sci. Veselin Totev Videv*
Dept. Mathematics and Informatics, Trakia University, 6000 Stara Zagora, Bulgaria, Europe Union.

Proposition 1[4]: *Let M be a four-dimensional Einstein Lorentzian manifold. Then at any point* $p\in M$*, there exist a Lorentzian basis* $e_1, e_2, e_3, e_4 (e_4\in S^{-}_{p}M)$ *in the tangent space* M_p*, such that the matrix of the curvature operator* $\mathfrak{R}$ *in bivector space* $\wedge^2 M_p$*, with respect to the orthonormal basis* $e_1\wedge e_2, e_1\wedge e_3, e_1\wedge e_4, e_3\wedge e_4, e_4\wedge e_2, e_2\wedge e_3$*, has the form*:

$$\begin{pmatrix} \mathcal{M} & \mathcal{N} \\ -\mathcal{N} & \mathcal{M} \end{pmatrix},$$

where M and N are one of the following three types:

$$\mathcal{M} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}, \qquad \text{I}$$

$$\alpha_1+\alpha_2+\alpha_3=\lambda \quad ,$$
$$\beta_1+\beta_2+\beta_3=0 \quad ;$$

$$\mathcal{M} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2+1 & 0 \\ 0 & 0 & \alpha_2-1 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 1 \\ 0 & 1 & \beta_2 \end{pmatrix}, \qquad \text{II}$$

$$\alpha_1+2\alpha_2=\lambda \quad ,$$
$$\beta_1+2\beta_2=0 \quad ;$$

$$\mathcal{M} = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad 3\alpha=\lambda . \qquad \text{III}$$

We call this basis Petrov basis of type I, II or III.

The Jacobi operator R_X is a symmetric linear operator in the tangent space M_p, at a point $p\in M$, defined by the equality [3]:

$$R_X(u)=R(u, X, X) , \qquad X\in S^{\pm}_{p}M .$$

Since X is an eigenvector of R_X, with the corresponding eigenvalue 0, then the characteristic equation of R_X has the form:

$$c(c^{n-1} - J_1 c^{n-1} + J_2 c^{n-2} + ... + (-1)^{n-2} J_{n-2} c + (-1)^{n-1} J_{n-1}) = 0 , \qquad (1)$$

where

$$J_1(p; X) = \rho(X,X), \qquad X\in S^{\pm}_{p}M. \qquad (2)$$

If trace $J_1(p; X)$ is a pointwise constant, for any tangent vector $X\in S^{\pm}_{p}M$, at any point $p\in M$, then from (2) it follows that M is an Einstein Lorentzian manifold. An n-dimensional Lorentzian manifold M is called Osserman Lorentzian manifold if at any point $p\in M$, the characteristic coefficients of the Jacobi operator R_X are a constants for any tangent vector $X\in S^{\pm}_{p}M$, at any point $p\in M$[2].

Proposition 2[2]: *An n-dimensional* ($n \geq 3$) *Lorentzian manifold M is an Osserman manifold if and only if M is a space of constant sectional curvature.*

Further we consider the case when M is a four-dimensional Lorentzian manifold, then the characteristic equation of the Jacobi operator has the form

$$c(c^3 - J_1 c^2 + J_2 c - J_3) = 0 \quad .$$

If the characteristic coefficient $J_3(p; X) = 0$, for any tangent vector $X\in S^{\pm}_{p}M$, at any point $p\in M$, then we have the following:

Theorem: *M is a four-dimensional Einstein Lorentzian manifold such that the characteristic coefficient* $J_3(p; X) = 0$*, for any Jacobi operator* R_X*,* $X\in S^{\pm}_{p}M$*, at any point* $p\in M$*, if and only if one of the following cases is true*:

a) *M is a space of constant sectional curvature*;
b) *M is a reducible space with metric which is reduce to the following two quadratic forms*:

$$ds^2 = dx_1^2 + \cos^2\left(\sqrt{\lambda}x_1\right)dx_2^2 + dx_3^2 - \cos^2\left(\sqrt{\lambda}x_3\right)dx_4^2 ; \quad \lambda>0 \qquad (3)$$

$$ds^2 = dx_1^2 + ch^2\left(\sqrt{-\lambda}x_1\right)dx_2^2 + dx_3^2 - ch^2\left(\sqrt{-\lambda}x_3\right)dx_4^2 ; \quad \lambda<0 \qquad (4)$$

c) *M is a space of maximal mobility with metric*:

$$ds^2 = dx_1^2 + sh^2\left(x_1 - x_4\right)dx_2^2 + \sin\left(x_1 - x_4\right)dx_3^2 - dx_4^2. \quad (5)$$

