International Journal of Mathematical Archive-9(2), 2018, 115-119
Available online through www.ijma.info ISSN 2229 – 5046

Certain Subclasses of Bi-Univalent Functions Associated with Salagean Operator

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(Received On: 25-11-17; Revised & Accepted On: 16-01-18)

ABSTRACT

The purpose of the present paper is to introduce new subclasses of the function class of bi-univalent functions by using Salagean operator. Furthermore, we obtain estimates on the coefficients and for functions of these subclasses. Relevant connections of well-known results are briefly indicated.

Keywords: *Analytic functions, univalent functions, Bi-univalent functions, Salagean derivative.*

2010 AMS Subject Classification: *30C45, 30C50.*

1. INTRODUCTION

Let *A* denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the unit disc $U = \{z : |z| < 1\}$ and let S denote the subclass of A consisting function of the type (1.1) which are normalized and univalent in U.

It is well known that every $f \in S$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z, \quad z \in U$$

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots\ldots$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1). Examples of functions in the class Σ are $\frac{z}{1-z}, \log(1-z), \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and so on.

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in S such as $z - \frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are also not members of Σ.

Lewin [3] investigated the Bi-univalent function class Σ showed that $|a_2| < 1.5$.

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Subsequently by Brannan and Clunie [1] conjectured that $|a_2| \le \sqrt{2}$.

The study of Bi-univalent functions is a recent topic of study in Geometric Function Theory. Several researchers for example see ([5], [7], [8]) introduced the various new subclass of the bi-univalent function class $\sum$ and obtained non sharp bounds on the initial coefficients $|a_2|$ and $|a_3|$.

To prove our main results, we shall require the following lemma due to [2].

Lemma 1.1: If $h \in P$ then $|c_k| \le 2$ for each k, where P is the family of all functions h analytic in U for which $\operatorname{Re}\{h(z)\} > 0,\ h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ for $z \in U$.

2. MAIN RESULT

Definition 2.1: A function $f(z)$ given by (1.1) is said to be in the class $\Re^{\alpha}_{\Sigma,n}$ if the following conditions are satisfied $f \in \Sigma$ and

$$\left|\arg\left(\frac{D^{n+1} f(z)}{z}\right)\right| < \frac{\alpha\pi}{2}\ (z \in U,\ 0 < \alpha \le 1) \tag{2.1}$$

$$\left|\arg\left(\frac{D^{n+1} g(w)}{z}\right)\right| < \frac{\alpha\pi}{2}\ (w \in U,\ 0 < \alpha \le 1) \tag{2.2}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (a_2^3 - 5a_2 a_3 + a_4) w^4 + ... \tag{2.3}$$

and D^n stands for the differential operator introduced by Salagean [6].

We first state and prove the following result.

Theorem 2.1: Let $f(z)$ given by (1.1) be in the function class $\Re^{\alpha}_{\Sigma,n}$. Then

$$|a_2| \le \alpha\sqrt{\frac{2}{\alpha 3^{n+1} + (1-\alpha) 2^{2n+1}}} \text{ and } |a_3| \le 2\alpha \frac{(3^{n+1}\alpha + 2^{2n+1})}{3^{n+1}.2^{2n+1}}. \tag{2.4}$$

Proof: We can write the argument inequalities in (2.1) and (2.2) equivalently as follows:

$$\frac{D^{n+1} f(z)}{z} = [Q(z)]^{\alpha} \text{ and } \frac{D^{n+1} g(w)}{z} = [L(w)]^{\alpha}, \tag{2.5}$$

respectively, where $Q(z)$ and $L(w)$ satisfy the following inequalities

$$\Re\{Q(z)\} > 0\ (z \in U) \text{ and } \Re\{L(w)\} > 0\ (w \in U). \tag{2.6}$$

