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THE (MOD AND INTEGRAL) SUM LABELING OF CERTAIN GRAPHS

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ABSTRACT

This paper establishes that (mod and integral) sum graph labeling for the friendship graph $f_{3,n}$ and also find (optimal, mod and integral) sum number for $f_{3,n}$. Moreover an attempt has been made and also (mod and integral) sum numbers for the ladder graphs L_n, for $n \geq 1$, the crown graphs $P_n \odot K_1$ for $n \geq 2$ and $C_n \odot K_1$ for $n \geq 3$ are determined in this paper.

Keywords: *The (optimal, mod and integral) sum graph; the sequential numbering method; Recurrence Relation method; Friendship Graph; ladders graph; crowns graph.*

1. INTRODUCTION

All graphs considered in this paper are simple, finite and undirected. We follow in general the graph-theoretic notation and terminology of Ref [1] unless otherwise specified. The notion of a sum graph was introduced by Harary [2] in 1990. A graph $G(V,E)$ is called a *sum graph* if there is a bijection λ from $V(G)$ to a set of positive integers $S \subset N$ such that $uv \in E(G)$ if and only if $\lambda(u)+\lambda(v) \in S$. The *sum number* $\sigma(G)$ of a connected graph is the least nonnegative m of isolated vertices mK_1 such that $G \bigcup mK_1$ is a sum graph. Let $\delta(G)$ be the smallest degree of vertices of the G. It is obvious that $\sigma(G) \geq \delta(G)$. In case $\sigma(G) = \delta(G)$, then the graph G is called $\delta-$ optimum summable graph. In 1993 Harary [3] extended the sum graph concept to allow the selection of vertex labels from all the integers. A graph $G(V,E)$ is called an *integral sum graph* if there is a bijection λ from $V(G)$ to a set of integers $S \subset Z$ such that $uv \in E(G)$ if and only if $\lambda(u)+\lambda(v) \in S$. The *integral sum number* $\zeta(G)$ of a graph is the smallest nonnegative number m of isolated vertices such that $G \bigcup mK_1$ is an integral sum graph. It is obvious that $\zeta(G) \leq \sigma(G)$ for any graph G and a graph G is an integral sum graph if and only if $\zeta(G) = 0$.

The concept of mod sum graph was introduced by Boland, Laskar, Turner and Domke [4] in 1990. A graph $G(V,E)$ is a *mod sum graph* if there exist a positive integer z and a labeling of the vertices with distinct elements from $\{1,2,3,...,z-1\}$ such that $uv \in E_p$ if and only if $\lambda(u)+\lambda(v) (\text{mod } z)$ is the label of vertex of G. We similarly define the *mod sum number* $\rho(G)$ of a graph is the smallest nonnegative number m of isolated vertices such that $G \bigcup mK_1$ is a mod sum graph(MSG). A mod sum graph may be connected, so that the mod sum number may be zero. All sum graphs are mod sum graphs by choosing a sufficiently large modulus z but the converse is not true. For an updated survey on the results on (integral, mod) sum graphs one may refer to [6, 7 and 8]. Here we give (optimal, mod and integral) sum labeling to certain classes of graphs find their (mod and integral) sum numbers.

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Definition 1.1 [5] (Friendship graph $f_{3,n}$): The *friendship graph* $f_{3,n}$ is a graph with vertex set $V = V\left(f_{3,n}\right) = \left\{c, v_1, v_2, v_3, v_4, \ldots v_{2n-1}, v_{2n}\right\}$ and edges set $E\left(f_{3,n}\right) = \left\{(c, v_i)(v_{2m-1}, v_{2m}) \mid i = 1,2,3,\ldots 2n, m = 1,2,\ldots,n\right\}$. The vertex c is called *centre* of $f_{3,n}$ and each edge (c, v_i) for $i = 1,2,3,\ldots,2n$ is called a *spoke* and each edge (v_{2m-1}, v_{2m}) for $m = 1,2,3,\ldots,n$ is called a *rim*. In this paper, we determine the (mod and integral) sum number of the graph $f_{3,n}$

Definition 1.2 (Ladder graph L_n): It is a simple graph with vertex set $V = V(L_n) = \{a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n\}$ and edges set $E = E(L_n) = \{a_i b_i : 1 \le i \le n\} \bigcup \{a_i a_{i+1} : 1 \le i \le n-1\} \bigcup \{b_i b_{i+1} : 1 \le i \le n-1\}$ .Hence L_n has *2n* vertices and $(3n-2)$ edges.

