

SUM CORDIAL LABELING OF ZERO-DIVISOR GRAPHS

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ABSTRACT

Let $G = (V, E)$ be a simple graph with n vertices. A function $f : V(G) \rightarrow \{0, 1\}$ is said to be a sum cordial labeling if for each edge $e = uv$, the induced map $f^*(uv) = (f(u) + f(v)) \pmod{2}$ satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ are the number of vertices and edges with label i , $i \in \{0, 1\}$ respectively. A graph G is said to be sum cordial if it has a sum cordial labeling. In this paper, we prove that certain classes of zero-divisor graphs of commutative rings are sum cordial.

Keywords: zero-divisor graph, cordial labeling, sum cordial labeling.

Subject Classification: Primary: 05C78, 05C25.

1. INTRODUCTION

All graphs considered in this paper are finite, simple and undirected. A detailed study of applications on graph labeling is carried out Bloom and Golomb [3]. The complete summary of graceful and harmonious graphs and the results along with some open problems can be found in J. Gallian's [5] dynamic survey of graph labeling. The notion of cordial graphs were introduced by I. Cahit [4] in 1987, as a weaker version of both graceful and harmonious graphs. J. Shiama, introduced the concept of sum cordial [7] in 2012. Let $G = (V, E)$ be a simple graph with n vertices. A function $f : V(G) \rightarrow \{0, 1\}$ is said to be a sum cordial labeling if for each edge $e = uv$, the induced map $f^*(uv) = (f(u) + f(v)) \pmod{2}$ satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ are the number of vertices and edges with label i , $i \in \{0, 1\}$ respectively. A graph G is said to be sum cordial graph if it has a sum cordial labeling. The idea of a graph associated to zero-divisors of a commutative ring was introduced by I. Beck [2] in 1988. Later it was modified by D.F Anderson and P.S Livingston [1] and accordingly, for a commutative ring R , the zero-divisor graph $\Gamma(R)$ is the simple undirected graph with vertex set $Z(R)^*$, the set of all non-zero divisors in R and two distinct vertices x and y are adjacent if $xy = 0$. The complement \overline{G} of the graph G is the graph with vertex set $V(G)$ and two vertices are adjacent in \overline{G} if and only if they are not adjacent in G . For graph theoretic terminology and standard notation we follow F. Harray [6]. The join $G_1 + G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E[G_1 + G_2] = E(G_1) \cup E(G_2) \cup [uv : u \in V(G_1) \text{ and } v \in V(G_2)]$.

In this paper, we prove that certain zero-divisor graphs of commutative rings are sum cordial.

2. SUM CORDIAL LABELING

In this section, we prove that certain classes of zero-divisor graphs are sum cordial.

Theorem 2.1: For any prime number $p > 2$, $\Gamma(\mathbb{Z}_{2p})$ is sum cordial.

Proof: Let $p > 2$ be a prime number. Then the vertex set of $\Gamma(\mathbb{Z}_{2p})$ is

$$V(\Gamma(\mathbb{Z}_{2p})) = \{v_1, \dots, v_{p-1}, v_p\} = \{2, 4, \dots, 2(p-1), p\}$$

and the edge set

$$E(\Gamma(\mathbb{Z}_{2p})) = \{v_i v_p / 1 \leq i \leq p-1\}.$$

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Define $f : V(G) \rightarrow \{0, 1\}$ by $f(v_i) = i \pmod{2}$, for $1 \leq i \leq p-1$ and $f(v_p) = 1$.

$$\text{Clearly } v_f(0) = \frac{p-1}{2}; v_f(1) = \frac{p-1}{2} + 1.$$

Then the induced edge labeling function $f^* : E(G) \rightarrow \{0, 1\}$ is given by

$$f^*(v_i v_p) = \begin{cases} 0 & \text{if } i \text{ is odd;} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

From this we have that $e_f(0) = \frac{p-1}{2} = e_f(1)$ and so $\Gamma(Z_{2p})$ is sum cordial.

Theorem 2.2: For any prime number $p > 3$, the zero-divisor graph $\Gamma(Z_{3p})$ is sum cordial.

