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SUM CORDIAL LABELING OF ZERO-DIVISOR GRAPHS

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ABSTRACT

Let G = (V, E) be a simple graph with n vertices. A function $f : V(G) \to \{0, 1\}$ is said to be a sum cordial labeling if for each edge e = uv, the induced map $f^*(uv) = (f(u) + f(v)) \pmod 2$ satisfies the conditions $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ where $v_f(i)$ and $e_f(i)$ are the number of vertices and edges with label i, $i \in \{0, 1\}$ respectively. A graph G is said to be sum cordial if it has a sum cordial labeling. In this paper, we prove that certain classes of zero-divisor graphs of commutative rings are sum cordial.

Keywords: *zero-divisor graph, cordial labeling, sum cordial labeling.*

Subject Classification: *Primary: 05C78, 05C25.*

1. INTRODUCTION

All graphs considered in this paper are finite, simple and undirected. A detailed study of applications on graph labeling is carried out Bloom and Golomb [3]. The complete summary of graceful and harmonious graphs and the results along with some open problems can be found in J. Gallian's [5] dynamic survey of graph labeling. The notion of cordial graphs were introduced by I. Cahit [4] in 1987, as a weaker version of both graceful and harmonious graphs. J. Shiama, introduced the concept of sum cordial [7] in 2012. Let G = (V, E) be a simple graph with n vertices. A function $f : V(G) \to \{0, 1\}$ is said to be a sum cordial labeling if for each edge e = uv, the induced map $f^*(uv) = (f(u) + f(v))$ (mod 2) satisfies the conditions $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ where $v_f(i)$ and $e_f(i)$ are the number of vertices and edges with label i, $i \in \{0, 1\}$ respectively. A graph G is said to be sum cordial graph if it has a sum cordial labeling. The idea of a graph associated to zero-divisors of a commutative ring was introduced by I. Beck [2] in 1988. Later it was modified by D.F Anderson and P.S Livingston [1] and accordingly, for a commutative ring R, the zero-divisor graph $\Gamma(R)$ is the simple undirected graph with vertex set $Z(R)^*$, the set of all non-zero divisors in R and two distinct vertices x and y are adjacent if xy = 0. The complement $\overline{G}$ of the graph G is the graph with vertex set V(G) and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. For graph theoretic terminology and standard notation we follow F. Harray [6]. The join $G_1 + G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E[G_1 + G_2] = E(G_1) \cup E(G_2) \cup [uv : u \in V(G_1) \text{ and } v \in V(G_2)]$.

In this paper, we prove that certain zero-divisor graphs of commutative rings are sum cordial.

2. SUM CORDIAL LABELING

In this section, we prove that certain classes of zero-divisor graphs are sum cordial.

Theorem 2.1: For any prime number p > 2, $\Gamma(\mathbb{Z}_{2p})$ is sum cordial.

Proof: Let p > 2 be a prime number. Then the vertex set of $\Gamma(\mathbb{Z}_{2p})$ is

$$V(\Gamma(\mathbb{Z}_{2p})) = \{v_1, ..., v_{p-1}, v_p\} = \{2, 4, \dots, 2(p-1), p\}$$

and the edge set

$$E(\Gamma(\mathbb{Z}_{2p})) = \{v_i\, v_p \,/\, 1 \le i \le p-1\}.$$

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Define $f : V(G) \to \{0, 1\}$ by $f(v_i) = i \pmod 2$, for $1 \le i \le p-1$ and $f(v_p) = 1$.

Clearly $v_f(0) = \frac{p-1}{2}$; $v_f(1) = \frac{p-1}{2}+1$.

Then the induced edge labeling function $f^* : E(G) \to \{0, 1\}$ is given by

$$f^*(v_i v_p) = \begin{cases} 0 & \text{if } i \text{ is odd;} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

From this we have that $e_f(0) = \frac{p-1}{2} = e_f(1)$ and so $\Gamma(\mathbb{Z}_{2p})$ is sum cordial.