Definition 1.3 (Crown graph): A graph obtained by attaching a single pendant edge to each vertex of a path $P_n = a_1 a_2 \ldots a_n$ is called a *crown graph* $P_n \odot K_1$ which is obtained from the path by joining a vertex $a_i \text{ to } b_i, 1 \le i \le n$ .The edges are labeled as $1 \le i \le n, e_{2i-1} = a_i b_i \text{ and } e_{2i} = a_i a_{i+1}$ .Hence $P_n \odot K_1$ has $2n$ vertices and $(2n-1)$ edges. *The crown* $C_n \odot K_1$ is the graph obtained by taking one copy of C_n and n copies of K_1 and joining the i^{th} vertex of C_n with an edge to every vertex in the i^{th} copy of K_1. Hence $C_n \odot K_1$ has $2n$ vertices and $2n$ edges.

2. RESULTS

Although the following theorems are existing results but we determine the sum number, mod sum number and integral sum number of the graph $f_{3,n}$ by the *sequential numbering method.*

Theorem 2.1: $f_{3,n}$ is an optimal sum graph and $\sigma\left(f_{3,n}\right) = 2$, for $n \ge 2$.

Proof: Label of the centre vertex $c = 1$ and label of the remaining vertices of $f_{3,n} \bigcup 2K_1$ are of the following scheme.

- $v_{2i-1} = 2n + i - 1, i = 1,2,\ldots,n \text{ and } v_{2i} = 4n - i, i = 1,2,\ldots,n.$
- $c_1 = c + v_2 \text{ and } c_2 = v_{2i-1} + v_{2i} \text{ for } i = 1,2,\ldots,n.$

Let $A = \{c, v_1, v_2, v_3, \ldots v_{2n}\}, C = \{c_1, c_2\}, S = A \bigcup C.$ it is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $c + v_i \in S \text{ for } i = 1,2,\ldots,2n.$
3. $v_{2i-1} + v_{2j} \notin S \text{ for } i, j = 1,2,\ldots,m \text{ and } i \ne j.$

Hence the above projected labeling is an optimal sum labeling of $f_{3,n}$ and $\sigma\left(f_{3,n}\right) = 2$, for $n \ge 2$. □

Theorem 2.2: $f_{3,n}$ is an integral sum graph and $\zeta\left(f_{3,n}\right) = 0$, for $n \ge 2$.

Proof: Label of the centre vertex $c = 0$ and label of the remaining vertices of $f_{3,n}$ are of the following scheme.

- $v_{2i-1} = -(2)^i, \text{ for } i = 1,2,\ldots,n \text{ and } v_{2i} = (2)^i, \text{ for } i = 1,2,\ldots,n.$
- $c + v_i = v_i, \text{ for } i = 1,2,\ldots,2n \text{ and } v_{2i-1} + v_{2i} = 0 \text{ for } i = 1,2,\ldots,n.$

Let $A = \{c, v_1, v_2, v_3, \ldots v_{2n}\}$. it is easily verified the following assertions are true and valid.

1. The vertices of A are distinct.
2. $v_{2i-1} + v_{2i} \notin A, \text{ for } i = 1,2,\ldots,n \text{ and } i \ne j.$

Hence the above projected labeling is an integral sum labeling of $f_{3,n}$ and, $\zeta\left(f_{3,n}\right) = 0$ for $n \ge 2$. □

Theorem 2.3: $f_{3,n}$ is a mod sum graph and $\rho\left(f_{3,n}\right)=0$, for $n \geq 2$.

Proof: Label of the centre vertex $c=3n+3$ and label of the remaining vertices of $f_{3,n}$ are of the following scheme.

- $v_{2i-1}=3+(i-1)(3n+3)$, $for\ i=1,2,..,n$.
- $v_{2i}=3n+(n-i)(3n+3)$, $for\ i=1,2,..,n$ and with modulus $z=n(3n+3)$.

Hence the above projected labeling is a mod sum labeling of $f_{3,n}$ and, $\rho\left(f_{3,n}\right)=0$ for $n \geq 2$. □

In the next couple of results we determine the sum number, mod sum number and integral sum number of the graph L_n by the sequential numbering method.

Lemma 2.4: The sum number for ladder graph L_2 is given by $\sigma\left(L_2\right)=3$.

