

On Hilbert Modules over Locally m -Convex H^* – Algebras

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ABSTRACT

In this paper a Hilbert E – module W is defined where $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ is a locally m -convex H^* – algebra. With each $\lambda \in \Lambda$, we associate a Hilbert module \widehat{W}_λ over an H^* – algebra \widehat{E}_λ . We obtain relationship between these spaces and the initial space. Moreover the existence of orthonormal bases in a Hilbert E – module is proved. We topologized the space of bounded E – linear operators via suitable family of seminorms.

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1. Introduction

A locally multiplicatively convex algebra (l.m.c.a in short) is a topological algebra (E, τ) whose topology τ is determined by a directed family $(|\cdot|_\lambda)_{\lambda \in \Lambda}$ of submultiplicative seminorms. Such an algebra will usually denoted by $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$. If, in addition, E is endowed with an involution $x \mapsto x^*$ such that $|x|_\lambda = |x^*|_\lambda$, for any $x \in E, \lambda \in \Lambda$, then $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ is called an l.m.c.* – algebra. Let $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ be a complete l.m.c.a. It is known that $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ is the inverse limit of the normed algebras $(E_\lambda, (|\cdot|'_\lambda)_{\lambda \in \Lambda})$, where $E_\lambda = E / N_\lambda$ with $N_\lambda = \{x \in E : |x|_\lambda = 0\}$, and $|\bar{x}|'_\lambda = |x|_\lambda$. An element x of E is written $x = (x_\lambda)_\lambda = (\pi_\lambda(x))_\lambda$, where $\pi_\lambda : E \rightarrow E_\lambda$ is the canonical surjection. The algebra $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ is also the inverse limit of the Banach algebras \widehat{E}_λ , the completion of E'_λ 's. The norm in \widehat{E}_λ will also be denoted by $|\cdot|'_\lambda$.

In the following we define the locally m -convex H^* – algebra spaces. This notion was introduced in [5] as a natural extension of the classical H^* – algebras of W. Ambrose ([1]). Here we consider the case where the algebra is complete and it is endowed with a continuous involution.

Definition: 1.1 A locally m -convex H^* – algebra (l.m.c. H^* – algebra in short) is a complete l.m.c.*-algebra $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ on which is defined a family $(\langle \cdot, \cdot \rangle_\lambda)_{\lambda \in \Lambda}$ of positive semi-definite pseudo-inner products such that the following properties hold for all $x, y, z \in E$ and $\lambda \in \Lambda$:

- (i) $|x|_\lambda^2 = \langle x, x \rangle_\lambda$,
- (ii) $\langle xy, z \rangle_\lambda = \langle y, x^* z \rangle_\lambda$,
- (iii) $\langle yx, z \rangle_\lambda = \langle y, zx^* \rangle_\lambda$.

For every $\lambda \in \Lambda$, the quotient space $E_\lambda = E / N_\lambda$ is an inner product space under $\langle x_\lambda, y_\lambda \rangle_\lambda = \langle x, y \rangle_\lambda$. The

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underlying Banach space \widehat{E}_λ is a Hilbert space. Moreover, the involutive Banach algebra $(\widehat{E}_\lambda, |\cdot|'_\lambda)$ is an H^* - algebra. The algebra $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ is the inverse limit of the Banach H^* - algebras $(\widehat{E}_\lambda, |\cdot|'_\lambda)$, ([5], Theorem 2.3). So there exists a unique homomorphism $\phi: E \rightarrow \lim_{\leftarrow \lambda} \widehat{E}_\lambda$ in which $\chi_\mu \circ \phi = \pi_{\lambda, \mu} \circ \pi_\lambda$, where $|\cdot|_\lambda \geq |\cdot|_\mu$ and $\chi_\mu: \lim_{\leftarrow \lambda} \widehat{E}_\lambda \rightarrow \widehat{E}_\mu$ is the natural projection. One can see that ϕ is an isomorphism and $\lim_{\leftarrow \lambda} \widehat{E}_\lambda \cong E$ (See also the remarks following Satz 1.1 in [7]). One of the most useful consequences of this isomorphism is that every coherent sequence in $\{\widehat{E}_\lambda: \lambda \in \Lambda\}$ determines an element of E .