Theorem 2.2: For any prime number $p > 3$, the zero-divisor graph $\Gamma(\mathbb{Z}_{3p})$ is sum cordial.

Proof: Let $p > 3$ be a prime number. Then

$$Z^*(\mathbb{Z}_{3p}) = \{u_1, u_2, v_1, ..., v_{p-1}\}$$

where $u_1 = p$, $u_2 = 2p$ and $v_i = 3i$ for $1 \le i \le p-1$. From this, we have that

$$E(\Gamma(\mathbb{Z}_{3p})) = \{u_1v_i, u_2v_i, 1 \le i \le p-1\}.$$

Note that $|V(\Gamma(\mathbb{Z}_{3p}))| = p + 1$. $|E(\Gamma(\mathbb{Z}_{3p}))| = 2p - 2$. let us define $f : V(G) \to \{0, 1\}$

by $f(u_1) = 0$, $f(u_2) = 1$ and $f(v_i) = i \pmod 2$, $1 \le i \le p-1$.

It is clear that $v_f(0) = \frac{p+1}{2} = v_f(1)$.

Then the induced edge labeling function $f^*: E(G) \to \{0, 1\}$ is given by

$$f^*(u_1u_j) = \begin{cases} 1 & \text{if } j \text{ is odd;} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

and $$f^*(u_2u_j) = \begin{cases} 0 & \text{if } j \text{ is odd;} \\ 1 & \text{if } j \text{ is even.} \end{cases}$$

Also we have that $e_f(0) = \frac{p-1}{2} + \frac{p-1}{2} = p-1$, $e_f(1) = \frac{p-1}{2} + \frac{p-1}{2} = p - 1$ and so $\Gamma(\mathbb{Z}_{3p})$ is sum cordial graph.

Theorem 2.3: For any prime number $p \ge 3$, the zero-divisor graph $\Gamma(\mathbb{Z}_{4p})$ is sum cordial.

Proof: One can partition the vertex set of $\Gamma(\mathbb{Z}_{4p})$ into $V_1 = \{p, 2p, 3p\} = \{u_1, u_2, u_3\}$ and

$$V_2 = \{2, 4, \ldots, 2(p-1), 2(p+1), ..., 2(2p-1)\} = \{v_1, v_2, ..., v_{p-1}, v_{p+1}, ..., v_{2p-1}\}.$$

Note that the edge set of $\Gamma(\mathbb{Z}_{4p})$ is nothing but

$$E(\Gamma(\mathbb{Z}_{4p})) = \{u_1v_2, u_1v_4, ..., u_1v_{p-1}, u_1v_{p+1}, ..., u_1v_{2p-2}, u_2v_1, u_2v_2, u_2v_3, ..., u_2v_{p-1}, u_2v_{p+1}, ..., u_2v_{2p-1}, u_3v_2, u_3v_4, ..., u_3v_{p-1}, u_3v_{p+1}, ..., u_3v_{2p-2}\}.$$

Clearly $|V| = 2p + 1$, $|E| = 2p - 2 + \frac{2(2p-2)}{2} = 4p - 4$.

Define $f : V(\Gamma(\mathbb{Z}_{4p})) \to \{0, 1\}$ by $f(u_1) = 0$, $f(u_2) = 1$, $f(u_3) = 0$;

$f(v_j) = j \pmod 2$, $1 \le j \le p-1$ and

$f(v_j) = (j-1) \pmod 2$, $p+1 \le j \le 2p-1$.

From this we have that $v_f(0) = p - 1 + 2 = p + 1$; $v_f(1) = p - 1 + 1 = p$.