Proof: The sum labeling of the graph $L_2 \bigcup 3K_1$ is considered as stated below:

$$a_1=1, a_2=2, b_1=3, b_2=5, c_1=4, c_2=7, c_3=8.$$

Hence the above projected labeling is a sum labeling of $L_2 \bigcup 3K_1$ and $\sigma\left(L_2\right)=3$. □

Theorem 2.5: The sum number for ladder graph L_n, $\sigma\left(L_n\right)=n$, for $n \geq 3$.

Proof: The sum labeling of the graph $L_n \bigcup nK_1$ is considered as stated below.

$$a_i=i, 1 \leq i \leq n, b_i=2n-i+1, 1 \leq i \leq n, c_k=2n+2k-1, 1 \leq k \leq n.$$

Let $V=V\left(L_n\right)=\left\{a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n\right\}$, $C=\left\{c_1, c_2, \ldots, c_n\right\}$ be the vertex set of nK_1 and $S=V \bigcup C$. It is easily verified the following assertions are true and valid.

1. The vertices in S are distinct positive integers.
2. $c_i+c_j \notin S$ for any $c_i, c_j \in C$ $(i \neq j)$.
3. $a_i+a_j \in S$ for any $a_i, a_j \in V$ $(i \neq j)$.
4. $b_i+b_j \in S$ for any $b_i, b_j \in V$ $(i \neq j)$.
5. $a_i+c_j \notin S$ for any $a_i \in V$ and for any $c_j \in C$.
6. $b_i+c_j \notin S$ for any $a_i \in V$ and for any $c_j \in C$.

Thus the above projected labeling is a sum labeling of $L_n \bigcup nK_1$ and $\sigma\left(L_n\right)=n$, for $n \geq 3$. □

Theorem 2.6 An integral sum number for L_n, $\zeta\left(L_n\right)=3$, for $n \geq 2$.

Proof: The integral sum labeling of the graph $L_n \bigcup 3K_1$ is considered as stated below:

- $a_1=-1, a_2=3, a_i=a_{i-2}-a_{i-1}$, $for\ i=3,4,\ldots,n;$
- $b_1=1, b_2=-3, b_i=b_{i-2}-b_{i-1}$, $for\ i=3,4,\ldots,n; c_1=0, c_2=2, c_3=-2.$

Let $V=V\left(L_n\right)=\left\{a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n\right\}$, $C=\left\{c_1, c_2 c_3\right\}$ be the vertex set of $3K_1$ and $S=V \bigcup C$. It is easily verified the following assertions are true and valid.

The vertices in S are distinct integers.

1. $c_i+c_j \in S$ for any $c_i, c_j \in C$ $(i \neq j)$.
2. $a_i+a_j \in S$ for any $a_i, a_j \in V$ $(i \neq j)$.
3. $b_i+b_j \in S$ for any $b_i, b_j \in V$ $(i \neq j)$.

Hence the above projected labeling is an integral sum labeling of $L_n \bigcup 3K_1$ and $\zeta\left(L_n\right)=3$, for $n \geq 3$. □

Theorem 2.7: The mod sum number for ladder graph L_n, $\rho(L_n)=1$, for $n \ge 3$.

Proof: The mod sum labeling of the graph $L_n \cup K_1$ is considered as stated below.
$a_i = i,\ for\ i=1,2,...,n, b_i = 2n-i+1,\ for\ i=1,2,...,n$,with modulus $z = 2n+2$. The isolated vertex is $c = 2n+1$.
Let $V = V(L_n) = \{a_1, a_2, ...a_n, b_1, b_2, ...b_n\}$, $C=\{c\}$ be the vertex set of K_1 and $S = V \cup C$. It is easily verified the following assertions are true and valid.

1. The vertices in S are distinct.
2. $a_i + a_j \in S$ for any $a_i, a_j \in V\ (i \ne j)$.
3. $b_i + b_j \in S$ for any $b_i, b_j \in V\ (i \ne j)$.

Hence the above projected labeling is mod sum labeling of $L_n \cup K_1$ and $\rho(L_n)=1$, for $n \ge 3$. □
In the next couple of results we determine the sum number, mod sum number and integral sum number of the graphs $P_n \odot K_1$ and $C_n \odot K_1$ by iterative scheme and the Recurrence Relation method.