Given an l.m.c. H^* - algebra $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$. Since $*$ is an involution, E is proper, namely $\text{lan}(E) = \{0\}$, where $\text{lan}(E) = \{x \in E: xE = \{0\}\}$ is the left annihilator of E and so each \widehat{E}_λ , for every $\lambda \in \Lambda$. The trace class $\tau(E)$ of E is defined as the set $\tau(E) = \{ab: a, b \in E\}$. Clearly, $\tau(E)$ is an ideal of E which is complete $*$ - algebra in the topology τ determined by suitable submultiplicative seminorms $\tau_\lambda(\cdot)$, $\lambda \in \Lambda$ related to given seminorms on E by $\tau_\lambda(a^*a) = |a|_\lambda^2$, for all a in E . For every $\lambda \in \Lambda$, there exists a canonical continuous linear form on $\tau(E)$ called the trace $-\lambda$ of E and we denote it by tr_λ which is related with the semi-definite pseudo-inner product $\langle \cdot, \cdot \rangle_\lambda$ of E by $tr_\lambda(ab) = \langle a, b^* \rangle_\lambda$ for all $a, b \in E$. The trace class in the H^* - algebra \widehat{E}_λ , $\lambda \in \Lambda$ is defined as the set $\tau(\widehat{E}_\lambda) = \{[a + N_\lambda][b + N_\lambda]: a, b \in E\}$. It is known that $\tau(\widehat{E}_\lambda)$ is an ideal of \widehat{E}_λ , which is Banach $*$ - algebra under a suitable norm $\widehat{\tau}_\lambda(\cdot)$. The norm $\widehat{\tau}_\lambda$ is related to given norm $|\cdot|'_\lambda$ on \widehat{E}_λ by $\widehat{\tau}_\lambda([a^*a + N_\lambda]) = |[a + N_\lambda]|_\lambda^2$, for all $a \in E$. The trace class $\tau(E_\lambda)$ of E_λ , $\lambda \in \Lambda$ is defined similarly. Obviously $\tau(E_\lambda)$ is an ideal of E_λ which is norm $*$ - algebra under a suitable norm $\tau_\lambda(\cdot)$, in which $\tau_\lambda(a^*a + N_\lambda) = |a + N_\lambda|_\lambda^2$ for all $a \in E$. For $\lambda \in \Lambda$, there exists a continuous linear form \widehat{tr}_λ on $\tau(\widehat{E}_\lambda)$ satisfying $\widehat{tr}_\lambda([a + N_\lambda][b + N_\lambda]) = \widehat{tr}_\lambda([b + N_\lambda][a + N_\lambda]) = \langle a, b^* \rangle_\lambda$. Similarly there exists a continuous linear form tr_λ on the $\tau(E_\lambda)$, $\lambda \in \Lambda$, satisfying $tr_\lambda(ab + N_\lambda) = tr_\lambda(ba + N_\lambda) = \langle a, b^* \rangle_\lambda$.

Now suppose that $\chi_\lambda: \tau(E) \rightarrow \tau(E_\lambda)$ defined by $\chi_\lambda(ab) = ab + N_\lambda$ and $\pi_{\lambda, \mu}: \tau(E_\lambda) \rightarrow \tau(E_\mu)$ defined by $\pi_{\lambda, \mu}(a + N_\lambda)(b + N_\lambda) = (a + N_\mu)(b + N_\mu)$, where $|\cdot|_\lambda \geq |\cdot|_\mu$. Then $\{\tau(E), \tau; \chi_\lambda\}$ is the inverse limit of the inverse system $\{\tau(E_\lambda), \tau_\lambda; \pi_{\lambda, \mu}, \lambda, \mu \in \Lambda, |\cdot|_\lambda \geq |\cdot|_\mu\}$ and it is also inverse limit of the inverse system $\{\tau(\widehat{E}_\lambda), \widehat{\tau}_\lambda; \pi_{\lambda, \mu}, \lambda, \mu \in \Lambda, |\cdot|_\lambda \geq |\cdot|_\mu\}$.

In this paper we will introduce a Hilbert module W over an l.m.c. H^* - algebra $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ and for each $\lambda \in \Lambda$ we will associate a Hilbert module \widehat{W}_λ over an H^* - algebra \widehat{E}_λ . We shall see that $W \cong \lim_{\leftarrow \lambda} \widehat{W}_\lambda$. Then we will discuss about orthonormal bases in these spaces. Also we will topologize $L_E(V, W)$ and $B_E(V, W)$, the set of adjointable E - linear operators and the set of bounded E - linear operators from Hilbert E - module V into Hilbert E - module W , respectively, via suitable families of seminorms. Throughout this paper E is an l.m.c. H^* - algebra and W is a Hilbert E - module except some results about unitary operators in Hilbert H^* - modules at the end of the paper. The paper is organized as follows.

In section 2 we will introduce Hilbert modules over l.m.c. H^* - algebras and their properties are studied. The existence of orthonormal bases in these spaces is proved.

In section 3 we topologize the space of bounded E - linear operators and the space of all adjointable E - linear operators. Also, more properties of these spaces are detected.

2. Hilbert modules over l.m.c. H^* -algebras and orthonormal bases

Hilbert modules over l.m.c. H^* -algebras generalize the notion of Hilbert H^* -modules by allowing the $\tau(E)$ -valued product in an l.m.c. H^* -algebra.

