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ANALYTICAL APPROACH TO THE ONE-DIMENSIONAL ADVECTION-DIFFUSION EQUATION USING HOMOTOPY PERTURBATION METHOD

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ABSTRACT

In this paper, a user friendly algorithm based on the homotopy perturbation method (HPM) is proposed to solve the one-dimensional advection-diffusion equations. The advantage of this method that is its ability and flexibility to provide the analytical or approximate solutions to linear and nonlinear equations without linearization or discretization makes it reliable for solving the one-dimensional advection-diffusion equations. The aim of this article is to employ HPM to solve of one-dimensional advection-diffusion equations and comparing the results obtained with exact solution. The numerical results are presented to show the efficiency of this method.

Keywords and Phrases: *Homotopy perturbation method; one-dimensional advection-diffusion equations*

Mathematics Subject Classification: *47J30, 49S05, 35G25.*

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1. INTRODUCTION

The idea of homotopy perturbation method was first introduced by He in 1998 [1]. By the homotopy technique in topology, a homotopy can be constructed with an embedding parameter $p \in [0,1]$, which is considered as a small one. In this method, the solution is considered as the summation of a series which rapidly converge to the solution. Homotopy perturbation method is a combination of the perturbation and homotopy methods. This method can take the advantages of the conventional perturbation method while eliminating its restrictions. Recently the homotopy perturbation method is used to solve the sine-Gordon Equation [2]. In [3], He's homotopy perturbation method is employed to solve non-linear systems of second-order boundary value problems. Interested readers are referred to [4-6], for some applications of the method. In this article, homotopy perturbation method has been used to solve the one-dimensional advection-diffusion equations.

Consider the one-dimensional advection-diffusion equations:

$$\frac{\partial u}{\partial t}(x,t)+\beta\frac{\partial u}{\partial x}(x,t)=\alpha\frac{\partial^2 u}{\partial x^2}(x,t), \qquad (x,t)\in[0,L]\times[0,T] \tag{1.1}$$

With

$$u(x,0)=\phi(x) \tag{1.2}$$

Where β is an arbitrary constant (as we see for $\beta=0$ the heat equation is obtained) which shows the speed of convection and the diffusion coefficient. i.e. α is appositive constant.

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2. BASIC IDEAS OF HOMOTOPY PERTURBATION METHOD

To illustrate the basic concepts of homotopy perturbation method, consider the following non-linear functional equation:

$$A(u) = f(r), \quad r \in \Omega, \tag{2.1}$$

With the following boundary conditions:

$$B\left(u, \partial u / \partial n\right) = 0, \ \ r \in \Gamma. \tag{2.2}$$

Where A is a functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking, the operator A can be decomposed into two parts L and N, where L is a linear and N is a non-linear operator. Therefore, Eq. (2.1) can be rewritten as the following:

$$L(u) + N(u) - f(r) = 0. \tag{2.3}$$

We construct a homotopy $U(r, p) : \Omega \times [0,1] \to R,$ which satisfies:

$$H(U, p) = (1-p)\left[L(U) - L(u_0)\right] + p\left[A(U) - f(r)\right] = 0, \ \ p \in [0,1], \ \ r \in \Omega, \tag{2.4}$$

Or

$$H(U, p) = L(U) - L(u_0) + pL(u_0) + p\left[N(U) - f(r)\right] = 0, \ \ p \in [0,1], \ \ r \in \Omega., \tag{2.5}$$

Where u_0 is an initial approximation to the solution of Eq. (1.1). In this method, homotopy perturbation parameter p is used to expand the solution, as a power series, say;

$$U = U_0 + pU_1 + p^2U_2 + \cdots \ , \tag{2.6}$$

Usually an approximation to the solution, will be obtained by taking the limit, as p tends to 1,

$$U = \lim_{p \to 1} U = U_0 + U_1 + U_2 \cdots \ , \tag{2.7}$$

The convergence and stability of this method is addressed in [7].

3. USING HOMOTOPY PERTURBATION METHOD TO SOLVE Eq. (1.1)

For solving Eq. (1.1), by homotopy perturbation method, we construct the following homotopy:

$$H(U, p) = (1-p)\left[\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p\left[\frac{\partial U}{\partial t} + \beta\frac{\partial U}{\partial x} - \alpha\frac{\partial^2 U}{\partial x^2}\right] = 0, \tag{3.1}$$

Or

$$H(U, p) = \frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p\left[\beta\frac{\partial U}{\partial x} - \alpha\frac{\partial^2 U}{\partial x^2} + \frac{\partial u_0}{\partial t}\right] = 0, \tag{3.2}$$