Theorem 2.8: The sum number for graph $P_n \odot K_1$, $\sigma(P_n \odot K_1)=1$, for $n \ge 2$.

Proof: The sum labeling of the graph $(P_n \odot K_1) \cup K_1$ is considered as stated below:

$$a_i = i, 1 \le i \le n; b_i = 2n-i+1, 1 \le i \le n; c = 2n+1$$

Let $V = V(P_n \odot K_1) = \{a_1, a_2, ...a_n, b_1, b_2, ...b_n\}, E(P_n \odot K_1) = \{a_i b_i : 1 \le i \le n\} \cup \{a_i a_{i+1} : 1 \le i \le n-1\}$.
Hence the above projected labeling is the sum labeling of $P_n \odot K_1$ and $\sigma(P_n \odot K_1)=1$, for $n \ge 2$. □

Theorem 2.9: An integral sum number for graph $P_n \odot K_1$, $\zeta(P_n \odot K_1)=1$, for $n \ge 3$.

Proof: An integral sum labeling of the graph $(P_n \odot K_1) \cup K_1$ is considered as stated below:

- $a_1 = 3, a_2 = -4, a_{i+1} = a_{i-1} - a_i,\ for\ i = 2,3,...,n-1;$
- $c = 2, b_i = c - a_i,\ for\ i = 1,2,...,n.$

Let $V = V(P_n \odot K_1) = \{a_1, a_2, ...a_n, b_1, b_2, ...b_n\}, E(P_n \odot K_1) = \{a_i b_i : 1 \le i \le n\} \cup \{a_i a_{i+1} : 1 \le i \le n-1\}$.
Hence the above projected labeling is an integral sum labeling of $P_n \odot K_1$ and $\zeta(P_n \odot K_1)=1$, for $n \ge 3$. □

Theorem 2.10: $P_n \odot K_1$ is a mod sum graph and $\rho(P_n \odot K_1)=0$, for $n \ge 2$.

Proof: We only give a mod sum labeling of the graph $(P_n \odot K_1)$.We can consider the following two cases.

Case-I: If n is an even.

$a_i = 2i,\ for\ i=1,2,...,n, b_i = 2i-1,\ for\ i=1,2,...,n$,with modulus $z = 2n+1$.

Case-II: If n is an odd.

$a_i = 2i,\ for\ i=1,2,...,n, b_i = 2i+1,\ for\ i=1,2,...,n$,with modulus $z = 2n+1$.

Let $V = V(P_n \odot K_1) = \{a_1, a_2, ...a_n, b_1, b_2, ...b_n\}, E(P_n \odot K_1) = \{a_i b_i : 1 \le i \le n\} \cup \{a_i a_{i+1} : 1 \le i \le n-1\}$.

Hence the above projected labeling is a mod sum labeling of $P_n \odot K_1$ and $\rho(P_n \odot K_1)=0$, for $n \ge 2$. □

Theorem 2.11: The sum number for graph $C_n \odot K_1$, $\sigma(C_n \odot K_1)=2$, for $n \ge 3$.

Proof: We only give a sum labeling of the graph $(C_n \odot K_1) \cup 2K_1$.We can consider the following two cases.

Case-1: $n = 2k + 1(k \geq 1)$.

- $f_1 = 1,\ f_2 = 2,\ f_{i+1} = f_i + f_{i-1}, i = 2,3,...,2k.$
- $a_i = f_{2i-1}, i = 1,2,...,k+1, b_i = f_{2i}, i = 1,2,...,k.$
- $d_1 = a_{k+1} + b_k, d_{i+1} = a_i + d_i, i = 1,2,...,k, e_1 = a_{k+1} + d_{k+1}, e_{i+1} = b_i + e_i, i = 1,2,...,k-1.$
- $c_1 = a_1 + a_{k+1}, c_2 = b_k + e_k.$