Definition: 2.1 Let $(E, (\cdot|\cdot)_\lambda)_{\lambda \in \Lambda}$ be a Hausdorff l.m.c. H^* -algebra. A pre-Hilbert E -module is a left module W over E provided with a mapping

$[\cdot|\cdot]: W \times W \rightarrow \tau(E)$ (called $\tau(E)$ -valued product) where $\tau(E) = \{ab : a, b \in E\}$ which satisfies the following conditions:

- (i) $[\alpha x | y] = \alpha[x | y] \forall \alpha \in E, \forall x, y \in W$,
- (ii) $[x + y | z] = [x | z] + [y | z] \forall x, y, z \in W$,
- (iii) $[ax | y] = a[x | y], \forall a \in E, \forall x, y \in W$,
- (iv) $[x | y]^* = [y | x] \forall x, y \in W$,
- (v) $\forall x \in W, x \neq 0, \exists a \in E, a \neq 0$, such that $[x | x] = a^*a$,
- (vi) for each $\lambda \in \Lambda, W$ is a semi-definite pseudo-inner product space with $(x, y)_\lambda = \langle a, b^* \rangle_\lambda$ (or $tr_\lambda(ab)$) where $[x | y] = ab \in \tau(E)$.

We say that W is a Hilbert E -module if it is complete with respect to the topology determined by the family of seminorms $\|x\|_\lambda = \sqrt{(x, x)_\lambda}, x \in W, \lambda \in \Lambda$.

Given a Hilbert E -module W , then for $\lambda \in \Lambda, \xi_\lambda = \{x \in W : \|x\|_\lambda = 0\}$ is a closed submodule of W . Indeed, for non zero x in ξ_λ and $a \in E$, if $[x | x] = b^*b$ for some non zero $b \in E$, then $\|ax\|_\lambda = \|ab^*\|_\lambda \leq \|a\|_\lambda \|b\|_\lambda = 0$. For $\lambda \in \Lambda, W_\lambda = W / \xi_\lambda$ is an inner product space with $(x, y)_\lambda = tr_\lambda[x + \xi_\lambda | y + \xi_\lambda]$ (or $\langle a, b^* \rangle_\lambda$ where $[x | y] = ab$) and its completion \widehat{W}_λ is a Hilbert space.

Example: 2.2 Let $(E, (\cdot|\cdot)_\lambda)_{\lambda \in \Lambda}$ be an l.m.c. H^* -algebra. Then E is a Hilbert module over itself with $\tau(E)$ -valued product defined by $[a | b] = ab^*$. Also it is easy to verify that a closed submodule of a Hilbert E -module is again a Hilbert E -module. Note that analogue of Lemma 2.2 of [1] holds for l.m.c. H^* -algebras. More precisely if x is a non zero element in an l.m.c. H^* -algebra E , then x^*x, xx^*, x^* are also non zero. The proof of the following proposition can be based on the direct application of previous comments about Hilbert E -module W .

Proposition: 2.3 Let W be a Hilbert module over an l.m.c. H^* -algebra $(E, (\cdot|\cdot)_\lambda)_{\lambda \in \Lambda}$. For each $\lambda \in \Lambda, \widehat{W}_\lambda$ is a Hilbert module over the proper H^* -algebra \widehat{E}_λ with $\pi_\lambda(a)(x + \xi_\lambda) = ax + \xi_\lambda$ and $[x + \xi_\lambda | y + \xi_\lambda] = \pi_\lambda([x | y])$ for every $a \in E$ and for every $x, y \in W$. Let σ_λ^W be the canonical map from W onto $\widehat{W}_\lambda, \lambda \in \Lambda$. For $\lambda_1, \lambda_2 \in \Lambda, \|\cdot\|_{\lambda_1} \geq \|\cdot\|_{\lambda_2}$, there is a canonical surjective linear map $\sigma_{\lambda_1 \lambda_2}^W : \widehat{W}_{\lambda_1} \rightarrow \widehat{W}_{\lambda_2}$ such that $\sigma_{\lambda_1 \lambda_2}^W(\sigma_{\lambda_1}^W(x)) = \sigma_{\lambda_2}^W(x)$. Also $\{\widehat{W}_\lambda, \widehat{E}_\lambda; \sigma_{\lambda_1 \lambda_2}^W, \|\cdot\|_{\lambda_1} \geq \|\cdot\|_{\lambda_2}, \lambda_1, \lambda_2 \in \Lambda\}$ is an inverse system of Hilbert H^* -modules in the following sense:

$$\sigma_{\lambda_1 \lambda_2}^W(\pi_{\lambda_1}(a)\sigma_{\lambda_1}^W(x)) = \pi_{\lambda_1 \lambda_2}(\pi_{\lambda_1}(a))\sigma_{\lambda_1 \lambda_2}^W(\sigma_{\lambda_1}^W(x))$$

and

$$(\sigma_{\lambda_1 \lambda_2}^W(\sigma_{\lambda_1}^W(x)), \sigma_{\lambda_1 \lambda_2}^W(\sigma_{\lambda_1}^W(y)))_{\lambda_2} = \langle \pi_{\lambda_1 \lambda_2}(\pi_{\lambda_1}(a)), \pi_{\lambda_1 \lambda_2}(\pi_{\lambda_1}(b^*)) \rangle_{\lambda_2}$$