Proof: Let $p > 3$ be a prime number. Then

$$Z^*(Z_{3p}) = \{u_1, u_2, v_1, \dots, v_{p-1}\}$$

where $u_1 = p, u_2 = 2p$ and $v_i = 3i$ for $1 \leq i \leq p-1$. From this, we have that

$$E(\Gamma(Z_{3p})) = \{u_1 v_i, u_2 v_i, 1 \leq i \leq p-1\}.$$

Note that $|V(\Gamma(Z_{3p}))| = p+1, |E(\Gamma(Z_{3p}))| = 2p-2$. let us define $f : V(G) \rightarrow \{0, 1\}$

by $f(u_1) = 0, f(u_2) = 1$ and $f(v_i) = i \pmod{2}, 1 \leq i \leq p-1$.

$$\text{It is clear that } v_f(0) = \frac{p+1}{2} = v_f(1).$$

Then the induced edge labeling function $f^* : E(G) \rightarrow \{0, 1\}$ is given by

$$f^*(u_1 u_j) = \begin{cases} 1 & \text{if } j \text{ is odd;} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

$$\text{and } f^*(u_2 u_j) = \begin{cases} 0 & \text{if } j \text{ is odd;} \\ 1 & \text{if } j \text{ is even.} \end{cases}$$

Also we have that $e_f(0) = \frac{p-1}{2} + \frac{p-1}{2} = p-1, e_f(1) = \frac{p-1}{2} + \frac{p-1}{2} = p-1$ and so $\Gamma(Z_{3p})$ is sum cordial graph.

Theorem 2.3: For any prime number $p \geq 3$, the zero-divisor graph $\Gamma(Z_{4p})$ is sum cordial.

Proof: One can partition the vertex set of $\Gamma(Z_{4p})$ into $V_1 = \{p, 2p, 3p\} = \{u_1, u_2, u_3\}$ and

$$V_2 = \{2, 4, \dots, 2(p-1), 2(p+1), \dots, 2(2p-1)\} = \{v_1, v_2, \dots, v_{p-1}, v_{p+1}, \dots, v_{2p-1}\}.$$

Note that the edge set of $\Gamma(Z_{4p})$ is nothing but

$$E(\Gamma(Z_{4p})) = \{u_1 v_2, u_1 v_4, \dots, u_1 v_{p-1}, u_1 v_{p+1}, \dots, u_1 v_{2p-2}, u_2 v_1, u_2 v_2, u_2 v_3, \dots, u_2 v_{p-1}, \\ u_2 v_{p+1}, \dots, u_2 v_{2p-1}, u_3 v_2, u_3 v_4, \dots, u_3 v_{p-1}, u_3 v_{p+1}, \dots, u_3 v_{2p-2}\}.$$

$$\text{Clearly } |V| = 2p+1, |E| = 2p-2 + \frac{2(2p-2)}{2} = 4p-4.$$

Define $f : V(\Gamma(Z_{4p})) \rightarrow \{0, 1\}$ by $f(u_1) = 0, f(u_2) = 1, f(u_3) = 0;$

$$f(v_j) = j \pmod{2}, 1 \leq j \leq p-1 \text{ and}$$

$$f(v_j) = (j-1) \pmod{2}, p+1 \leq j \leq 2p-1.$$

From this we have that $v_f(0) = p-1+2 = p+1; v_f(1) = p-1+1 = p$.

The induced edge labeling function $f^* : E \rightarrow \{0, 1\}$ is given by

$$f^*(u_1 v_j) = f^*(u_3 v_j) = 0, j \text{ is even}, 1 \leq j \leq p-1,$$

$$f^*(u_1 v_j) = f^*(u_3 v_j) = 1, j \text{ is odd}, p+1 \leq j \leq 2p-1 \text{ and}$$

$$f^*(u_2u_j) = \begin{cases} 1 & \text{if } j \text{ is odd;} \\ 0 & \text{if } j \text{ is even.} \end{cases}, \text{ for } j \neq p \text{ and } 1 \leq j \leq 2p-1$$

$$\Rightarrow e_f(0) = \frac{2p-2}{2} + \frac{2p-2}{2} = 2p-2 \text{ and } e_f(1) = 2p-2.$$

Hence $|e_f(0) - e_f(1)| \leq 1$ and so $\Gamma(Z_{4p})$ is sum cordial.