Proof: Let M be an Einstein Lorentzian manifold and let e_1, e_2, e_3, e_4 ($e_4 \in S^-_pM$) be a Petrov basis of type I. If a and b are an arbitrary real numbers with the property $a^2 - b^2 = 1$, then the orthonormal basis

$$ae_1+be_4,\ be_1+ae_4,\ e_2,\ e_3 \quad (6)$$

is a Lorentzian basis in M_p. Using the characteristic equation of the Jacobi operator $R_{ae_1+be_2}$, with respect to this basis, we obtain:

$$J_3(p; ae_1+be_2) = \alpha_3\left(\alpha_1\alpha_2 - a^2b^2\left(\left(\alpha_1-\alpha_2\right)^2 + \left(\beta_1-\beta_2\right)^2\right)\right) = 0, \quad (7)$$

and from here, at a=1, and b=0, we get

$$J_3(p;e_1) = \alpha_1\alpha_2\alpha_3 = 0. \quad (8)$$

If $\alpha_1=\alpha_2=\alpha_3=0$, then M is flat. If at least one of α_1, α_2, α_3 is different from zero, suppose α_3, then from (8) it follows that $\alpha_1\alpha_2=0$, and then from (7) we obtain

$$a^2b^2\left(\left(\alpha_1-\alpha_2\right)^2 + \left(\beta_1-\beta_2\right)^2\right) = 0.$$

From here it follows that $\alpha_1=\alpha_2=0$, $\beta_1=\beta_2$ and using second property of the curvature tensor R, we obtain $\beta_3=-2\beta_1$. That means that for the invariants of the Petrov basis of type I, we have:

$$\alpha_1 = const.,\quad \alpha_2=\alpha_3=0,\quad \beta_1=\beta_2,\quad \beta_3=-2\beta_1. \quad (9)$$

Let $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6$ be an eigenvector basis of the curvature operator $\mathfrak{R}$ in $\wedge^2M_p$, and let $k_1, k_2, k_3, \bar{k}_1, \bar{k}_2, \bar{k}_3$ are the corresponding eigenvalues. Let $k_j = \alpha_j + i\beta_j$, where $i^2=-1$, and j=1,2,3. From (9) it follows that the matrix of $\mathfrak{R}$, with respect to $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6$ has the form:

$$(\mathfrak{R}) = \begin{pmatrix} \alpha_1-2i\beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2i\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i\beta_1-\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\beta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2i\beta_1 \end{pmatrix}.$$

Since the set of $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6$ is a reducible basis, then there exist a Lorentzian basis v_1, v_2, v_3, v_4 ($v_4 \in S^-_pM$) in M_p, $p\in M$, with respect to which all non-zero curvature components are:

$$R(v_1, v_2, v_2, v_1) = -R(v_3, v_4, v_4, v_3) = \alpha_1 - 2i\beta_1,$$
$$R(v_1, v_3, v_3, v_1) = -R(v_2, v_4, v_2, v_4) = i\beta_1,$$
$$R(v_2, v_3, v_3, v_2) = -R(v_1, v_4, v_4, v_1) = -2i\beta_1.$$

Using the characteristic equation of the Jacobi operators R_{v_2} with respect to v_1, v_2, v_3, v_4 ($v_4 \in S^-_pM$) we obtain that R_{v_2} has an eigenvalues $\dfrac{-2i\beta_1}{g\left(v_1,v_1\right)}$, $\dfrac{2i\beta_1}{g\left(v_3,v_3\right)}$, $\dfrac{2i\beta_1}{g\left(v_4,v_4\right)}$.