Where $[x | y] = ab$, for every $x, y \in W$ and for every $a \in E$;

$\sigma_{\lambda_2 \lambda_3}^W \sigma_{\lambda_1 \lambda_2}^W = \sigma_{\lambda_1 \lambda_3}^W, | \cdot |_{\lambda_1} \geq | \cdot |_{\lambda_2} \geq | \cdot |_{\lambda_3}; \sigma_{\lambda \lambda}^W = id_{W_\lambda}, \lim_{\leftarrow \lambda} \widehat{W}_\lambda$ is a Hilbert E – module with

$$(\pi_\lambda(a))_\lambda (\sigma_\lambda^W(x))_\lambda = (\sigma_\lambda^W(ax))_\lambda,$$

and

$$\langle (\sigma_\lambda^W(x))_{\lambda \in \Lambda}, (\sigma_\lambda^W(y))_{\lambda \in \Lambda} \rangle = \langle (\pi_\lambda(a), \pi_\lambda(b^*))_{\lambda \in \Lambda} \rangle_{\lambda \in \Lambda}$$

Where $[x | y] = ab$. Indeed, $\lim_{\leftarrow \lambda} \widehat{W}_\lambda$ maybe identified with W . So every coherent sequence in $\{\widehat{W}_\lambda : \lambda \in \Lambda\}$ determines an element of W .

For the basic facts about Hilbert H^* – modules we refer to [2] and [4]. In particular with the assumption of the previous proposition, we have the following three relations in Hilbert \widehat{E}_λ – module $\widehat{W}_\lambda (\lambda \in \Lambda)$,

$$\|x + \xi_\lambda\|_\lambda^2 = \widehat{tr}_\lambda([x + \xi_\lambda | x + \xi_\lambda]) = \widehat{\tau}_\lambda([x + \xi_\lambda | x + \xi_\lambda]).$$

$$|[x + \xi_\lambda | y + \xi_\lambda]|'_\lambda \leq \widehat{\tau}_\lambda([x + \xi_\lambda | y + \xi_\lambda]) \leq \|x + \xi_\lambda\|_\lambda \|y + \xi_\lambda\|_\lambda.$$

$$\|(a + N_\lambda)(x + \xi_\lambda)\|_\lambda \leq a + N_\lambda \quad |'_\lambda \|x + \xi_\lambda\|_\lambda.$$

As an immediate consequence of the above relations and the previous comments we obtain:

$$\|x\|_\lambda^2 = tr_\lambda([x | x]) = \tau_\lambda([x | x]).$$

$$|[x | y]|'_\lambda \leq \tau_\lambda([x | y]) \leq \|x\|_\lambda \|y\|_\lambda.$$

$$\|ax\|_\lambda \leq a \quad |'_\lambda \|x\|_\lambda.$$

Proposition: 2.4. Let W be a Hilbert module over an l.m.c. H^* – algebra E and $b(E) = \{a \in E : \|a\|_\infty = \sup_\lambda \|a\|_\lambda < \infty\}$ and $b(W) = \{x \in W : \|x\|_\infty = \sup_\lambda \|x\|_\lambda < \infty\}$. Then $b(E)$ is an H^* – algebra and $b(W)$ is a $b(E)$ – Hilbert module.

Proof: Clearly, the sets $b(E)$ and $b(W)$ are complex vector spaces and $b(W)$ is a left $b(E)$ – module. Because, when W is identified with $\lim_{\leftarrow \lambda} \widehat{W}_\lambda$, we see that $b(W)$ corresponds to the set of bounded coherent sequences. The Cauchy-Schwarz inequality, applied to Hilbert \widehat{E}_λ – module \widehat{W}_λ , yields for $x, y \in b(W)$, the inequality $\|(x, y)\|_\infty^2 \leq \|(x, x)\|_\infty \|(y, y)\|_\infty$, so that the restriction of $b(W)$ of the $\tau(E)$ – valued product on W is a $\tau(b(E))$ – valued product on $b(W)$. Obviously, $(b(E), (\langle \cdot, \cdot \rangle_\lambda |_{b(E) \times b(E)})_{\lambda \in \Lambda})$ and $(b(W), [\cdot, \cdot] |_{b(W) \times b(W)})$ take more properties being a subset of E and W respectively. To proof of completeness in [7], Satz 3.1, also applied here and show that $b(E)$ and $b(W)$ are complete for norm $\|a\|_\infty = \|\langle a, a \rangle\|_\infty^{\frac{1}{2}}$ and $\|x\|_\infty = \|(x, x)\|_\infty^{\frac{1}{2}}$, respectively, for every $a \in E, x \in W$. Q.E.D.