The induced edge labeling function $f^* : E \to \{0, 1\}$ is given by

$f^*(u_1v_j) = f^*(u_3v_j) = 0$, j is even, $1 \le j \le p-1$,

$f^*(u_1v_j) = f^*(u_3v_j) = 1$, j is odd, $p+1 \le j \le 2p-1$ and

$$f^*(u_2u_j) = \begin{cases} 1 & \text{if } j \text{ is odd;} \\ 0 & \text{if } j \text{ is even.} \end{cases}, \text{ for } j \neq p \text{ and } 1 \leq j \leq 2p-1$$

$$\Rightarrow e_f(0) = \frac{2p-2}{2} + \frac{2p-2}{2} = 2p-2 \text{ and } e_f(1) = 2p-2.$$

Hence $|e_f(0) - e_f(1)| \leq 1$ and so $\Gamma(\mathbb{Z}_{4p})$ is sum cordial.

Theorem 2.4: For two distinct primes p and q with $p < q$, the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$ is sum cordial.

Proof: The vertex set of $\Gamma(\mathbb{Z}_{pq})$ can be partitioned into V_1 and V_2 where

$V_1 = \{p, 2p, 3p, ..., (q-1)p\} = \{u_1, u_2, ..., u_{q-1}\}$ and

$V_2 = \{q, 2q, 3q, ..., (p-1)q\} = \{v_1, v_2, ..., v_{p-1}\}$.

The edge set of $\Gamma(\mathbb{Z}_{pq})$ is given by

$E(\Gamma(\mathbb{Z}_{pq})) = \{u_i v_j : u_i \in V_1 \text{ and } v_j \in V_2, 1 \leq i \leq q-1, 1 \leq j \leq p-1\}$.

Consider the vertex labeling $f : V(\Gamma(\mathbb{Z}_{pq})) \to \{0, 1\}$ defined by

$f(u_i) = i \pmod 2$ for $1 \leq i \leq q-1$ and

$f(v_j) = j \pmod 2$ for $1 \leq j \leq p-1$.

Further note that $|E| = (p-1)(q-1)$ and the induced edge labeling $f^*: E \to \{0, 1\}$ is given by

$$f^*(u_iv_j) = \begin{cases} 0 & \text{both } i \,\&\, j \text{ are odd, } 1 \leq i < q-1, 1 \leq j \leq p-1; \text{ and both } i\&j \text{ are even;} \\ 1 & \text{otherwise} \end{cases}$$

Clearly $e_f(0) = \frac{(p-1)(q-1)}{2}$ and $e_f(1) = \frac{(p-1)(q-1)}{2}$.

Hence $|e_f(0) - e_f(1)| \leq 1$ and so $\Gamma(\mathbb{Z}_{pq}))$ is sum cordial.

Theorem 2.5: For any prime number $p > 2$, the join graph $\Gamma(\mathbb{Z}_{2p}) + \Gamma(\mathbb{Z}_4)$ is sum cordial.

Proof: Let $G = \Gamma(\mathbb{Z}_{2p}) + \Gamma(\mathbb{Z}_4)$.

The vertex set of the graph G,

$V(G) = \{u_1, u_2, ... u_{p-1}, u_p, x\} = \{2, 4, ..., 2(p-1), p, x\}$, where $x = 2 \in \mathbb{Z}_4$.

Also the edge set of G,

$E(G) = \{u_iu_p, u_ix, u_px \,/\, 1 \leq i \leq p-1\}$.

Note that $|V| = p+1$ and $|E| = p - 1 + p - 1 + 1 = 2p - 1$.

Define the vertex labeling $f : V(G) \to \{0, 1\}$ by

$f(u_k) = k \pmod 2$, $1 \leq k \leq p$ and $f(x) = 0$.

It is clear that $v_f(0) = \frac{p+1}{2} = v_f(1)$

Then the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by

$$f^*(u_iu_p) = \begin{cases} 0 & \text{if } i \text{ is odd;} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

$$f^*(u_ix) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

and $f^*(u_px) = 1$.

From the above $e_f(0) = \frac{(p-1)}{2} + \frac{(p-1)}{2} = p-1$, $e_f(1) = \frac{(p-1)}{2} + \frac{(p-1)}{2} + 1 = p$

and $|e_f(0) - e_f(1)| \leq 1$.

