

THE (MOD AND INTEGRAL) SUM LABELING OF CERTAIN GRAPHS

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ABSTRACT

This paper establishes that (mod and integral) sum graph labeling for the friendship graph $f_{3,n}$ and also find (optimal, mod and integral) sum number for $f_{3,n}$. Moreover an attempt has been made and also (mod and integral) sum numbers for the ladder graphs L_n , for $n \geq 1$, the crown graphs $P_n \odot K_1$ for $n \geq 2$ and $C_n \odot K_1$ for $n \geq 3$ are determined in this paper.

Keywords: The (optimal, mod and integral) sum graph; the sequential numbering method; Recurrence Relation method; Friendship Graph; ladders graph; crowns graph.

1. INTRODUCTION

All graphs considered in this paper are simple, finite and undirected. We follow in general the graph-theoretic notation and terminology of Ref [1] unless otherwise specified. The notion of a sum graph was introduced by Harary [2] in 1990. A graph $G(V, E)$ is called a *sum graph* if there is a bijection λ from $V(G)$ to a set of positive integers $S \subset \mathbb{N}$ such that $uv \in E(G)$ if and only if $\lambda(u) + \lambda(v) \in S$. The *sum number* $\sigma(G)$ of a connected graph is the least nonnegative m of isolated vertices mK_1 such that $G \cup mK_1$ is a sum graph. Let $\delta(G)$ be the smallest degree of vertices of the G . It is obvious that $\sigma(G) \geq \delta(G)$. In case $\sigma(G) = \delta(G)$, then the graph G is called δ – optimum summable graph. In 1993 Harary [3] extended the sum graph concept to allow the selection of vertex labels from all the integers. A graph $G(V, E)$ is called an *integral sum graph* if there is a bijection λ from $V(G)$ to a set of integers $S \subset \mathbb{Z}$ such that $uv \in E(G)$ if and only if $\lambda(u) + \lambda(v) \in S$. The *integral sum number* $\zeta(G)$ of a graph is the smallest nonnegative number m of isolated vertices such that $G \cup mK_1$ is an integral sum graph. It is obvious that $\zeta(G) \leq \sigma(G)$ for any graph G and a graph G is an integral sum graph if and only if $\zeta(G) = 0$.

The concept of mod sum graph was introduced by Boland, Laskar, Turner and Domke [4] in 1990. A graph $G(V, E)$ is a *mod sum graph* if there exist a positive integer z and a labeling of the vertices with distinct elements from $\{1, 2, 3, \dots, z-1\}$ such that $uv \in E_p$ if and only if $\lambda(u) + \lambda(v) \pmod{z}$ is the label of vertex of G . We similarly define the *mod sum number* $\rho(G)$ of a graph is the smallest nonnegative number m of isolated vertices such that $G \cup mK_1$ is a mod sum graph(MSG). A mod sum graph may be connected, so that the mod sum number may be zero. All sum graphs are mod sum graphs by choosing a sufficiently large modulus z but the converse is not true. For an updated survey on the results on (integral, mod) sum graphs one may refer to [6, 7 and 8]. Here we give (optimal, mod and integral) sum labeling to certain classes of graphs find their (mod and integral) sum numbers.

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Definition 1.1 [5] (Friendship graph $f_{3,n}$): The friendship graph $f_{3,n}$ is a graph with vertex set $V = V(f_{3,n}) = \{c, v_1, v_2, v_3, v_4, \dots, v_{2n-1}, v_{2n}\}$ and edges set $E(f_{3,n}) = \{(c, v_i)(v_{2m-1}, v_{2m}) | i = 1, 2, 3, \dots, 2n, m = 1, 2, \dots, n\}$. The vertex c is called *centre* of $f_{3,n}$ and each edge (c, v_i) for $i = 1, 2, 3, \dots, 2n$ is called a *spoke* and each edge (v_{2m-1}, v_{2m}) for $m = 1, 2, 3, \dots, n$ is called a *rim*. In this paper, we determine the (mod and integral) sum number of the graph $f_{3,n}$.