Definition: 2.5 A non zero projection e in an l.m.c. H^* – algebra E is called minimal, if $eEe = Ce$. Also element u in a Hilbert E – module W is said to be a basic element if there exists a minimal projection $e \in E$ such that $[u | u] = e$. An orthonormal system in W is a family of basic elements $\{u_\alpha\}_\alpha, \alpha \in I$ satisfying $[u_\alpha | u_\beta] = 0$ for all $\alpha, \beta \in I, \alpha \neq \beta$. An orthonormal basis in W is an orthonormal system generating a dense submodule of W .

If V is a subset of a Hilbert E – module W , we define $V^\perp = \{w \in W | [w | v] = 0 \forall v \in V\}$. Clearly V^\perp is a closed submodule of W . If V is a submodule of W then $V^\perp = \{x \in W | (x, v)_\lambda = 0, \forall v \in V, \forall \lambda \in \Lambda\}$.

Lemma 2.6. Let W be a Hilbert E - module and let u in W be such that $e = [u | u]$ is an idempotent in E . If the closed submodule generated by u is complemented or e does not belong to N_λ , for each $\lambda \in \Lambda$ then $[w | u] = [w | u]e$ for all $w \in W$.

We can also generalized Corollary 1.4 of [4] to Hilbert modules over l.m.c. H^* - algebras.

Corollary: 2.7 If $\{u_\alpha\}_{\alpha \in I}$ is an orthonormal system in a Hilbert E - module W in which for every $\alpha \in I$ the closed submodule generated by u_α is complemented or u_α does not belong to ξ_λ , for each $\lambda \in \Lambda$, then

$$[w - \sum_{\alpha \in J} [w | u_\alpha] u_\alpha | w - \sum_{\alpha \in J} [w | u_\alpha] u_\alpha] = [w | w] - \sum_{\alpha \in J} [w | u_\alpha] [w | u_\alpha]^*$$

for every finite subset J of I and for all w in W .

Our next result is a generalization of Proposition 1.5 of [4].

Proposition: 2.8 Let W be a Hilbert E - module, let u be a basic element in W , and let M denote the closed submodule of W generated by u . If M is orthogonally complemented in W then the mapping $w \rightarrow [w | u]u$ is the orthogonal projection from W onto M . As a consequence we have $M = Eu$.

Our next result is a generalization of Theorem 1.6 of [4] which provides a very useful characterization of orthonormal bases in a Hilbert E - module W .

Theorem: 2.9 Let $\{u_\alpha\}_{\alpha \in I}$ be an orthonormal system in a Hilbert E - module W in which for every $\alpha \in I$, the closed submodule generated by u_α is complemented, then the following statement are equivalent.

- (i) For all w_1, w_2 in W the family $\{[w_1 | u_\alpha][w_2 | u_\alpha]^*\}_{\alpha \in I}$ is summable in the space $(\tau(E), (\tau_\lambda)_{\lambda \in \Lambda})$, with sum equal to $[w_1 | w_2]$.
- (ii) For every w in W , we have $[w | w] = \sum_{\alpha \in I} [w | u_\alpha][w | u_\alpha]^*$ (Parseval's identity) in the space $(\tau(E), (\tau_\lambda)_{\lambda \in \Lambda})$.
- (iii) For every w in W , we have $w = \sum_{\alpha \in I} [w | u_\alpha] u_\alpha$ (Fourier expansion).
- (iv) $\{u_\alpha\}_{\alpha \in I}$ is an orthonormal basis in W .

Remark: 2.10 Parseval's identity leads to the equality $\sum_{\alpha \in I} |[w | u_\alpha]|_\lambda^2 = \|w\|_\lambda^2$ for every $\lambda \in \Lambda$. Indeed,

$$\sum_{\alpha \in I} |[w | u_\alpha]|_\lambda^2 = \sum_{\alpha \in I} \text{tr}_\lambda([w | u_\alpha][w | u_\alpha]^*) = \text{tr}_\lambda(\sum_{\alpha \in I} ([w | u_\alpha][w | u_\alpha]^*)) = \text{tr}_\lambda([w | w]) = \|w\|_\lambda^2.$$

The existence of basic elements in a Hilbert module over an l.m.c. H^* - algebra can be guaranteed by an argument similar to Proposition 1.7 of [4]. A slightly modification of Theorem 1.9 of [4] gives the following theorem in a Hilbert E - module.

Theorem: 2.11 Let S be a subset of a Hilbert E - module W . If S is a maximal orthonormal system then it is an orthonormal basis in W and converse is true when each closed submodule generated by every element of S is complemented in W .

Proof: Let S be a maximal orthonormal system in W and let M be the closed submodule generated by S . If $M \neq W$, then there exists $x \in W - M$. It implies that $\|x\|_{\lambda_0} \neq 0$, for some $\lambda_0 \in \Lambda$. Now if $M^\perp = \{0\}$ then $(M + \xi_{\lambda_0})^\perp \subseteq M^\perp = \{0\}$. Hence in the Hilbert \widehat{E}_{λ_0} - module \widehat{W}_{λ_0} , we have $M + \xi_{\lambda_0} = W + \xi_{\lambda_0}$ which is a

contradiction. So $M^\perp \neq \{0\}$ and therefore there exists a basic element u in M^\perp . Obviously $S \cup \{u\}$ is an orthonormal system strictly containing S , which is a contradiction. The converse is obvious by Fourier expansion. Q.E.D.