Therefore G is sum cordial.

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Theorem 2.6: For any prime number $p > 2$, the join graph $\Gamma(\mathbb{Z}_{2p}) + \Gamma(\mathbb{Z}_9)$ is sum cordial.

Proof: Let $G = \Gamma(\mathbb{Z}_{2p}) + \Gamma(\mathbb{Z}_9)$. The vertex set of the graph G is

$$V(G) = \{u_1, \dots u_{p-1}, u_p, x, y\}$$
$$= \{2, 4, \dots, 2(p-1), p, x, y\}$$, where $x = 3$ and $y = 6 \in \mathbb{Z}_9$ and the edge set of G is

$E(G) = \{u_iu_p, u_ix, u_iy, u_px, u_py, xy \,/\, 1 \le i \le p-1\}$.

Note that $|V| = p+2$ and $|E| = p - 1 + p - 1 + p - 1 + 3 = 3p$.

Define the vertex labeling $f : V(G) \to \{0, 1\}$ by $f(u_k) = k \pmod 2$ for $1 \le k \le p$,
$f(x) = 0$ and $f(y) = 1$.

Clearly $v_f(0) = \frac{p-1}{2} + 1$ and $v_f(1) = \frac{p-1}{2} + 2$.

Then the induced edge labeling $f^*: E(G) \to \{0, 1\}$ is given by

$$f^*(u_iu_p) = \begin{cases} 0 & \text{if } i \text{ is odd;} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

$$f^*(u_ix) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

$$f^*(u_iy) = \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

$f^*(u_px) = 1$, $f^*(u_py) = 0$ and $f^*(xy) = 1$.

From the above $e_f(0) = \frac{3(p-1)}{2} + 1$, $e_f(1) = \frac{3(p-1)}{2} + 2$ and $|e_f(-1) - e_f(1)| \le 1$. Hence G is sum cordial.

Theorem 2.7: For any prime number $p > 2$, the join graph $\Gamma(\mathbb{Z}_{2p}) + \Gamma(\mathbb{Z}_6)$ is sum cordial.

Proof: Let $G = \Gamma(\mathbb{Z}_{2p}) + \Gamma(\mathbb{Z}_6)$.

The vertex set of the graph G,

$$V(G) = \{u_1, \dots u_{p-1}, u_p, x, y, z\}$$
$$= \{2, 4, \dots, 2(p-1), p, x, y, z\}$$, where $x = 2$, $y = 3$ and $z = 3 \in \mathbb{Z}_6$ and the edge set of G

$E(G) = \{u_iu_p, u_ix, u_iy, u_iz, u_px, u_py, u_pz, xy, yz \,/\, 1 \le i \le p-1\}$.

Note that $|V| = p + 3$ and $|E| = p - 1 + p - 1 + p - 1 + p - 1 + 5 = 4p + 1$.

Define the vertex labeling $f: V(G) \to \{0, 1\}$ by
$f(u_k) = k \pmod 2$ for $1 \le k \le p$, $f(x) = 0$, $f(y) = 0$ and $f(z) = 1$.

Note that $v_f(0) = \frac{p-1}{2} + 2$; $v_f(1) = \frac{p-1}{2} + 2$.

Then the induced edge labeling $f^*: E(G) \to \{0, 1\}$ is given by

$$f^*(u_iu_p) = \begin{cases} 0 & \text{if } i \text{ is odd;} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

$$f^*(u_ix) = f^*(u_iy) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

$$f^*(u_iz) = \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

$f^*(u_px) = 1$, $f^*(u_py) = 1$, $f^*(u_pz) = 0$, $f^*(xy) = 0$ and $f^*(yz) = 1$.

From the above $e_f(0) = \dfrac{4(p-1)}{2} + 2$

$e_f(1) = \dfrac{4(p-1)}{2} + 3$ and

$|e_f(0) - e_f(1)| \le 1.$

Therefore G is sum cordial.