Theorem 2.4: For two distinct primes p and q with $p < q$, the zero-divisor graph $\Gamma(Z_{pq})$ is sum cordial.

Proof: The vertex set of $\Gamma(Z_{pq})$ can be partitioned into V_1 and V_2 where

$$V_1 = \{p, 2p, 3p, \dots, (q-1)p\} = \{u_1, u_2, \dots, u_{q-1}\} \text{ and}$$

$$V_2 = \{q, 2q, 3q, \dots, (p-1)q\} = \{v_1, v_2, \dots, v_{p-1}\}.$$

The edge set of $\Gamma(Z_{pq})$ is given by

$$E(\Gamma(Z_{pq})) = \{u_i v_j : u_i \in V_1 \text{ and } v_j \in V_2, 1 \leq i \leq q-1, 1 \leq j \leq p-1\}.$$

Consider the vertex labeling $f : V(\Gamma(Z_{pq})) \rightarrow \{0, 1\}$ defined by

$$f(u_i) = i \pmod{2} \text{ for } 1 \leq i \leq q-1 \text{ and}$$

$$f(v_j) = j \pmod{2} \text{ for } 1 \leq j \leq p-1.$$

Further note that $|E| = (p-1)(q-1)$ and the induced edge labeling $f^* : E \rightarrow \{0, 1\}$ is given by

$$f^*(u_i v_j) = \begin{cases} 0 & \text{both } i \text{ \& } j \text{ are odd, } 1 \leq i < q-1, 1 \leq j \leq p-1; \text{ and both } i \text{ \& } j \text{ are even;} \\ 1 & \text{otherwise} \end{cases}$$

$$\text{Clearly } e_f(0) = \frac{(p-1)(q-1)}{2} \text{ and } e_f(1) = \frac{(p-1)(q-1)}{2}.$$

Hence $|e_f(0) - e_f(1)| \leq 1$ and so $\Gamma(Z_{pq})$ is sum cordial.

Theorem 2.5: For any prime number $p > 2$, the join graph $\Gamma(Z_{2p}) + \Gamma(Z_4)$ is sum cordial.

Proof: Let $G = \Gamma(Z_{2p}) + \Gamma(Z_4)$.

The vertex set of the graph G ,

$$V(G) = \{u_1, u_2, \dots, u_{p-1}, u_p, x\} = \{2, 4, \dots, 2(p-1), p, x\}, \text{ where } x = 2 \in Z_4.$$

Also the edge set of G ,

$$E(G) = \{u_i u_p, u_i x, u_p x / 1 \leq i \leq p-1\}.$$

Note that $|V| = p+1$ and $|E| = p-1 + p-1 + 1 = 2p-1$.

Define the vertex labeling $f : V(G) \rightarrow \{0, 1\}$ by

$$f(u_k) = k \pmod{2}, 1 \leq k \leq p \text{ and } f(x) = 0.$$

$$\text{It is clear that } v_f(0) = \frac{p+1}{2} = v_f(1)$$

Then the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by

$$f^*(u_i u_p) = \begin{cases} 0 & \text{if } i \text{ is odd;} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

$$f^*(u_i x) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

$$\text{and } f^*(u_p x) = 1.$$

$$\text{From the above } e_f(0) = \frac{(p-1)}{2} + \frac{(p-1)}{2} = p-1, e_f(1) = \frac{(p-1)}{2} + \frac{(p-1)}{2} + 1 = p$$

and $|e_f(0) - e_f(1)| \leq 1$.

Therefore G is sum cordial.

Theorem 2.6: For any prime number $p > 2$, the join graph $\Gamma(Z_{2p}) + \Gamma(Z_9)$ is sum cordial.