Corollary: 2.12 Every non zero Hilbert module over an l.m.c. H^* - algebra has an orthonormal basis.

Theorem: 2.13. In a Hilbert E - module W , $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal system if and only if $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal system in the Hilbert \widehat{E}_λ - module \widehat{W}_λ when v_α does not belong to ξ_λ for each $\alpha \in I$ and for each $\lambda \in \Lambda$. Also if $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal basis in W and v_α does not belong to ξ_λ for each $\alpha \in I$ then $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal basis in the Hilbert \widehat{E}_λ - module \widehat{W}_λ . So we have an analogue of Fourier expansion and Parseval's identity associated to this orthonormal basis in Hilbert \widehat{E}_λ - module \widehat{W}_λ .

Proof: Suppose that v is a basic element in W . If for some $\lambda_0 \in \Lambda$, $v \in \xi_{\lambda_0}$, then $v + \xi_{\lambda_0}$ is not a basic element in \widehat{W}_{λ_0} . It is clear that for $\mu \in \Lambda$ in which, $|\cdot|_{\lambda_0} \geq |\cdot|_\mu$, $v \in \xi_\mu$ and $v + \xi_\mu$ is not a basic element in \widehat{W}_μ . Now if v does not belong to ξ_λ , $\lambda \in \Lambda$ then we have

$$\begin{aligned} [v + \xi_\lambda | v + \xi_\lambda] \widehat{E}_\lambda [v + \xi_\lambda | v + \xi_\lambda] &= \pi_\lambda([v | v]) \widehat{E}_\lambda \pi_\lambda([v | v]) \\ &= \pi_\lambda([v | v] E [v | v]) \\ &= \pi_\lambda(C[v | v]) \\ &= C[v + \xi_\lambda | v + \xi_\lambda]. \end{aligned}$$

Hence $v + \xi_\lambda$ is a basic element in \widehat{W}_λ . Conversely, if for each $\lambda \in \Lambda$, $v + \xi_\lambda$ is a basic element in \widehat{W}_λ then we have

$$[v + \xi_\lambda | v + \xi_\lambda] \widehat{E}_\lambda [v + \xi_\lambda | v + \xi_\lambda] = C[v + \xi_\lambda | v + \xi_\lambda]$$

and this implies that

$$\pi_\lambda([v | v] E [v | v] - C[v | v]) = 0,$$

for every $\lambda \in \Lambda$. So $[v | v] E [v | v] - C[v | v] \subseteq \widehat{N}_\lambda$ for every $\lambda \in \Lambda$ and therefore $[v | v] E [v | v] - C[v | v] = 0$. It is easy to verify that, $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal system in W if and only if $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal system in \widehat{W}_λ , when v_α does not belong to ξ_λ for each $\alpha \in I$ and for each $\lambda \in \Lambda$. Now suppose that $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal basis in W and for each $\alpha \in I$, v_α does not belong to ξ_λ then as we mentioned before, $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal system in \widehat{W}_λ . We are going to show that it generates a dense submodule of \widehat{W}_λ . For this, let $x + \xi_\lambda \in \widehat{W}_\lambda$. Since $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal basis in W , So we have

$$\exists \gamma_{i_k} \in C, \exists v_{\alpha_{i_k}} \in \{v_\alpha\}_{\alpha \in I}; k \in \mathbb{N}, \sum_{k=1}^n \gamma_{i_k} v_{\alpha_{i_k}} \rightarrow x,$$

as n tends to ∞ . It implies that

$$\sum_{k=1}^n \gamma_{i_k} (v_{\alpha_{i_k}} + \xi_\lambda) \rightarrow x + \xi_\lambda,$$

when n tends to ∞ . Thus if for all $\alpha \in I$, v_α does not belong to ξ_λ then $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal basis in \widehat{W}_λ . Q.E.D.

3. Space of bounded operators

Definition: 3.1 Let V and W be two Hilbert modules over an l.m.c. H^* – algebra $(E, (\|\cdot\|_\lambda)_{\lambda \in \Lambda})$. An operator $T : V \rightarrow W$ is called E – linear if it is linear and satisfies $T(ax) = aT(x)$ for all $a \in E$ and for all $x \in V$. We say that E – linear operator T is bounded if for each $\lambda \in \Lambda$, and for each $x \in V$ there exists $K_\lambda > 0$ in which $\|T(x)\|_\lambda \leq K_\lambda \|x\|_\lambda$.