Suppose that the solution of Eq. (3.2) to be in the following form

$$U = U_0 + pU_1 + p^2U_2 + \ldots \tag{3.3}$$

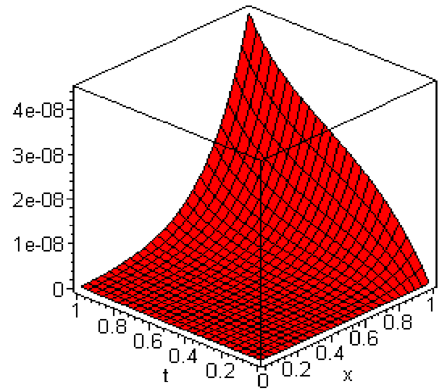

Figure: 1 Absolute errors of $u_{20}(x,t)$ obtained by HPM in Example.1

Substituting Eq. (3.3) into Eq. (3.2), and equating the coefficients of the terms with the identical powers of p, we obtain:

$$\begin{aligned}
&p^0: \frac{\partial U_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\
&p^1: \frac{\partial U_1}{\partial t} + \beta \frac{\partial U_0}{\partial x} - \alpha \frac{\partial^2 U_0}{\partial x^2} + \frac{\partial u_0}{\partial t} = 0, \\
&p^2: \frac{\partial U_2}{\partial t} + \beta \frac{\partial U_1}{\partial x} - \alpha \frac{\partial^2 U_1}{\partial x^2} = 0, \\
&p^3: \frac{\partial U_3}{\partial t} + \beta \frac{\partial U_2}{\partial x} - \alpha \frac{\partial^2 U_2}{\partial x^2} = 0, \\
&\vdots \\
&p^j: \frac{\partial U_j}{\partial t} + \beta \frac{\partial U_{j-1}}{\partial x} - \alpha \frac{\partial^2 U_{j-1}}{\partial x^2} = 0, \\
&\vdots
\end{aligned} \tag{3.4}$$

For simplicity we take

$$U_0(x,t) = u_0(x,t) = u(x,0) = \phi(x) \tag{3.5}$$

Having this assumption we get the following iterative equation

$$U_1 = \int_0^t \left(-\beta \frac{\partial U_0}{\partial x} + \alpha \frac{\partial^2 U_0}{\partial x^2} - \frac{\partial u_0}{\partial t} \right) dt, \tag{3.6}$$

$$U_j = \int_0^t \left(-\beta \frac{\partial U_{j-1}}{\partial x} + \alpha \frac{\partial^2 U_{j-1}}{\partial x^2} \right) dt, \qquad j = 2, 3, \ldots \tag{3.7}$$

Therefore, the solutions of Eq. (1.1) can be obtained, by setting $p = 1$

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + \ldots \tag{3.8}$$

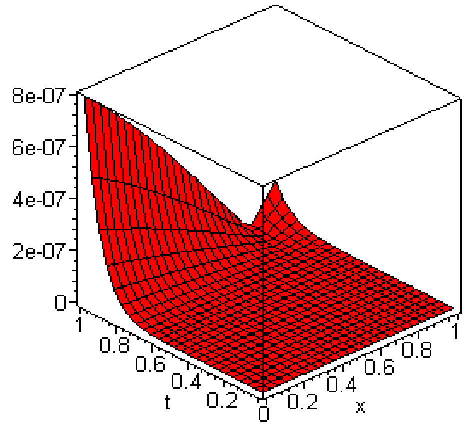

Figure: 2 Absolute errors of $u_{10}(x,t)$ obtained by HPM at $\alpha=1, \beta=1$ in Example.2

4. NUMERICAL RESULTS

To illustrate capability, reliability and simplicity of the method, three examples for different cases of the equation will be discussed here.

Example: 1 We consider the advection–diffusion equation (1.1) with $\beta=1,\ \alpha=0.1$ and the following initial condition:

$$\phi(x)=\exp(5x)\left[\cos\left(\frac{\pi}{2}x\right)+0.25\sin\left(\frac{\pi}{2}x\right)\right]. \tag{4.1}$$

The exact solution is given by

$$u(x,t)=\exp(5(x-t/2))\exp(-\frac{\pi^2}{40}t)\left[\cos\left(\frac{\pi}{2}x\right)+0.25\sin\left(\frac{\pi}{2}x\right)\right]. \tag{4.2}$$

A homotopy can be readily constructed as follows:

$$\frac{\partial U(x,t)}{\partial t}-\frac{\partial u_0(x,t)}{\partial t}+p\left(\frac{\partial U}{\partial x}-0.1\frac{\partial^2 U}{\partial x^2}+\frac{\partial u_0}{\partial t}\right)=0. \tag{4.3}$$