Furthermore, the functions $Q(z)$ and $L(w)$ have the forms:

$$Q(z) = 1 + c_1 z + c_2 z^2 + ... \tag{2.7}$$

and

$$L(w) = 1 + l_1 w + l_2 w^2 + ... \tag{2.8}$$

respectively. Now, equating the coefficients of $\frac{D^{n+1} f(z)}{z}$ with $[Q(z)]^{\alpha}$ and the coefficient of $\frac{D^{n+1} g(w)}{z}$ with $[L(w)]^{\alpha}$

$$2^{n+1} a_2 = \alpha c_1 \tag{2.9}$$

$$3^{n+1} a_3 = \alpha c_2 + \frac{\alpha(\alpha-1)}{2} c_1^{\ 2} \tag{2.10}$$

$$-2^{n+1}a_2=\alpha l_1 \tag{2.11}$$

$$3^{n+1}\left(2a_2^2-a_3\right)=\alpha l_2+\frac{\alpha(\alpha-1)}{2}l_1^2 \tag{2.12}$$

From (2.9) and (2.11), we get

$$c_1=-l_1$$

and

$$2^{2n+3}a_2^2=\alpha^2\left(l_1^2+c_1^2\right) \tag{2.13}$$

Also, from (2.10), (2.12) and (2.13), we find that

$$2\left(3^{n+1}\right)a_2^2=\alpha\left(l_2+c_2\right)+\frac{\alpha(\alpha-1)}{2}\left(l_1^2+c_1^2\right)$$

$$2\left(3^{n+1}\right)a_2^2=\alpha\left(l_2+c_2\right)+\frac{(\alpha-1)}{2\alpha}2^{2n+3}a_2^2.$$

Therefore, we have

$$a_2^2=\frac{\alpha^2}{2\left(\alpha 3^{n+1}+(1-\alpha)2^{2n+1}\right)}\left(l_2+c_2\right)$$

Which, in conjunction with the following well-known inequalities (see [2], page 41)

$$|c_2|\le 2 \text{ and } |l_2|\le 2,$$

gives us the desired estimate on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$ by subtracting (2.12) from (2.10), we get

$$2\left(3^{n+1}a_3-3^{n+1}a_2^2\right)=\alpha c_2+\frac{\alpha(\alpha-1)}{2}c_1^2-\left(\alpha l_2+\frac{\alpha(\alpha-1)}{2}l_1^2\right).$$

Upon substituting the values of a_2^2 from (2.13) and observing that

$$c_1^2=l_1^2.$$

It follows that

$$a_3=\frac{1}{4^{n+1}}\alpha^2c_1^2+\frac{1}{2\left(3^{n+1}\right)}\alpha\left(c_2-l_2\right)$$

The familiar inequalities (cf. [2], page 41)

$$|c_2|\le 2 \text{ and } |l_2|\le 2$$

would now yield

$$|a_3|\le\frac{1}{4^{n+1}}\alpha^2.4+\frac{1}{2\left(3^{n+1}\right)}\alpha.4=2\alpha\frac{\left(3^{n+1}\alpha+2^{2n+1}\right)}{3^{n+1}.2^{2n+1}}.$$

This completes the proof of Theorem 2.1.

3. COEFFICIENT BOUNDS

Coefficient Bounds for the function class $\Re_{\Sigma,n}(\beta)$

Definition 3.1: A function $f(z)$ given by (1.1) is said to be in the class $\Re_{\Sigma,n}(\beta)$ $(0 \le \beta < 1)$ if the following conditions are satisfied $f \in \Sigma$ and

$$R\left(\frac{D^{n+1}f(z)}{z}\right) > \beta \left(w \in U,\, 0 \le \beta < 1\right), \tag{3.1}$$

$$R\left(\frac{D^{n+1}g(w)}{z}\right) > \beta \left(w \in U,\, 0 \le \beta < 1\right), \tag{3.2}$$

where the function g is given by (2.3).