Put $\overline{P}_\lambda^V(x) = (x, x)_\lambda^{\frac{1}{2}}$ and $\overline{P}_\lambda^W(Tx) = (Tx, Tx)_\lambda^{\frac{1}{2}}$, where $(x, x)_\lambda$ and $(Tx, Tx)_\lambda$ are denoted positive semi-definite pseudo-inner products in V and W respectively. So the E – linear operator T is bounded if for each $\lambda \in \Lambda$, and for each $x \in V$ there exists $K_\lambda > 0$ in which $\overline{P}_\lambda^W(Tx) \leq K_\lambda \overline{P}_\lambda^V(x)$.

The set of all bounded E – linear operators from Hilbert module V into W is denoted by $B_E(V, W)$ and when $V = W$ is denoted by $B_E(V)$. It is easy to see that the map $\tilde{P}_\lambda, \lambda \in \Lambda$, defined by $\tilde{P}_\lambda(T) = \sup\{\overline{P}_\lambda^W(Tx) : x \in V, \overline{P}_\lambda^V(x) \leq 1\}$ is a seminorm on $B_E(V, W)$.

Theorem: 3.2 Let V and W be Hilbert modules over an l.m.c. H^* – algebra $(E, (\|\cdot\|_\lambda)_{\lambda \in \Lambda})$. Then

- (i) $B_E(V, W)$ is a complete locally convex space with topology determined by the family of seminorms $\{\tilde{P}_\lambda\}_{\lambda \in \Lambda}$.
- (ii) $B_E(V)$ is a locally C^* – algebra with the topology determined by the family of seminorms $\{\tilde{P}_\lambda\}_{\lambda \in \Lambda}$.

Proof: Suppose that $\lambda_1, \lambda_2 \in \Lambda, \|\cdot\|_{\lambda_1} \geq \|\cdot\|_{\lambda_2}, S \in B_{\hat{E}_{\lambda_1}}(\hat{V}_{\lambda_1}, \hat{W}_{\lambda_1})$, set of bounded \hat{E}_{λ_1} – linear operators. We have

$$(\sigma_{\lambda_1 \lambda_2}^{\hat{W}}(S(\sigma_{\lambda_1}^{\hat{V}}(x))), \sigma_{\lambda_1 \lambda_2}^{\hat{W}}(S(\sigma_{\lambda_1}^{\hat{V}}(x))))_{\lambda_2} \leq \|S\|_{\lambda_1}^2 (\sigma_{\lambda_2}^{\hat{V}}(x), \sigma_{\lambda_2}^{\hat{V}}(x))_{\lambda_2}.$$

Therefore the following map is a bounded \hat{E}_{λ_2} -bounded.

$$(\pi_{\lambda_1 \lambda_2})_*(S) : \hat{V}_{\lambda_2} \rightarrow \hat{W}_{\lambda_2}$$

$$\sigma_{\lambda_2}^{\hat{V}}(x) \mapsto \sigma_{\lambda_1 \lambda_2}^{\hat{W}}(S(\sigma_{\lambda_1}^{\hat{V}}(x))).$$

So we yield a bounded operator $(\pi_{\lambda_1 \lambda_2})_*$ from $B_{\hat{E}_{\lambda_1}}(\hat{V}_{\lambda_1}, \hat{W}_{\lambda_1})$ into $B_{\hat{E}_{\lambda_2}}(\hat{V}_{\lambda_2}, \hat{W}_{\lambda_2})$. Also

$\{B_{\hat{E}_\lambda}(\hat{V}_\lambda, \hat{W}_\lambda), (\pi_{\lambda_1 \lambda_2})_*; \|\cdot\|_{\lambda_1} \geq \|\cdot\|_{\lambda_2}, \lambda_1, \lambda_2 \in \Lambda\}$ is an inverse system of Banach spaces. We are going to show that $B_E(V, W)$ and $\lim_{\leftarrow \lambda} B_{\hat{E}_\lambda}(\hat{V}_\lambda, \hat{W}_\lambda)$ are isomorphic. Suppose that $\lambda \in \Lambda, T \in B_E(V, W)$. One can see that $T(\xi_\lambda^V) \subseteq \xi_\lambda^W$ and so there exists a unique operator $T_\lambda : V_\lambda \rightarrow W_\lambda$ in which $\sigma_\lambda^W \circ T = T_\lambda \circ \sigma_\lambda^V$. Moreover T_λ is a bounded E_λ – linear operator. It has a continuous extension $\hat{T}_\lambda : \hat{V}_\lambda \rightarrow \hat{W}_\lambda$. Thus we can define the following continuous linear operator

$$(\pi_\lambda)_* : B_E(V, W) \rightarrow B_{\hat{E}_\lambda}(\hat{V}_\lambda, \hat{W}_\lambda)$$

$$T \mapsto \hat{T}_\lambda$$

where $\sigma_\lambda^{\hat{W}} \circ T = \hat{T}_\lambda \circ \sigma_\lambda^{\hat{V}}$. Also, for $\lambda_1, \lambda_2 \in \Lambda, \|\cdot\|_{\lambda_1} \geq \|\cdot\|_{\lambda_2}$ we have $(\pi_{\lambda_1 \lambda_2})_* \circ (\pi_{\lambda_1})_* = (\pi_{\lambda_2})_*$. Now we can define the following isomorphism operator

$$\begin{aligned} \phi : B_E(V, W) &\rightarrow \lim_{\leftarrow \lambda} B_{\hat{E}_\lambda}(\hat{V}_\lambda, \hat{W}_\lambda) \\ \phi(T) &= ((\pi_\lambda)_*(T))_\lambda. \end{aligned}$$