Definition 1.2 (Ladder graph L_n): It is a simple graph with vertex set $V = V(L_n) = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$ and edges set $E = E(L_n) = \{a_i b_i : 1 \leq i \leq n\} \cup \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{b_i b_{i+1} : 1 \leq i \leq n-1\}$. Hence L_n has $2n$ vertices and $(3n - 2)$ edges.

Definition 1.3 (Crown graph): A graph obtained by attaching a single pendant edge to each vertex of a path $P_n = a_1 a_2 \dots a_n$ is called a *crown graph* $P_n \odot K_1$ which is obtained from the path by joining a vertex a_i to $b_i, 1 \leq i \leq n$. The edges are labeled as $1 \leq i \leq n, e_{2i-1} = a_i b_i$ and $e_{2i} = a_i a_{i+1}$. Hence $P_n \odot K_1$ has $2n$ vertices and $(2n - 1)$ edges. The crown $C_n \odot K_1$ is the graph obtained by taking one copy of C_n and n copies of K_1 and joining the i^{th} vertex of C_n with an edge to every vertex in the i^{th} copy of K_1 . Hence $C_n \odot K_1$ has $2n$ vertices and $2n$ edges.

2. RESULTS

Although the following theorems are existing results but we determine the sum number, mod sum number and integral sum number of the graph $f_{3,n}$ by the *sequential numbering method*.

Theorem 2.1: $f_{3,n}$ is an optimal sum graph and $\sigma(f_{3,n}) = 2$, for $n \geq 2$.

Proof: Label of the centre vertex $c = 1$ and label of the remaining vertices of $f_{3,n} \cup 2K_1$ are of the following scheme.

- $v_{2i-1} = 2n + i - 1, i = 1, 2, \dots, n$ and $v_{2i} = 4n - i, i = 1, 2, \dots, n$.
- $c_1 = c + v_2$ and $c_2 = v_{2i-1} + v_{2i}$ for $i = 1, 2, \dots, n$.

Let $A = \{c, v_1, v_2, v_3, \dots, v_{2n}\}, C = \{c_1, c_2\}, S = A \cup C$. it is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $c + v_i \in S$ for $i = 1, 2, \dots, 2n$.
3. $v_{2i-1} + v_{2j} \notin S$ for $i, j = 1, 2, \dots, n$ and $i \neq j$.

Hence the above projected labeling is an optimal sum labeling of $f_{3,n}$ and $\sigma(f_{3,n}) = 2$, for $n \geq 2$. □

Theorem 2.2: $f_{3,n}$ is an integral sum graph and $\zeta(f_{3,n}) = 0$, for $n \geq 2$.

Proof: Label of the centre vertex $c = 0$ and label of the remaining vertices of $f_{3,n}$ are of the following scheme.

- $v_{2i-1} = -(2)^i$, for $i = 1, 2, \dots, n$ and $v_{2i} = (2)^i$, for $i = 1, 2, \dots, n$.
- $c + v_i = v_i$, for $i = 1, 2, \dots, 2n$ and $v_{2i-1} + v_{2i} = 0$ for $i = 1, 2, \dots, n$.

Let $A = \{c, v_1, v_2, v_3, \dots, v_{2n}\}$. it is easily verified the following assertions are true and valid.

1. The vertices of A are distinct.
2. $v_{2i-1} + v_{2i} \notin A$, for $i = 1, 2, \dots, n$ and $i \neq j$.

Hence the above projected labeling is an integral sum labeling of $f_{3,n}$ and, $\zeta(f_{3,n}) = 0$ for $n \geq 2$. □

Theorem 2.3: $f_{3,n}$ is a mod sum graph and $\rho(f_{3,n}) = 0$, for $n \geq 2$.

Proof: Label of the centre vertex $c = 3n + 3$ and label of the remaining vertices of $f_{3,n}$ are of the following scheme.

- $v_{2i-1} = 3 + (i-1)(3n+3)$, for $i = 1, 2, \dots, n$.
- $v_{2i} = 3n + (n-i)(3n+3)$, for $i = 1, 2, \dots, n$ and with modulus $z = n(3n+3)$.