Let $A = \{a_1, a_2, ..., a_k, a_{k+1}\}, B = \{b_1, b_2, ..., b_k\}, D = \{d_1, d_2, ..., d_k, d_{k+1}\}$,
$E = \{e_1, e_2, ...e_k\}, C = \{c_1, c_2\}, S = A \cup B \cup C \cup D \cup E$.It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. There only are $a_{k+1} + a_1 = c_1 \in S$ for any $a_i, a_j \in A(i \neq j)$;
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B(i \neq j)$;
4. $d_i + d_j \notin S$ for any $d_i, d_j \in D(i \neq j)$;
5. $e_i + e_j \notin S$ for any $e_i, e_j \in E(i \neq j)$;
6. $a_i + b_j \in S$ if and only if a_i is adjacent to b_j.
7. $a_i + d_j \in S$ if and only if $i = j$; and $a_1 b_1 a_2 b_2 a_3 b_3 ....b_k a_{k+1} a_1$ is a cycle C_{2K+1}.
8. $a_i + e_j \notin S$ for any $a_i \in A$ and for any $e_j \in E$;
9. $b_i + d_j \notin S$ for any $b_i \in B$ and for any $d_j \in D$;
10. $b_i + e_j \in S$ if and only if $i = j$;
11. $d_i + e_j \notin S$ for any $d_i \in D$ and for any $e_j \in E$;
12. $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C$;
13. $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$;
14. $d_i + c_j \notin S$ for any $d_i \in D$ and for any $c_j \in C$;
15. $e_i + c_j \notin S$ for any $e_i \in E$ and for any $c_j \in C$;
16. $c_i + c_j \notin S$ for any $c_i, c_j \in C(i \neq j)$.

Case-2: $n = 2k(k \geq 2)$.

- $f_1 = 1,\ f_2 = 2,\ f_{i+1} = f_i + f_{i-1}, i = 2,3,...,2k-1.$
- $a_i = f_{2i-1}, i = 1,2,...,k, b_i = f_{2i}, i = 1,2,...,k.$
- $d_1 = a_k + b_k, d_{i+1} = a_i + d_i, i = 1,2,...,k-1, e_1 = a_k + d_k, e_{i+1} = b_i + e_i, i = 1,2,...,k-1.$
- $c_1 = a_1 + b_k, c_2 = b_k + e_k$

Let $A = \{a_1, a_2, ..., a_k\}, B = \{b_1, b_2, ..., b_k\}, D = \{d_1, d_2, ..., d_k\}$,
$E = \{e_1, e_2, ...e_k\}, C = \{c_1, c_2\}, S = A \cup B \cup C \cup D \cup E$.It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. There only are $b_k + a_1 = c_1 \in S$ for any $a_i, a_j \in A(i \neq j)$;
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B(i \neq j)$;
4. $d_i + d_j \notin S$ for any $d_i, d_j \in D(i \neq j)$;
5. $e_i + e_j \notin S$ for any $e_i, e_j \in E(i \neq j)$;
6. $a_i + b_j \in S$ if and only if a_i is adjacent to b_j.
7. $a_i + d_j \in S$ if and only if $i = j$; and $a_1 b_1 a_2 b_2 a_3 b_3 ....a_k b_k a_1$ is a cycle C_{2K}.

8. $a_i + e_j \notin S$ for any $a_i \in A$ and for any $e_j \in E$;
9. $b_i + d_j \notin S$ for any $b_i \in B$ and for any $d_j \in D$;
10. $b_i + e_j \in S$ if and only if $i = j$;
11. $d_i + e_j \notin S$ for any $d_i \in D$ and for any $e_j \in E$;
12. $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C$;
13. $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$;
14. $d_i + c_j \notin S$ for any $d_i \in D$ and for any $c_j \in C$;
15. $e_i + c_j \notin S$ for any $e_i \in E$ and for any $c_j \in C$;
16. $c_i + c_j \notin S$ for any $c_i, c_j \in C(i \neq j)$.

Hence the above projected labeling is a sum labeling of graph $(C_n \odot K_1) \bigcup 2K_1$ and $\sigma(C_n \odot K_1) = 2$, for $n \geq 3$ □

Theorem 2.12: An integral sum number for graph $C_n \odot K_1, \zeta(C_n \odot K_1) = 2$, for $n \geq 3$.

Proof: We only give an integral sum labeling of the graph $(C_n \odot K_1) \bigcup 2K_1$. We can consider the following two cases.

Case-1: $n = 2k + 1 (k \geq 1)$.

- $f_1 = -1, f_2 = 3, f_{i+1} = f_{i-1} - f_i, i = 2, 3, ..., 2k.$
- $a_i = f_{2i-1} i = 1, 2, ..., k+1, b_i = f_{2i}, i = 1, 2, ..., k.$
- $d_1 = a_{k+1} + a_1, d_{i+1} = a_i + d_i, i = 1, 2, ..., k, e_1 = a_{k+1} + d_{k+1}, e_{i+1} = b_i + e_i, i = 1, 2, ..., k-1.$
- $c_1 = a_1 + b_1, c_2 = b_k + e_k$.

