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AN ANALYTICAL TECHNIQUE FOR FINDING EXACT SOLUTION OF TWO-DIMENSIONAL DIFFUSION EQUATIONS

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ABSTRACT

The variational iteration method (VIM) is a powerful tool for solving large amount of problems. In this article, the variational iteration method has been used to obtain exact solutions of the two-dimensional diffusion equation. The idea of variational iteration method was first introduced by He in 1997 [1]. The variational iteration method, a correction functional is constructed by a general Lagrange multiplier which can be identified via a variational theory. The variational iteration method has successfully been applied to many situations [2-5]. Tow illustrative examples are given to demonstrate the effectiveness of the present method.

Keywords and Phrases: Homotopy perturbation method; Two-dimensional diffusion equation

Mathematics Subject Classification: *47J30, 49S05, 35G25.*

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1. INTRODUCTION

Consider the following two-dimensional diffusion equation with the following initial condition

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{1.1}$$

With initial condition

$$u(x,y,0)=f(x,y), \qquad 0\le x,y\le 1. \tag{1.2}$$

In this article, we use the variational iteration method to solve this kind of equations. To illustrate its basic idea of the method, we consider the following general nonlinear equation

$$Lu(t)+Nu(t)=g(t). \tag{1.3}$$

Where, L is a linear operator, N is a nonlinear operator and $g(t)$ an inhomogeneous term.

According to the variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(\xi)\left(Lu_n(\xi)+N\tilde{u}_n(\xi)-g(\xi)\right)d\xi, \tag{1.4}$$

Where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n=0$ [6].

It is obvious now that the main steps of He's variational iteration method require first the determination of the Lagrangian multiplier λ that will be identified optimally. Having determined the Lagrange multiplier, the successive approximations $u_{n+1}, n\ge 0$. Of the solution u will be readily obtained upon using any selective

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function u_0. Consequently, the solution

$$u(x)=\lim_{n\to\infty}u_n(x). \tag{1.5}$$

In other words, the correction functional (1.4) will give several approximations, and therefore the exact solution is obtained at the limit of the resulting successive approximations.

2. ANALYSIS OF TWO-DIMENSIONAL DIFFUSION EQUATION:

Now for solving Eq. (1.1), by VIM, we construct a correction functional in the following form

$$u_{n+1}(x,y,t)=u_n(x,y,t)+\int_0^t\lambda(\xi)\left\{\frac{\partial u_n}{\partial\xi}(x,y,\xi)-\alpha\left(\frac{\partial^2\tilde{u}_n}{\partial x^2}(x,y,\xi)+\frac{\partial^2\tilde{u}_n}{\partial y^2}(x,y,\xi)\right)\right\}d\xi, \tag{2.1}$$

After some calculations, we obtain the following stationary conditions:

$$\lambda'(\xi)=0,\quad 1+\lambda(\xi)\big|_{\xi=x}=0, \tag{2.2}$$

The Lagrange multiplier, can be easily identified as $\lambda=-1$.

Substituting the identified multiplier in to equation (2.1), we would have the following iteration formula

$$u_{n+1}(x,y,t)=u_n(x,y,t)-\int_0^t\left\{\frac{\partial u_n}{\partial\xi}(x,y,\xi)-\alpha\left(\frac{\partial^2 u_n}{\partial x^2}(x,y,\xi)+\frac{\partial^2 u_n}{\partial y^2}(x,y,\xi)\right)\right\}d\xi, \tag{2.3}$$