Hence the above projected labeling is a mod sum labeling of $f_{3,n}$ and, $\rho(f_{3,n}) = 0$ for $n \geq 2$. \square

In the next couple of results we determine the sum number, mod sum number and integral sum number of the graph L_n by the sequential numbering method.

Lemma 2.4: The sum number for ladder graph L_2 is given by $\sigma(L_2) = 3$.

Proof: The sum labeling of the graph $L_2 \cup 3K_1$ is considered as stated below:

$$a_1 = 1, a_2 = 2, b_1 = 3, b_2 = 5, c_1 = 4, c_2 = 7, c_3 = 8.$$

Hence the above projected labeling is a sum labeling of $L_2 \cup 3K_1$ and $\sigma(L_2) = 3$. \square

Theorem 2.5: The sum number for ladder graph L_n , $\sigma(L_n) = n$, for $n \geq 3$.

Proof: The sum labeling of the graph $L_n \cup nK_1$ is considered as stated below.

$$a_i = i, 1 \leq i \leq n, b_i = 2n - i + 1, 1 \leq i \leq n, c_k = 2n + 2k - 1, 1 \leq k \leq n.$$

Let $V = V(L_n) = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$, $C = \{c_1, c_2, \dots, c_n\}$ be the vertex set of nK_1 and $S = V \cup C$. It is easily verified the following assertions are true and valid.

1. The vertices in S are distinct positive integers.
2. $c_i + c_j \notin S$ for any $c_i, c_j \in C$ ($i \neq j$).
3. $a_i + a_j \in S$ for any $a_i, a_j \in V$ ($i \neq j$).
4. $b_i + b_j \in S$ for any $b_i, b_j \in V$ ($i \neq j$).
5. $a_i + c_j \notin S$ for any $a_i \in V$ and for any $c_j \in C$.
6. $b_i + c_j \notin S$ for any $b_i \in V$ and for any $c_j \in C$.

Thus the above projected labeling is a sum labeling of $L_n \cup nK_1$ and $\sigma(L_n) = n$, for $n \geq 3$. \square

Theorem 2.6 An integral sum number for L_n , $\zeta(L_n) = 3$, for $n \geq 2$.

Proof: The integral sum labeling of the graph $L_n \cup 3K_1$ is considered as stated below:

- $a_1 = -1, a_2 = 3, a_i = a_{i-2} - a_{i-1}$, for $i = 3, 4, \dots, n$;
- $b_1 = 1, b_2 = -3, b_i = b_{i-2} - b_{i-1}$, for $i = 3, 4, \dots, n$; $c_1 = 0, c_2 = 2, c_3 = -2$.

Let $V = V(L_n) = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$, $C = \{c_1, c_2, c_3\}$ be the vertex set of $3K_1$ and $S = V \cup C$. It is easily verified the following assertions are true and valid.

The vertices in S are distinct integers.

1. $c_i + c_j \in S$ for any $c_i, c_j \in C$ ($i \neq j$).
2. $a_i + a_j \in S$ for any $a_i, a_j \in V$ ($i \neq j$).
3. $b_i + b_j \in S$ for any $b_i, b_j \in V$ ($i \neq j$).

Hence the above projected labeling is an integral sum labeling of $L_n \cup 3K_1$ and $\zeta(L_n) = 3$, for $n \geq 3$. \square

Theorem 2.7: The mod sum number for ladder graph L_n , $\rho(L_n) = 1$, for $n \geq 3$.

Proof: The mod sum labeling of the graph $L_n \cup K_1$ is considered as stated below.

$a_i = i$, for $i = 1, 2, \dots, n$, $b_i = 2n - i + 1$, for $i = 1, 2, \dots, n$, with modulus $z = 2n + 2$. The isolated vertex is $c = 2n + 1$.

Let $V = V(L_n) = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$, $C = \{c\}$ be the vertex set of K_1 and $S = V \cup C$. It is easily verified the following assertions are true and valid.

