

FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS IN COMPLETE METRIC SPACE

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ABSTRACT

In this paper, we present fixed point results for generalization on spaces with two metrics. The focus is on continuation results for such type of mappings.

Key words: Metric Space, Complete Metric Space, Self Mapping, Fixed Point, Commute Mapping.

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INTRODUCTION

The study of common fixed point of mapping contractive type condition has been a very active field of research activity during the last three decades. The most general of the common fixed point pertain to two or three mapping of a metric space (X, d) and use either a Banach type contractive condition or other contractive condition. Many, Hardy [1], Rajput [2], Yadav [3], Sengupta [4] and so many authors work in this field and prove more interesting result. Throughout this section (X, d') denotes a complete metric space and d be an metric on X . if $x_0 \in X$ and $r > 0$ denote by $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ and by $\text{clos. } B(x_0, r)^{d'}$ the d' -closer of $B(x_0, r)$.

Fixed point results for Banach Generalized contractions

Theorem: 1 Let (X, d') be a complete metric space, d another metric on X , $x_0 \in X$, $r > 0$ and T be the mapping from, $\text{clos. } B(x_0, r)^{d'}$ into X , satisfying the following conditions;

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \tag{1.1}$$

Where non negative α , such that, $0 \leq \alpha < 1$

In addition assume the following three properties hold:

$$d(x_0, Tx_0) < (1 - \alpha) r \tag{1.2}$$

$$\text{If } d \neq d' \text{ then } T \text{ is uniformly continuous from } (B(x_0, r), d) \text{ into } (X, d') \tag{1.3}$$

$$\text{if } d \neq d' \text{ then } T \text{ is continuous from } (\text{clos. } B(x_0, r)^{d'}, d') \text{ into } (X, d') \tag{1.4}$$

then T has fixed point, that is there exists $x \in \text{clos. } B(x_0, r)^{d'}$ with $Tx = x$.

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Proof: Let $x_1 = Tx_0$ then from (1.2), we have

$$d(x_0, x_1) = d(x_0, Tx_0) < (1 - \alpha) r \leq r$$

So, that, $x_1 \in B(x_0, r)$

Next let $x_2 = Tx_1$ then we note that,

$$d(x_1, x_2) = d(Tx_0, Tx_1)$$

From (1.1)

$$d(Tx_0, Tx_1) \leq \alpha d(x_0, x_1)$$

$$d(Tx_0, Tx_1) \leq \alpha (1 - \alpha) r$$

Now

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ d(x_0, x_2) &\leq (1 - \alpha) r + \alpha (1 - \alpha) r \\ d(x_0, x_2) &\leq (1 - \alpha) r (1 + \alpha) \\ d(x_0, x_2) &< (1 - \alpha) r (1 + \alpha + \alpha^2 + \alpha^3 + \dots \dots \dots) \\ d(x_0, x_2) &< (1 - \alpha) r (1 - \alpha)^{-1} \\ d(x_0, x_2) &< r \end{aligned}$$

So that, $x_2 \in B(x_0, r)$

Proceeding inductively we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \alpha^n d(x_0, x_1) \\ d(x_0, x_{n+1}) &< (1 - \alpha)^n r (1 - \alpha)^{-1} \end{aligned}$$

It follows $d(x_0, x_{n+1}) < r$ and $x_{n+1} \in B(x_0, r)$

In this way we construct a sequence $\{x_n\}$ of elements of X , such that $\{x_n\}$ is a Cauchy sequence with respect to, d , which converges to x .

We claim that $\{x_n\}$ is a Cauchy sequence with respect to d' .

If $d \geq d'$ then this is trivial.

Next we suppose that, $d \not\geq d'$

Let $\varepsilon > 0$ be given. Now from (1.3) that there exists $\delta > 0$ such that,

$$d'(Tx, Ty) < \varepsilon \text{ Whenever } x, y \in B(x_0, r) \text{ and } d(x, y) < \delta \tag{1.5}$$

From the above the sequence $\{x_n\}$ is a Cauchy sequence with respect to d , so we know that there exists N with

$$d(x_n, x_m) < \delta \text{ for all } n, m \geq N \tag{1.6}$$

Now from (1.5) and (1.6) implies

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon \text{ whenever } n, m \geq N$$

Which proves that $\{x_n\}$ is a Cauchy sequence with respect to d'

Now since (X, d') is complete there exists $x \in \text{clos. } B(x_0, r)^{d'}$ with

$$d'(x_n, x) \rightarrow 0 \text{ and } n \rightarrow \infty.$$

$$\text{We claim that, } x = Tx \tag{1.7}$$

First consider the case, when $d \neq d'$.

$$d'(x, Tx) \leq d(x, x_n) + d(x_n, Tx) = d(x, x_n) + d(Tx_{n-1}, Tx)$$

Let $n \rightarrow \infty$ and using (1.4), we obtain

$$d'(x, Tx) \leq d(x, x) + d(Tx, Tx)$$

$$d'(x, Tx) = 0$$

And thus (8.7) is true,

Next we suppose that $d = d'$ then

$$d'(x, Tx) \leq d(x, x_n) + d(Tx_{n-1}, Tx)$$

From (1.1),

$$d'(x, Tx) \leq d(x, x_n) + \alpha d(x_{n-1}, Tx)$$

As $n \rightarrow \infty$, $Tx_n = x = Tx$ and above inequality can be written as,

$$(1 - \alpha)d(x, Tx) \leq 0$$

So that, $d(x, Tx) = 0$ and (1.7) holds.