Using that R_{v_2} is diagonalizable and under the assumption $J_3(p;v_2)=0$, we get $\beta_1=0$ and according to (9), for the invariants of the Petrov basis of type I we have

$$\alpha_1 = \text{const., and}\quad \alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=0. \quad (10)$$

If $\alpha_1=0$, then M is flat. If $\alpha_1\neq 0$, then M is a reducible space with a metric form given by (3) and (4) [4]. Conversely if M is a reducible Einstein Lorentzian manifold, such that at any point $p\in M$, there exist a Petrov basis in the tangent space M_p, which invariants fulfill (10). If $X\in S_p^{\pm}M$ is an arbitrary tangent vector in the tangent space M_p, at a point $p\in M$, for which

$$X = \sum_{i=1}^{4} a_i e_i \quad , \tag{11}$$

where a_i are an arbitrary real numbers, then the characteristic equation of R_X, has the form:

$$c^2\left(c^2 - \alpha_3 c + \left(a_1^2 - a_4^2\right)\left(a_2^2 + a_3^2\right)\left(9\beta_1^2 + \alpha_3^2\right)\right) = 0,$$

and from here it follows that $J_3(p;X) = 0$.

If e_1, e_2, e_3, e_4 ($e_4 \in S^{-}_{p}M$) is a Petrov basis of type II, then using the characteristic equations of the Jacobi operators $R_{ae_1+be_4}$ and $R_{ae_2+be_4}$, with respect to this basis, and the conditions $J_3(p;ae_1+be_4) = J_3(p;ae_2+be_4) = 0$, we obtain the system:

$$(\alpha_2-1)\alpha_2^2 + (\alpha_2+1)9\beta_2^2 = 0,$$
$$(\alpha_2+1)\alpha_2^2 + (\alpha_2-1)9\beta_2^2 = 0.$$

From here it follows that $\alpha_2=\beta_2=0$ and then using the characteristic equation of the Jacobi operator R_{e_1} with respect to the same basis, and the condition $J_3(p;e_1) = 0$, we obtain $\alpha_1=0$. Since $\beta_2=0$, then from the second property of the curvature tensor R, we get $\beta_1=0$. That means that for the invariants of the Petrov basis of type II, holds $\alpha_1=\alpha_2=\beta_1=\beta_2=0$, which means that M is a space of maximal mobility with metric of the form (5)[4]. Conversely if M is a four-dimensional Lorentzian manifold, with metric of the form (5), then for any tangent vector $X \in S^{\pm}_{p}M$, given by (11), and for the corresponding Jacobi operator R_X, we have:

$$J_3(p;X) = \begin{vmatrix} (a_3+a_4)^2 & 0 & -a_1(a_3+a_4) & -a_1(a_3+a_4) \\ 0 & -(a_3+a_4)^2 & -a_2(a_3+a_4) & a_2(a_3+a_4) \\ -a_1(a_3+a_4) & -a_2(a_3+a_4) & a_1^2 - a_2^2 & a_2^2 - a_1^2 \\ -a_1(a_3+a_4) & -a_2(a_3+a_4) & a_1^2 - a_2^2 & a_2^2 - a_1^2 \end{vmatrix} = 0.$$

Finally if e_1, e_2, e_3, e_4 ($e_4 \in S^{-}_{p}M$) is a Petrov basis of type III, then using characteristic equation of the Jacobi operator R_{e_1}, with respect to this basis, we obtain $J_3(p;e_1) = 0$, which means that M must to be a symmetric space, and which according to the results of Petrov is impossible[4].

REFERENCES

1. Besse A., *Geometrie Riemanniene et dimension* 4, Cedic/Fernand Nathan Paris (1981)
2. Blazic N., Gilkey P., Bokan N., A note of Osserman Lorentzian manifolds, Bull. Lond. Math. Soc. 28 (1997), 227-230
3. Chi Q.-Sh., A curvature characterization of certain locally rank-one symmetric spaces, J. Diff. Geom. 28 (1988), 187-202
4. Petrov A.Z., Einstein spaces, Oxford: Pergamon Press, (1969).

Source of support: Nil, Conflict of interest: None Declared.