Corollary 2.8: For any prime number p > 2, the join graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_4)$ is sum cordial.

Proof: Since the graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_{2p})$, by Theorem 2.1, $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_4)$ is sum cordial.

Theorem 2.9: For any prime number p > 2, the join graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_9)$ is sum cordial.

Proof: Let G = $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_9)$. The vertex set of the graph G is

$V(G) = \{u_1, \dots u_{p-1}, x, y\}$

$= \{p, 2p, \dots, (p-1)p, x, y\}$, where x = 3 and y = 6 $\in Z_9$ and the edge set of G is

$E(G) = \{u_i x, u_i y, xy \,/\, 1 \le i \le p-1\}.$

Note that $|V| = p+1$ and $|E| = p - 1 + p - 1 + 1 = 2p - 1$.

Define the vertex labeling $f : V(G) \to \{0, 1\}$ by $f(u_k) = k \pmod 2$, for $1 \le k \le p-1$, $f(x) = 0$ and $f(y) = 1$.

Clearly $v_f(0) = \dfrac{p-1}{2} + 1$ and v_f(1) = $\dfrac{p-1}{2} + 1$.

Then the induced edge labeling $f^*: E(G) \to \{0, 1\}$ is given by

$$f^*(u_i x) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

$$f^*(u_i y) = \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

$f^*(xy) = 1.$

From the above

$e_f(0) = \dfrac{(p-1)}{2} + \dfrac{(p-1)}{2} = p - 1,$

$e_f(1) = \dfrac{(p-1)}{2} + \dfrac{(p-1)}{2} + 1 = p$ and satisfies $|e_f(0) - e_f(1)| \le 1$.

Hence G is sum cordial.

Theorem 2.10: For any prime number p > 2, the join graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_6)$ is a signed product cordial.

Proof: Let G = $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_6)$.

The vertex set of the graph G is

$V(G) = \{u_1, \dots u_{p-1}, x, y, z\}$

$= \{p, 2p, \dots, (p-1)p, x, y, z\}$, where x = 2, y = 3 and z = 3 $\in \mathbb{Z}_6$.

Further the edge set of G is

$E(G) = \{u_i x, u_i y, u_i z, xy, yz \,/\, 1 \le i \le p-1\}.$

Note that $|V| = p+2$ and $|E| = p - 1 + p - 1 + p - 1 + 2 = 3p - 1$.

Define the vertex labeling $f : V(G) \to \{0, 1\}$ by $f(u_k) = k \pmod 2$ for $1 \le k \le p-1$, $f(x) = 0$, $f(y) = 0$ and $f(z) = 1$.

Clearly $v_f(0) = \frac{p-1}{2} + 2$; $v_f(1) = \frac{p-1}{2} + 1$.

Then the induced edge labeling $f^*: E(G) \to \{0, 1\}$ is given by

$$f^*(u_i x) = f^*(u_i y) = \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

$$f^*(u_i z) = \begin{cases} 1 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

$f^*(xy) = 0$ and $f^*(yz) = 1$

From the above

$e_f(0) = \frac{(p-1)}{2} + \frac{(p-1)}{2} + \frac{(p-1)}{2} + 1 = \frac{3(p-1)}{2} + 1$,

$e_f(1) = \frac{(p-1)}{2} + \frac{(p-1)}{2} + \frac{(p-1)}{2} + 1 = \frac{3(p-1)}{2} + 1$ and satisfies

$|e_f(0) - e_f(1)| \le 1$. Hence G is sum cordial.

Corollary 2.11: For any prime number $p > 2$, the join graph $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_4)$ is sum cordial.

Proof: Since the graph $\overline{\Gamma(\mathbb{Z}_4)} = \Gamma(\mathbb{Z}_4)$, by Corollary 2.8. $\overline{\Gamma(\mathbb{Z}_{p^2})} + \Gamma(\mathbb{Z}_4)$ is sum cordial.

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