Let $A = \{a_1, a_2, ..., a_k, a_{k+1}\}, B = \{b_1, b_2, ..., b_k\}, D = \{d_1, d_2, ..., d_k, d_{k+1}\}$, $E = \{e_1, e_2, ...e_k\}, C = \{c_1, c_2\}, S = A \bigcup B \bigcup C \bigcup D \bigcup E$. It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $a_1 + b_1 = c_1 \in S$ for $a_1 \in A$ and for $b_1 \in B$.
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B(i \neq j)$;
4. $d_i + d_j \notin S$ for any $d_i, d_j \in D(i \neq j)$;
5. $e_i + e_j \notin S$ for any $e_i, e_j \in E(i \neq j)$;
6. $a_i + b_j \in S$ if and only if a_i is adjacent to b_j.
7. $a_i + d_j \in S$ if and only if $i = j$; and $a_1 b_1 a_2 b_2 a_3 b_3 .... b_k a_{k+1} a_1$ is a cycle C_{2K+1}.
8. $a_i + e_j \notin S$ for any $a_i \in A$ and for any $e_j \in E$;
9. $b_i + d_j \notin S$ for any $b_i \in B$ and for any $d_j \in D$;
10. $b_i + e_j \in S$ if and only if $i = j$;
11. $d_i + e_j \notin S$ for any $d_i \in D$ and for any $e_j \in E$;
12. $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C$;
13. $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$;
14. $d_i + c_j \notin S$ for any $d_i \in D$ and for any $c_j \in C$;
15. $e_i + c_j \notin S$ for any $e_i \in E$ and for any $c_j \in C$;
16. $c_i + c_j \notin S$ for any $c_i, c_j \in C(i \neq j)$.

Case-2: $n = 2k\left(k \geq 2\right)$.

- $f_1 = -1,\ f_2 = 3,\ f_{i+1} = f_{i-1} - f_i, i = 2,3,...,2k-1.$
- $a_i = f_{2i-1} i = 1,2,...,k, b_i = f_{2i}, i = 1,2,...,k.$
- $d_1 = a_1 + b_k, d_{i+1} = a_i + d_i, i = 1,2,...,k-1, e_1 = a_k + d_k, e_{i+1} = b_i + e_i i = 1,2,...,k-1.$
- $c_1 = a_1 + b_1, c_2 = b_k + e_k$.

Let $A = \left\{a_1, a_2, ..., a_k\right\}, B = \left\{b_1, b_2, ..., b_k\right\}, D = \left\{d_1, d_2, ..., d_k\right\}$,
$E = \left\{e_1, e_2, ...e_k\right\}, C = \left\{c_1, c_2\right\}, S = A \bigcup B \bigcup C \bigcup D \bigcup E$.It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $a_1 + b_1 = c_1 \in S$ for $a_1 \in A$ and $b_1 \in B$
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B\left(i \neq j\right)$;
4. $d_i + d_j \notin S$ for any $d_i, d_j \in D\left(i \neq j\right)$;
5. $e_i + e_j \notin S$ for any $e_i, e_j \in E\left(i \neq j\right)$;
6. $a_i + b_j \in S$ if and only if a_i is adjacent to b_j.
7. $a_i + d_j \in S$ if and only if $i = j$; and $a_1 b_1 a_2 b_2 a_3 b_3 ....a_k b_k a_1$ is a cycle C_{2K}.
8. $a_i + e_j \notin S$ for any $a_i \in A$ and for any $e_j \in E$;
9. $b_i + d_j \notin S$ for any $b_i \in B$ and for any $d_j \in D$;
10. $b_i + e_j \in S$ if and only if $i = j$;
11. $d_i + e_j \notin S$ for any $d_i \in D$ and for any $e_j \in E$;
12. $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C$;
13. $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$;
14. $d_i + c_j \notin S$ for any $d_i \in D$ and for any $c_j \in C$;
15. $e_i + c_j \notin S$ for any $e_i \in E$ and for any $c_j \in C$;
16. $c_i + c_j \notin S$ for any $c_i, c_j \in C\left(i \neq j\right)$.