1. The vertices in S are distinct.
2. $a_i + a_j \in S$ for any $a_i, a_j \in V$ ($i \neq j$).
3. $b_i + b_j \in S$ for any $b_i, b_j \in V$ ($i \neq j$).

Hence the above projected labeling is mod sum labeling of $L_n \cup K_1$ and $\rho(L_n) = 1$, for $n \geq 3$. \square

In the next couple of results we determine the sum number, mod sum number and integral sum number of the graphs $P_n \odot K_1$ and $C_n \odot K_1$ by iterative scheme and the Recurrence Relation method.

Theorem 2.8: The sum number for graph $P_n \odot K_1$, $\sigma(P_n \odot K_1) = 1$, for $n \geq 2$.

Proof: The sum labeling of the graph $(P_n \odot K_1) \cup K_1$ is considered as stated below:

$$a_i = i, 1 \leq i \leq n; b_i = 2n - i + 1, 1 \leq i \leq n; c = 2n + 1$$

$$\text{Let } V = V(P_n \odot K_1) = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}, E(P_n \odot K_1) = \{a_i b_i : 1 \leq i \leq n\} \cup \{a_i a_{i+1} : 1 \leq i \leq n-1\}.$$

Hence the above projected labeling is the sum labeling of $P_n \odot K_1$ and $\sigma(P_n \odot K_1) = 1$, for $n \geq 2$. \square

Theorem 2.9: An integral sum number for graph $P_n \odot K_1$, $\zeta(P_n \odot K_1) = 1$, for $n \geq 3$.

Proof: An integral sum labeling of the graph $(P_n \odot K_1) \cup K_1$ is considered as stated below:

- $a_1 = 3, a_2 = -4, a_{i+1} = a_{i-1} - a_i$, for $i = 2, 3, \dots, n-1$;
- $c = 2, b_i = c - a_i$, for $i = 1, 2, \dots, n$.

$$\text{Let } V = V(P_n \odot K_1) = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}, E(P_n \odot K_1) = \{a_i b_i : 1 \leq i \leq n\} \cup \{a_i a_{i+1} : 1 \leq i \leq n-1\}.$$

Hence the above projected labeling is an integral sum labeling of $P_n \odot K_1$ and $\zeta(P_n \odot K_1) = 1$, for $n \geq 3$. \square

Theorem 2.10: $P_n \odot K_1$ is a mod sum graph and $\rho(P_n \odot K_1) = 0$, for $n \geq 2$.

Proof: We only give a mod sum labeling of the graph $(P_n \odot K_1) \cup K_1$. We can consider the following two cases.

Case-I: If n is an even.

$$a_i = 2i, \text{ for } i = 1, 2, \dots, n, b_i = 2i - 1, \text{ for } i = 1, 2, \dots, n, \text{ with modulus } z = 2n + 1.$$

Case-II: If n is an odd.

$$a_i = 2i, \text{ for } i = 1, 2, \dots, n, b_i = 2i + 1, \text{ for } i = 1, 2, \dots, n, \text{ with modulus } z = 2n + 1.$$

$$\text{Let } V = V(P_n \odot K_1) = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}, E(P_n \odot K_1) = \{a_i b_i : 1 \leq i \leq n\} \cup \{a_i a_{i+1} : 1 \leq i \leq n-1\}.$$

Hence the above projected labeling is a mod sum labeling of $P_n \odot K_1$ and $\rho(P_n \odot K_1) = 0$, for $n \geq 2$. \square

Theorem 2.11: The sum number for graph $C_n \odot K_1$, $\sigma(C_n \odot K_1) = 2$, for $n \geq 3$.

Proof: We only give a sum labeling of the graph $(C_n \odot K_1) \cup 2K_1$. We can consider the following two cases.

Case-1: $n = 2k + 1 (k \geq 1)$.