This the proof of the theorem.

Theorem: 2 Let (X, d') be a complete metric space, d another metric on X , $x_0 \in X$, $r > 0$ and T be the mapping from, $\text{clos. } B(x_0, r)^{d'}$ into X , satisfying the following conditions;

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \quad (2.1)$$

Where non negative α, β, γ , such that, $0 \leq \alpha + \beta + \gamma < 1$

In addition assume the following three properties hold:

$$d(x_0, Tx_0) < \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r \quad (2.2)$$

If $d \not\geq d'$

then T is uniformly continuous from $(B(x_0, r), d)$ into (X, d') (2.3)

if $d \neq d'$ then T is continuous from $(\text{clos. } B(x_0, r)^{d'}, d')$ into (X, d') (2.4)

then T has fixed point, that is there exists $x \in \text{clos. } B(x_0, r)^{d'}$ with $Tx = x$.

Proof: Let $x_1 = Tx_0$ then from (2.2), we have

$$d(x_0, x_1) = d(x_0, Tx_0) < \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r \leq r$$

So that, $x_1 \in B(x_0, r)$

Next let $x_2 = Tx_1$ then we note that,

$$d(x_1, x_2) = d(Tx_0, Tx_1)$$

From (2.1)

$$d(Tx_0, Tx_1) \leq \alpha d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma d(x_0, x_2)$$

$$d(Tx_0, Tx_1) \leq \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right) \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r$$

Now

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ d(x_0, x_2) &\leq \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r + \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right) \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \\ d(x_0, x_2) &\leq \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \left(1 + \frac{\alpha+\beta}{1-\beta-\gamma}\right) \\ d(x_0, x_2) &< \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \left(1 + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right] + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^2 + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^3 + \dots \dots \dots\right) \\ d(x_0, x_2) &< \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right) r \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1} \\ d(x_0, x_2) &< r \end{aligned}$$

So that, $x_2 \in B(x_0, r)$

Proceeding inductively we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^n d(x_0, x_1) \\ d(x_0, x_{n+1}) &< \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^n r \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1} \end{aligned}$$

It follows $d(x_0, x_{n+1}) < r$ and $x_{n+1} \in B(x_0, r)$

In this way we construct a sequence $\{x_n\}$ of elements of X , such that $\{x_n\}$ is a Cauchy sequence with respect to, d , which converges to x .

We claim that $\{x_n\}$ is a Cauchy sequence with respect to d' .

If $d \geq d'$ then this is trivial.

Next we suppose that, $d > \neq d'$

Let $\varepsilon > 0$ be given Now from (1.3) that there exists $\delta > 0$ such that,

$$d'(Tx, Ty) < \varepsilon \text{ whenever } x, y \in B(x_0, r) \text{ and } d(x, y) < \delta \tag{2.5}$$

From the above the sequence $\{x_n\}$ is a Cauchy sequence with respect to d , so we know that there exists N with

$$d(x_n, x_m) < \delta \text{ for all } n, m \geq N \tag{2.6}$$

Now from (2.5) and (2.6) implies

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon \text{ whenever } n, m \geq N$$

Which proves that $\{x_n\}$ is a Cauchy sequence with respect to d' .

Now since (X, d') is complete there exists $x \in \text{clos. } B(x_0, r)^{d'}$ with

$$d'(x_n, x) \rightarrow 0 \text{ and } n \rightarrow \infty.$$

We claim that, $x = Tx$

(2.7)

First consider the case, when $d \neq d'$

$$d'(x, Tx) \leq d(x, x_n) + d(x_n, Tx) = d(x, x_n) + d(Tx_{n-1}, Tx)$$

Let $n \rightarrow \infty$ and using (2.4), we obtain

$$d'(x, Tx) \leq d(x, x) + d(Tx, Tx)$$

$$d'(x, Tx) = 0$$

And thus (2.7) is true,

Next we suppose that $d = d'$ then

$$d'(x, Tx) \leq d(x, x_n) + d(Tx_{n-1}, Tx)$$

From (2.1),

$$d'(x, Tx) \leq d(x, x_n) + \alpha d(x_{n-1}, Tx) + \beta [d(x_{n-1}, Tx_{n-1}) + d(x, Tx)] + \gamma [d(x_{n-1}, Tx) + d(x, Tx_{n-1})]$$

As $n \rightarrow \infty$, $Tx_n = x = Tx$ and above inequality can be written as,

$$\left(1 - \left[\frac{\alpha + \beta}{1 - \beta - \gamma}\right]\right) d(x, Tx) \leq 0$$

So that, $d(x, Tx) = 0$ and (2.7) holds.

This complete proof of the theorem

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