Proof: Let $G = \Gamma(Z_{2p}) + \Gamma(Z_9)$. The vertex set of the graph G is

$$\begin{aligned} V(G) &= \{u_1, \dots, u_{p-1}, u_p, x, y\} \\ &= \{2, 4, \dots, 2(p-1), p, x, y\}, \text{ where } x = 3 \text{ and } y = 6 \in Z_9, \text{ and the edge set of } G \text{ is} \end{aligned}$$

$$E(G) = \{u_i u_p, u_i x, u_i y, u_p x, u_p y, xy / 1 \leq i \leq p-1\}.$$

Note that $|V| = p+2$ and $|E| = p-1 + p-1 + p-1 + 3 = 3p$.

Define the vertex labeling $f : V(G) \rightarrow \{0, 1\}$ by $f(u_k) = k \pmod{2}$ for $1 \leq k \leq p$, $f(x) = 0$ and $f(y) = 1$.

$$\text{Clearly } v_f(0) = \frac{p-1}{2} + 1 \text{ and } v_f(1) = \frac{p-1}{2} + 2.$$

Then the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by

$$\begin{aligned} f^*(u_i u_p) &= \begin{cases} 0 & \text{if } i \text{ is odd;} \\ 1 & \text{if } i \text{ is even.} \end{cases} \\ f^*(u_i x) &= \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd.} \end{cases} \\ f^*(u_i y) &= \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \\ f^*(u_p x) &= 1, f^*(u_p y) = 0 \text{ and } f^*(xy) = 1. \end{aligned}$$

From the above $e_f(0) = \frac{3(p-1)}{2} + 1$, $e_f(1) = \frac{3(p-1)}{2} + 2$ and $|e_f(-1) - e_f(1)| \leq 1$. Hence G is sum cordial.

Theorem 2.7: For any prime number $p > 2$, the join graph $\Gamma(Z_{2p}) + \Gamma(Z_6)$ is sum cordial.

Proof: Let $G = \Gamma(Z_{2p}) + \Gamma(Z_6)$.

The vertex set of the graph G ,

$$\begin{aligned} V(G) &= \{u_1, \dots, u_{p-1}, u_p, x, y, z\} \\ &= \{2, 4, \dots, 2(p-1), p, x, y, z\}, \text{ where } x = 2, y = 3 \text{ and } z = 3 \in Z_6 \text{ and the edge set of } G \end{aligned}$$

$$E(G) = \{u_i u_p, u_i x, u_i y, u_i z, u_p x, u_p y, u_p z, xy, yz / 1 \leq i \leq p-1\}.$$

Note that $|V| = p + 3$ and $|E| = p-1 + p-1 + p-1 + p-1 + 5 = 4p + 1$.

Define the vertex labeling $f : V(G) \rightarrow \{0, 1\}$ by

$$f(u_k) = k \pmod{2} \text{ for } 1 \leq k \leq p, f(x) = 0, f(y) = 0 \text{ and } f(z) = 1.$$

$$\text{Note that } v_f(0) = \frac{p-1}{2} + 2; v_f(1) = \frac{p-1}{2} + 2.$$

Then the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by

$$\begin{aligned} f^*(u_i u_p) &= \begin{cases} 0 & \text{if } i \text{ is odd;} \\ 1 & \text{if } i \text{ is even.} \end{cases} \\ f^*(u_i x) &= f^*(u_i y) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd.} \end{cases} \\ f^*(u_i z) &= \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \\ f^*(u_p x) &= 1, f^*(u_p y) = 1, f^*(u_p z) = 0, f^*(xy) = 0 \text{ and } f^*(yz) = 1. \end{aligned}$$

$$\begin{aligned} \text{From the above } e_f(0) &= \frac{4(p-1)}{2} + 2 \\ e_f(1) &= \frac{4(p-1)}{2} + 3 \text{ and} \\ |e_f(0) - e_f(1)| &\leq 1. \end{aligned}$$

Therefore G is sum cordial.

Corollary 2.8: For any prime number $p > 2$, the join graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_4)$ is sum cordial.