We start with the initial approximation of $u(x,y,0)$ given by Eq. (1.2). Using the above iteration formula, we can obtain the other components as follows:

$$\begin{aligned}
&u_0(x,y,t)=u(x,y,0)=f(x,y),\\
&u_1(x,y,t)=u_0(x,y,t)-\int_0^t\left\{\frac{\partial u_0}{\partial\xi}(x,y,\xi)-\alpha\left(\frac{\partial^2u_0}{\partial x^2}(x,y,\xi)+\frac{\partial^2u_0}{\partial y^2}(x,y,\xi)\right)\right\}d\xi,\\
&u_2(x,y,t)=u_1(x,y,t)-\int_0^t\left\{\frac{\partial u_1}{\partial\xi}(x,y,\xi)-\alpha\left(\frac{\partial^2u_1}{\partial x^2}(x,y,\xi)+\frac{\partial^2u_1}{\partial y^2}(x,y,\xi)\right)\right\}d\xi,\\
&\vdots
\end{aligned} \tag{2.4}$$

3. ILLUSTRATIVE EXAMPLES:

Example: 1 For the first test problem consider (1.1) with $\alpha=1$ and $f(x,y)=\exp(x+y)$, the exact solution is given with

$$u(x,y,t)=\exp(x+y+2t). \tag{3.1}$$

For solving by the variational iteration method we obtain the recurrence relation

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t \left\{ \frac{\partial u_n}{\partial \xi} - \left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} \right) \right\} d\xi, \tag{3.2}$$

Let us start with an initial approximation $u_0(x,y,t) = f(x,y) = \exp(x+y)$. applying the iteration formula (2.4). We can obtain directly the other components as

$$\begin{aligned}
&u_1(x,y,t) = \exp(x+y) + 2t\exp(x+y),\\
&u_2(x,y,t) = \exp(x+y) + 2t\exp(x+y) + 2t^2\exp(x+y),\\
&u_3(x,y,t) = \exp(x+y) + 2t\exp(x+y) + 2t^2\exp(x+y) + \frac{4}{3}t^3\exp(x+y),\\
&\vdots\\
&u_n(x,y,t) = \sum_{k=0}^{n} \frac{(2t)^k}{k!}\exp(x+y),\\
&\vdots
\end{aligned} \tag{3.3}$$

And exact solution will be as

$$u(x,y,t) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(2t)^k}{k!}\exp(x+y) = \exp(x+y+2t). \tag{3.4}$$

Example: 2 For the first test problem consider (1.1) with $\alpha = 1$ and $f(x,y) = (1+y)\exp(x)$, the exact solution is given with

$$u(x,y,t) = (1+y)\exp(x+t). \tag{3.5}$$

For solving by the variational iteration method we obtain the recurrence relation

$$\begin{aligned}
u_{n+1}(x,y,t) = {} & u_n(x,y,t)\\
& - \int_0^t \left\{ \frac{\partial u_n}{\partial \xi} - \left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} \right) \right\} d\xi,
\end{aligned} \tag{3.6}$$

Let us start with an initial approximation $u_0(x,y,t) = f(x,y) = (1+y)\exp(x)$. applying the iteration formula (2.4). We can obtain directly the other components as

$$\begin{aligned}
&u_1(x,y,t) = (1+y)\exp(x) + (1+y)\exp(x)t,\\
&u_2(x,y,t) = (1+y)\exp(x) + (1+y)\exp(x)t + \frac{1}{2}(1+y)\exp(x)t^2,\\
&u_3(x,y,t) = (1+y)\exp(x) + (1+y)\exp(x)t + \frac{1}{2}(1+y)\exp(x)t^2 + \frac{1}{6}(1+y)\exp(x)t^3,\\
&\vdots\\
&u_n(x,y,t) = u_n(x,y,t) = \sum_{k=0}^{n} \frac{t^k}{k!}(1+y)\exp(x),\\
&\vdots
\end{aligned} \tag{3.7}$$

And exact solution will be as

$$u(x,y,t) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k}{k!}(1+y)\exp(x) = (1+y)\exp(x+t). \tag{3.8}$$

4. CONCLUSION:

In this paper, variation iteration method (VIM) has been successfully applied to find the exact solution of two-dimensional diffusion equation. The obtained solution shows that the method is vary convenient and effective to solve wide classes of problems. The method was used in a direct way without using linearization, perturbation or restrictive assumptions. Also VIM provides more realistic series solutions that converge very rapidly in real physical problems.

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