- $f_1 = 1, f_2 = 2, f_{i+1} = f_i + f_{i-1}, i = 2, 3, \dots, 2k.$
- $a_i = f_{2i-1}, i = 1, 2, \dots, k+1, b_i = f_{2i}, i = 1, 2, \dots, k.$
- $d_1 = a_{k+1} + b_k, d_{i+1} = a_i + d_i, i = 1, 2, \dots, k, e_1 = a_{k+1} + d_{k+1}, e_{i+1} = b_i + e_i, i = 1, 2, \dots, k-1.$
- $c_1 = a_1 + a_{k+1}, c_2 = b_k + e_k.$

Let $A = \{a_1, a_2, \dots, a_k, a_{k+1}\}, B = \{b_1, b_2, \dots, b_k\}, D = \{d_1, d_2, \dots, d_k, d_{k+1}\},$

$E = \{e_1, e_2, \dots, e_k\}, C = \{c_1, c_2\}, S = A \cup B \cup C \cup D \cup E.$ It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. There only are $a_{k+1} + a_1 = c_1 \in S$ for any $a_i, a_j \in A (i \neq j);$
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j);$
4. $d_i + d_j \notin S$ for any $d_i, d_j \in D (i \neq j);$
5. $e_i + e_j \notin S$ for any $e_i, e_j \in E (i \neq j);$
6. $a_i + b_j \in S$ if and only if a_i is adjacent to $b_j.$
7. $a_i + d_j \in S$ if and only if $i = j;$ and $a_1 b_1 a_2 b_2 a_3 b_3 \dots b_k a_{k+1} a_1$ is a cycle $C_{2K+1}.$
8. $a_i + e_j \notin S$ for any $a_i \in A$ and for any $e_j \in E;$
9. $b_i + d_j \notin S$ for any $b_i \in B$ and for any $d_j \in D;$
10. $b_i + e_j \in S$ if and only if $i = j;$
11. $d_i + e_j \notin S$ for any $d_i \in D$ and for any $e_j \in E;$
12. $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C;$
13. $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C;$
14. $d_i + c_j \notin S$ for any $d_i \in D$ and for any $c_j \in C;$
15. $e_i + c_j \notin S$ for any $e_i \in E$ and for any $c_j \in C;$
16. $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j).$

Case-2: $n = 2k (k \geq 2).$

- $f_1 = 1, f_2 = 2, f_{i+1} = f_i + f_{i-1}, i = 2, 3, \dots, 2k-1.$
- $a_i = f_{2i-1}, i = 1, 2, \dots, k, b_i = f_{2i}, i = 1, 2, \dots, k.$
- $d_1 = a_k + b_k, d_{i+1} = a_i + d_i, i = 1, 2, \dots, k-1, e_1 = a_k + d_k, e_{i+1} = b_i + e_i, i = 1, 2, \dots, k-1.$
- $c_1 = a_1 + b_k, c_2 = b_k + e_k$

Let $A = \{a_1, a_2, \dots, a_k\}, B = \{b_1, b_2, \dots, b_k\}, D = \{d_1, d_2, \dots, d_k\},$

$E = \{e_1, e_2, \dots, e_k\}, C = \{c_1, c_2\}, S = A \cup B \cup C \cup D \cup E.$ It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. There only are $b_k + a_1 = c_1 \in S$ for any $a_i, a_j \in A (i \neq j);$
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j);$
4. $d_i + d_j \notin S$ for any $d_i, d_j \in D (i \neq j);$
5. $e_i + e_j \notin S$ for any $e_i, e_j \in E (i \neq j);$
6. $a_i + b_j \in S$ if and only if a_i is adjacent to $b_j.$
7. $a_i + d_j \in S$ if and only if $i = j;$ and $a_1 b_1 a_2 b_2 a_3 b_3 \dots a_k b_k a_1$ is a cycle $C_{2K}.$

8. $a_i + e_j \notin S$ for any $a_i \in A$ and for any $e_j \in E$;
9. $b_i + d_j \notin S$ for any $b_i \in B$ and for any $d_j \in D$;
10. $b_i + e_j \in S$ if and only if $i = j$;
11. $d_i + e_j \notin S$ for any $d_i \in D$ and for any $e_j \in E$;
12. $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C$;
13. $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$;
14. $d_i + c_j \notin S$ for any $d_i \in D$ and for any $c_j \in C$;
15. $e_i + c_j \notin S$ for any $e_i \in E$ and for any $c_j \in C$;
16. $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j)$.