Theorem 3.1: Let $f(z)$ given by (1.1) be in the function class $\Re_{\Sigma,n}(\beta)$ $(0 \le \beta < 1)$. Then

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{3^{n+1}}}$$

and

$$|a_3| \le (1-\beta)\left(\frac{1}{2^{2n}} + \frac{2}{3^{n+1}} - \beta\right).$$

Proof: First of all, the argument inequalities in (3.1) and (3.2) can easily be rewritten in their equivalent forms:

$$\frac{D^{n+1}f(z)}{z} = \beta + (1-\beta)Q(z) \quad and \quad \frac{D^{n+1}g(w)}{z} = \beta + (1-\beta)L(w), \tag{3.3}$$

respectively, where $Q(z)$ and $L(w)$ satisfy the following inequalities

$$\Re\{Q(z)\} > 0 \ (z \in U) \ and \ \Re\{L(w)\} > 0 \ (w \in U) \tag{3.4}$$

Moreover, the functions $Q(z)$ and $L(w)$ have the following forms:

$$Q(z) = 1 + c_1 z + c_2 z^2 + ... \tag{3.5}$$

and

$$L(w) = 1 + l_1 w + l_2 w^2 + ..., \tag{3.6}$$

As in the proof of Theorem 2.1, by suitably comparing coefficient, we get

$$2^{n+1}a_2 = (1-\beta)c_1 \tag{3.7}$$

$$3^{n+1}a_3 = (1-\beta)c_2 \tag{3.8}$$

$$-2^{n+1}a_2 = (1-\beta)l_1 \tag{3.9}$$

$$3^{n+1}\left(2a_2^2 - a_3\right) = (1-\beta)l_2 \tag{3.10}$$

From (3.7) and (3.9), we get

$$c_1 = -l_1$$

and

$$2^{2n+3}a_2^2 = (1-\beta)^2\left(l_1^2 + c_1^2\right) \tag{3.11}$$

Also, from (3.8) and (3.10), we find that

$$2\left(3^{n+1}\right)a_2^2 = 3^{n+1}a_3 + (1-\beta)l_2 = (1-\beta)c_2 + (1-\beta)l_2 = (1-\beta)(c_2 + l_2). \tag{3.12}$$

Therefore, we have

$$|a_2^2| \le \frac{(1-\beta)}{2\left(3^{n+1}\right)}\left(|l_2| + |c_2|\right) = \frac{(1-\beta)}{2\left(3^{n+1}\right)}.4 = \frac{2(1-\beta)}{\left(3^{n+1}\right)}.$$

This gives the bound on $|a_2|$, as asserted in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2\left(3^{n+1}a_3 - 3^{n+1}a_2^2\right) = (1-\beta)(c_2 - l_2),$$

which, upon substituting the values of a_2^2 from (3.11), yields

$$2\left(3^{n+1}\right)a_3 = \frac{3^{n+1}}{2^{2n+2}}(1-\beta)^2\left(l_1^2 + c_1^2\right) + (1-\beta)(c_2 - l_2),$$

This last equation, together with the well-known estimates:

$$|c_1| \le 2, |l_1| \le 2, |c_2| \le 2 \text{ and}, |l_2| \le 2$$

would lead us to the following inequality:

$$2\left(3^{n+1}\right)a_3 = \frac{3^{n+1}}{2^{2n+2}}(1-\beta)^2 .8 + (1-\beta).4.$$

Therefore, we have

$$|a_3| \le (1-\beta)\left(\frac{1}{2^{2n}} + \frac{2}{3^{n+1}} - \beta\right).$$

This completes the proof of theorem 3.1.

Remarks 1: If we put n = 0 then we obtain the corresponding results of Srivastava *et.al* [7].

ACKNOWLEDGEMENT

The authors are thankful to University Grants Commission, New Delhi for financial support and also to Prof. A.K. Mishra, Department of Mathematics, Berhampur University, Berhampur, Odisha and Prof. K.K. Dixit, Department of Mathematics, Janta College, Bakewar, Etawah, U.P. for their kind support.

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