Proof: Since the graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_{2p})$, by Theorem 2.1, $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_4)$ is sum cordial.

Theorem 2.9: For any prime number $p > 2$, the join graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_9)$ is sum cordial.

Proof: Let $G = \overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_9)$. The vertex set of the graph G is

$$\begin{aligned} V(G) &= \{u_1, \dots, u_{p-1}, x, y\} \\ &= \{p, 2p, \dots, (p-1)p, x, y\}, \text{ where } x = 3 \text{ and } y = 6 \in \mathbb{Z}_9 \text{ and the edge set of G is} \end{aligned}$$

$$E(G) = \{u_i x, u_i y, xy \mid 1 \leq i \leq p-1\}.$$

Note that $|V| = p+1$ and $|E| = p-1 + p-1 + 1 = 2p-1$.

Define the vertex labeling $f : V(G) \rightarrow \{0, 1\}$ by $f(u_k) = k \pmod{2}$, for $1 \leq k \leq p-1$, $f(x) = 0$ and $f(y) = 1$.

$$\text{Clearly } v_f(0) = \frac{p-1}{2} + 1 \text{ and } v_f(1) = \frac{p-1}{2} + 1.$$

Then the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by

$$\begin{aligned} f^*(u_i x) &= \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd.} \end{cases} \\ f^*(u_i y) &= \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \\ f^*(xy) &= 1. \end{aligned}$$

From the above

$$\begin{aligned} e_f(0) &= \frac{(p-1)}{2} + \frac{(p-1)}{2} = p-1, \\ e_f(1) &= \frac{(p-1)}{2} + \frac{(p-1)}{2} + 1 = p \text{ and satisfies } |e_f(0) - e_f(1)| \leq 1. \end{aligned}$$

Hence G is sum cordial.

Theorem 2.10: For any prime number $p > 2$, the join graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_6)$ is a signed product cordial.

Proof: Let $G = \overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_6)$.

The vertex set of the graph G is

$$\begin{aligned} V(G) &= \{u_1, \dots, u_{p-1}, x, y, z\} \\ &= \{p, 2p, \dots, (p-1)p, x, y, z\}, \text{ where } x = 2, y = 3 \text{ and } z = 3 \in \mathbb{Z}_6. \end{aligned}$$

Further the edge set of G is

$$E(G) = \{u_i x, u_i y, u_i z, xy, yz \mid 1 \leq i \leq p-1\}.$$

Note that $|V| = p+2$ and $|E| = p-1 + p-1 + p-1 + 2 = 3p-1$.

Define the vertex labeling $f : V(G) \rightarrow \{0, 1\}$ by $f(u_k) = k \pmod{2}$ for $1 \leq k \leq p-1$, $f(x) = 0$, $f(y) = 0$ and $f(z) = 1$.

$$\text{Clearly } v_f(0) = \frac{p-1}{2} + 2; v_f(1) = \frac{p-1}{2} + 1.$$

Then the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by

$$f^*(u_i, x) = f^*(u_i, y) = \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

$$f^*(u_i, z) = \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

$$f^*(xy) = 0 \text{ and } f^*(yz) = 1$$

From the above

$$e_f(0) = \frac{(p-1)}{2} + \frac{(p-1)}{2} + \frac{(p-1)}{2} + 1 = \frac{3(p-1)}{2} + 1,$$

$$e_f(1) = \frac{(p-1)}{2} + \frac{(p-1)}{2} + \frac{(p-1)}{2} + 1 = \frac{3(p-1)}{2} + 1 \text{ and satisfies}$$

$$|e_f(0) - e_f(1)| \leq 1. \text{ Hence } G \text{ is sum cordial.}$$

Corollary 2.11: For any prime number $p > 2$, the join graph $\overline{\Gamma(Z_{p^2})} + \Gamma(Z_4)$ is sum cordial.

Proof: Since the graph $\overline{\Gamma(Z_4)} = \Gamma(Z_4)$, by Corollary 2.8. $\overline{\Gamma(Z_{p^2})} + \Gamma(Z_4)$ is sum cordial.

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