Substituting Eq. (3.3) into Eq. (4.3), and equating the terms with identical powers of p , we have

$$\begin{aligned}
&p^0: U_0(x,t)=u(x,0)=\exp(5x)\left[\cos\left(\frac{\pi}{2}x\right)+0.25\sin\left(\frac{\pi}{2}x\right)\right],\\
&p^1: U_1(x,t)=\int_0^t\left(-\frac{\partial U_0}{\partial x}+0.1\frac{\partial U^2{}_0}{\partial x^2}\right)dt=-2.5\exp(5x)\left[\cos\left(\frac{\pi}{2}x\right)+0.25\sin\left(\frac{\pi}{2}x\right)\right]t\\
&\qquad +0.1\exp(5x)\left[-2.467401101\cos\left(\frac{\pi}{2}x\right)-0.6168502753\sin\left(\frac{\pi}{2}x\right)\right]t,\\
&\vdots\\
&p^j: U_j(x,t)=\int_0^t\left(-\frac{\partial U_{j-1}}{\partial x}+0.1\frac{\partial U^2{}_{j-1}}{\partial x^2}\right)dt=\ldots,
\end{aligned} \tag{4.4}$$

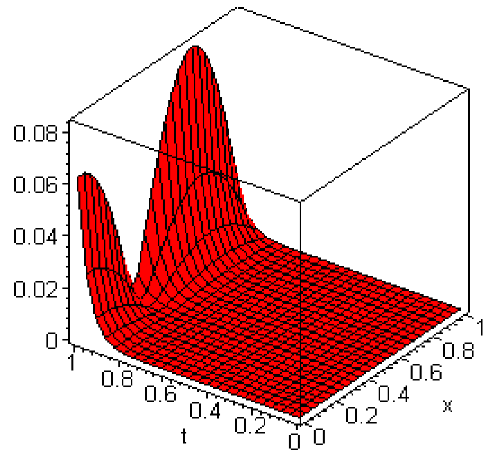

Figure 3: Absolute errors of $u_{20}(x,t)$ obtained by HPM at $\alpha = 1, \beta = 1$ in Example.3

Approximation to the solution of example 1 can be readily obtained by $u_{20} = \sum_{i=0}^{20} U_i$. The results corresponding absolute errors are presented in Figure 1.

Example: 2 Consider the advection–diffusion equation (1.1) with the following initial condition:

$$\phi(x) = \sin(x). \tag{4.5}$$

The exact solution is given by

$$u(x,t) = \exp(-\alpha t)\sin(x - \beta t). \tag{4.6}$$

A homotopy can be readily constructed as follows:

$$\frac{\partial U(x,t)}{\partial t} - \frac{\partial u_0(x,t)}{\partial t} + p\left(\beta\frac{\partial U}{\partial x} - \alpha\frac{\partial^2 U}{\partial x^2} + \frac{\partial u_0}{\partial t}\right) = 0 \tag{4.7}$$

Substituting Eq. (3.3) into Eq. (4.7), and equating the terms with identical powers of p , we have

$$\begin{aligned}
&p^0 : U_0(x,t) = u(x,0) = \sin(x),\\
&p^1 : U_1(x,t) = \int_0^t \left(-\beta\frac{\partial U_0}{\partial x} + \alpha\frac{\partial U^2{}_0}{\partial x^2}\right)dt = -\alpha\sin(x)t - \beta\cos(x)t\\
&p^2 : U_1(x,t) = \int_0^t \left(-\beta\frac{\partial U_1}{\partial x} + \alpha\frac{\partial U^2{}_1}{\partial x^2}\right)dt = \frac{1}{2}\left(\alpha\left(\alpha\sin(x) + \beta\cos(x)\right) - \beta\left(-\alpha\cos(x) + \beta\sin(x)\right)\right)t^2\\
&\vdots\\
&p^j : U_j(x,t) = \int_0^t \left(-\beta\frac{\partial U_{j-1}}{\partial x} + \alpha\frac{\partial U^2{}_{j-1}}{\partial x^2}\right)dt = \ldots,
\end{aligned} \tag{4.8}$$

Approximation to the solution of example 2 can be readily obtained by $u_{20} = \sum_{i=0}^{10} U_i$. The results corresponding absolute errors are presented in Figure 2.

Example: 3 We consider the advection–diffusion equation (1.1) with the following initial condition:

$$\phi(x) = \exp\left\{-\frac{(x-\beta)^2}{4\alpha}\right\}. \tag{4.9}$$

The exact solution is given by

$$u(x,t) = \frac{1}{\sqrt{1+t}}\exp\left\{-\frac{(x-(1+t)\beta)^2}{4\alpha(1+t)}\right\}. \tag{4.10}$$

According to the discussion presented in Example 2, Approximation to the solution of example 3 can be readily obtained by $u_{20} = \sum_{i=0}^{20} U_i$. The results corresponding absolute errors are presented in Figure 3.

5. CONCLUSION

Homotopy perturbation method is known to be a powerful device for solving many functional equations such as ordinary, partial differential equations, integral equations and so many other equations. In this article, we used homotopy perturbation method for solving the one-dimensional advection-diffusion equations. We can conclude that this method is powerful and meaningful for solving the one-dimensional advection-diffusion equations. Computations are performed using Maple 13.

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