Thus the above projected labeling is an integral sum labeling of graph $\left(C_n \odot K_1\right) \bigcup 2K_1$ and $\zeta\left(C_n \odot K_1\right) = 2$, for $n \geq 3$. □

Theorem 2.13: $C_n \odot K_1$ is a mod sum graph and $\rho\left(C_n \odot K_1\right) = 0$, for $n \geq 4$.

Proof: We only give a mod sum labeling of the graph $\left(C_n \odot K_1\right)$.We can consider the following two cases.

Case-1: $n = 2k+1\left(k \geq 2\right)$.

- $a_i = \left(k-i\right)N+1, i = 1,2,3,...,k, b_i = \left(k-i+1\right)N, i = 1,2,...,k.$
- $d_i = \left(k-i\right)N+2, i = 1,2,...,k, e_i = \left(k-i\right)N+3, i = 1,2,...,k.$
- $a_{k+1} = kN+1, d_{k+1} = kN+2$ and with modulus $z = kN$, where $N \geq 5$ is an integer.

Let $A = \left\{a_1, a_2, ..., a_k, a_{k+1}\right\}, B = \left\{b_1, b_2, ..., b_k\right\}, D = \left\{d_1, d_2, ..., d_k, d_{k+1}\right\}$,
$E = \left\{e_1, e_2, ...e_k\right\}, S = A \bigcup B \bigcup D \bigcup E$ .It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $a_i + b_j \bmod z \in S$ If and only if a_i is adjacent to b_j.
3. $a_i + d_j \bmod z \in S$ if and only if $i = j$.
4. $b_i + e_j \bmod z \in S$ if and only if $i = j$.
5. $a_1 b_1 a_2 b_2 a_3 b_3 ...b_k a_{k+1} a_1$ is a cycle C_{2k+1}.

Case-2: $n = 2k\left(k \geq 2\right)$.

- $a_i = \left(k-i\right)N+1, i=1,2,3,...,k, b_i = \left(k-i+1\right)N, i=1,2,...,k.$
- $d_i = \left(k-i\right)N+3, i=1,2,...,k, e_i = \left(k-i\right)N+4, i=1,2,...,k$, with modulus

$z = \left(n-k\right)N$,where $N \geq 5$ is an integer. Let $A = \left\{a_1, a_2, ..., a_k\right\}, B = \left\{b_1, b_2, ..., b_k\right\}$, $D = \left\{d_1, d_2, ...d_k\right\}, E = \left\{e_1, e_2, ...e_k\right\}, S = A \bigcup B \bigcup D \bigcup E$ .It is easily verified the following assertions are true and valid.

1. The vertices of S are distinct.
2. $a_i + b_j \bmod z \in S$ If and only if a_i is adjacent to b_j.
3. $a_i + d_j \bmod z \in S$ if and only if $i = j$.
4. $b_i + e_j \bmod z \in S$ if and only if $i = j$.
5. $a_1 b_1 a_2 b_2 a_3 b_3 ....a_k b_k a_1$ is a cycle C_{2K}.

Thus the above projected labeling is a mod sum labeling of graph $\left(C_n \odot K_1\right)$ .So $C_n \odot K_1$ is a mod sum graph and $\rho\left(C_n \odot K_1\right) = 0$, for $n \geq 3$. □

REFERENCES

1. A BONDY, U S R MURTY. Graph Theory with application [M].American Elsevier Publishing Co., Inc., New York, 1976.
2. F. Harary, Sum graphs and difference graphs, Congressus Numerantium 72 (1990) 101-108.
3. F. Harary, Sum graphs over all integers, Discrete Mathematics124 (1994) 99-105.
4. J.Bolland, R.Laskar, C. Turner, G.Domke, (1990) On mod sum graphs Congressus Numerantium 70; 131-135.
5. F. Fernau, J.F. Ryan, K.A. Sugeng (2009) A sum labeling for the generalized friendship graph, Discrete Math 308: 734-740.
6. J.Wu, J.Mao, D.Li , New types of integral sum graphs, Discrete Mathematics 260, (2003) 163-176.
7. W. Dou, J.Gao, The (mod, integral) sum numbers of fans and $K_{n,n} - E\left(nK_2\right)$, Discrete Mathematics 306 (2006) 2655-2669.
8. J.A. Gallian, A dynamic survey of graph labeling, Electronic Journal of Combinatorics, DS6, Nineteenth edition, December 26, 2016.

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