Hence the above projected labeling is a sum labeling of graph $(C_n \odot K_1) \cup 2K_1$ and $\sigma(C_n \odot K_1) = 2$, for $n \geq 3$ \square

Theorem 2.12: An integral sum number for graph $C_n \odot K_1, \zeta(C_n \odot K_1) = 2$, for $n \geq 3$.

Proof: We only give an integral sum labeling of the graph $(C_n \odot K_1) \cup 2K_1$. We can consider the following two cases.

Case-1: $n = 2k + 1 (k \geq 1)$.

- $f_1 = -1, f_2 = 3, f_{i+1} = f_{i-1} - f_i, i = 2, 3, \dots, 2k$.
- $a_i = f_{2i-1}, i = 1, 2, \dots, k+1, b_i = f_{2i}, i = 1, 2, \dots, k$.
- $d_1 = a_{k+1} + a_1, d_{i+1} = a_i + d_i, i = 1, 2, \dots, k, e_1 = a_{k+1} + d_{k+1}, e_{i+1} = b_i + e_i, i = 1, 2, \dots, k-1$.
- $c_1 = a_1 + b_1, c_2 = b_k + e_k$.

Let $A = \{a_1, a_2, \dots, a_k, a_{k+1}\}, B = \{b_1, b_2, \dots, b_k\}, D = \{d_1, d_2, \dots, d_k, d_{k+1}\},$

$E = \{e_1, e_2, \dots, e_k\}, C = \{c_1, c_2\}, S = A \cup B \cup C \cup D \cup E$. It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $a_1 + b_1 = c_1 \in S$ for $a_1 \in A$ and for $b_1 \in B$.
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j)$;
4. $d_i + d_j \notin S$ for any $d_i, d_j \in D (i \neq j)$;
5. $e_i + e_j \notin S$ for any $e_i, e_j \in E (i \neq j)$;
6. $a_i + b_j \in S$ if and only if a_i is adjacent to b_j .
7. $a_i + d_j \in S$ if and only if $i = j$; and $a_1 b_1 a_2 b_2 a_3 \dots b_k a_{k+1} a_1$ is a cycle C_{2K+1} .
8. $a_i + e_j \notin S$ for any $a_i \in A$ and for any $e_j \in E$;
9. $b_i + d_j \notin S$ for any $b_i \in B$ and for any $d_j \in D$;
10. $b_i + e_j \in S$ if and only if $i = j$;
11. $d_i + e_j \notin S$ for any $d_i \in D$ and for any $e_j \in E$;
12. $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C$;
13. $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$;
14. $d_i + c_j \notin S$ for any $d_i \in D$ and for any $c_j \in C$;
15. $e_i + c_j \notin S$ for any $e_i \in E$ and for any $c_j \in C$;
16. $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j)$.

Case-2: $n = 2k$ ($k \geq 2$).

- $f_1 = -1, f_2 = 3, f_{i+1} = f_{i-1} - f_i, i = 2, 3, \dots, 2k-1.$
- $a_i = f_{2i-1}, i = 1, 2, \dots, k, b_i = f_{2i}, i = 1, 2, \dots, k.$
- $d_1 = a_1 + b_k, d_{i+1} = a_i + d_i, i = 1, 2, \dots, k-1, e_1 = a_k + d_k, e_{i+1} = b_i + e_i, i = 1, 2, \dots, k-1.$
- $c_1 = a_1 + b_1, c_2 = b_k + e_k.$

Let $A = \{a_1, a_2, \dots, a_k\}, B = \{b_1, b_2, \dots, b_k\}, D = \{d_1, d_2, \dots, d_k\},$

$E = \{e_1, e_2, \dots, e_k\}, C = \{c_1, c_2\}, S = A \cup B \cup C \cup D \cup E.$ It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $a_i + b_1 = c_1 \in S$ for $a_i \in A$ and $b_1 \in B$
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j);$
4. $d_i + d_j \notin S$ for any $d_i, d_j \in D (i \neq j);$
5. $e_i + e_j \notin S$ for any $e_i, e_j \in E (i \neq j);$
6. $a_i + b_j \in S$ if and only if a_i is adjacent to $b_j.$
7. $a_i + d_j \in S$ if and only if $i = j;$ and $a_1 b_1 a_2 b_2 a_3 b_3 \dots a_k b_k a_1$ is a cycle $C_{2k}.$
8. $a_i + e_j \notin S$ for any $a_i \in A$ and for any $e_j \in E;$
9. $b_i + d_j \notin S$ for any $b_i \in B$ and for any $d_j \in D;$
10. $b_i + e_j \in S$ if and only if $i = j;$
11. $d_i + e_j \notin S$ for any $d_i \in D$ and for any $e_j \in E;$
12. $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C;$
13. $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C;$
14. $d_i + c_j \notin S$ for any $d_i \in D$ and for any $c_j \in C;$
15. $e_i + c_j \notin S$ for any $e_i \in E$ and for any $c_j \in C;$
16. $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j).$

Thus the above projected labeling is an integral sum labeling of graph $(C_n \odot K_1) \cup 2K_1$ and $\zeta(C_n \odot K_1) = 2,$ for $n \geq 3.$ \square

Theorem 2.13: $C_n \odot K_1$ is a mod sum graph and $\rho(C_n \odot K_1) = 0,$ for $n \geq 4.$

Proof: We only give a mod sum labeling of the graph $(C_n \odot K_1).$ We can consider the following two cases.

Case-1: $n = 2k + 1$ ($k \geq 2$).

- $a_i = (k-i)N + 1, i = 1, 2, 3, \dots, k, b_i = (k-i+1)N, i = 1, 2, \dots, k.$
- $d_i = (k-i)N + 2, i = 1, 2, \dots, k, e_i = (k-i)N + 3, i = 1, 2, \dots, k.$
- $a_{k+1} = kN + 1, d_{k+1} = kN + 2$ and with modulus $z = kN,$ where $N \geq 5$ is an integer.

Let $A = \{a_1, a_2, \dots, a_k, a_{k+1}\}, B = \{b_1, b_2, \dots, b_k\}, D = \{d_1, d_2, \dots, d_k, d_{k+1}\},$

$E = \{e_1, e_2, \dots, e_k\}, S = A \cup B \cup D \cup E.$ It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $a_i + b_j \bmod z \in S$ If and only if a_i is adjacent to $b_j.$
3. $a_i + d_j \bmod z \in S$ if and only if $i = j.$
4. $b_i + e_j \bmod z \in S$ if and only if $i = j.$
5. $a_1 b_1 a_2 b_2 a_3 b_3 \dots b_k a_{k+1} a_1$ is a cycle $C_{2k+1}.$

Case-2: $n = 2k$ ($k \geq 2$).

- $a_i = (k-i)N + 1, i = 1, 2, 3, \dots, k, b_i = (k-i+1)N, i = 1, 2, \dots, k.$
- $d_i = (k-i)N + 3, i = 1, 2, \dots, k, e_i = (k-i)N + 4, i = 1, 2, \dots, k$, with modulus

$z = (n-k)N$, where $N \geq 5$ is an integer. Let $A = \{a_1, a_2, \dots, a_k\}, B = \{b_1, b_2, \dots, b_k\},$

$D = \{d_1, d_2, \dots, d_k\}, E = \{e_1, e_2, \dots, e_k\}, S = A \cup B \cup D \cup E$. It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $a_i + b_j \bmod z \in S$ If and only if a_i is adjacent to b_j .
3. $a_i + d_j \bmod z \in S$ if and only if $i = j$.
4. $b_i + e_j \bmod z \in S$ if and only if $i = j$.
5. $a_1 b_1 a_2 b_2 a_3 b_3 \dots a_k b_k a_1$ is a cycle C_{2k} .

Thus the above projected labeling is a mod sum labeling of graph $(C_n \odot K_1)$. So $C_n \odot K_1$ is a mod sum graph and $\rho(C_n \odot K_1) = 0$, for $n \geq 3$. \square

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