For each $T \in B_E(V, W)$, $\|\phi(T)\|_\lambda = \tilde{P}_\lambda(T)$. Linear operator ϕ is surjective. Indeed, let $([T_\lambda])_{\lambda \in \Lambda} \in \lim_{\leftarrow \lambda} B_{\hat{E}_\lambda}(\hat{V}_\lambda, \hat{W}_\lambda)$. We define linear operator T as follows

$$T : V \rightarrow W$$

$$x \mapsto (\hat{T}_\lambda(\sigma_{\lambda}^{\hat{V}}(x)))_\lambda.$$

For $\lambda_1, \lambda_2 \in \Lambda$ that $|\cdot|_{\lambda_1} \geq |\cdot|_{\lambda_2}$ we have

$$\sigma_{\lambda_1 \lambda_2}^{\hat{V}}(\hat{T}_{\lambda_1}(\sigma_{\lambda_1}^{\hat{V}}(x))) = (\pi_{\lambda_1 \lambda_2})_*(\hat{T}_{\lambda_1})(\sigma_{\lambda_2}^{\hat{V}}(x)) = \hat{T}_{\lambda_2}(\sigma_{\lambda_2}^{\hat{V}}(x)).$$

So T is well-defined. Also it is a bounded E -module map and $\phi(T) = ((\pi_\lambda)_*(T))_{\lambda \in \Lambda}$. Completeness of $\lim_{\leftarrow \lambda} B_{\hat{E}_\lambda}(\hat{V}_\lambda, \hat{W}_\lambda)$ implies that $B_E(V, W)$ is complete.

For $\lambda \in \Lambda$, \tilde{P}_λ is a submultiplicative seminorm on $B_E(V)$ and $\{B_{\hat{E}_\lambda}(\hat{V}_\lambda), (\pi_{\lambda_1 \lambda_2})_*; |\cdot|_{\lambda_1} \geq |\cdot|_{\lambda_2}, \lambda_1, \lambda_2 \in \Lambda\}$ is an inverse system of C^* -algebras and linear operator

$$\tilde{\phi} : B_E(V) \rightarrow \lim_{\leftarrow \lambda} B_{\hat{E}_\lambda}(\hat{V}_\lambda)$$

$$T \mapsto ((\pi_\lambda)_*(T))_\lambda$$

is an isomorphism of topological algebras. Also, $\|\tilde{\phi}(T)\|_\lambda = \tilde{P}_\lambda(T)$ and since $B_{\hat{E}_\lambda}(\hat{V}_\lambda)$'s are C^* -algebras, so $B_E(V)$ is a locally C^* -algebra. Q.E.D.

Definition: 3.3 We say that E -linear operator T has an adjoint if there exists E -linear operator $T^* : W \rightarrow V$ in which $[Tx | y] = [x | T^*y]$ for each $x \in V$ and $y \in W$. The set of adjointable E -linear operators from Hilbert E -module V into Hilbert E -module W is denoted by $L_E(V, W)$ and for each $\lambda \in \Lambda$, the set of adjointable operators from V_λ into W_λ is denoted by $L_{E_\lambda}(V_\lambda, W_\lambda)$. Let $T \in L_E(V, W)$. For each $\lambda \in \Lambda$, since $T(\xi_\lambda^V) \subseteq \xi_\lambda^W$, we can define

$$(\pi_\lambda)_* : L_E(V, W) \rightarrow L_{E_\lambda}(V_\lambda, W_\lambda)$$

$$(\pi_\lambda)_*(T)(x + \xi_\lambda^V) = T(x) + \xi_\lambda^W,$$

Obviously $(\pi_\lambda)_*(T) \in L_{E_\lambda}(V_\lambda, W_\lambda)$ and $\|T\|_\lambda = \|(\pi_\lambda)_*(T)\|_{L_{E_\lambda}(V_\lambda, W_\lambda)}$ defines a seminorm on $L_E(V, W)$, where $\|\cdot\|_{L_{E_\lambda}(V_\lambda, W_\lambda)}$ is the operator norm in $L_{E_\lambda}(V_\lambda, W_\lambda)$.

We topologize $L_E(V, W)$ via these seminorms. By similar argument just like previous theorem $L_E(V, W)$ may be identified with $\lim_{\leftarrow \lambda} L_{E_\lambda}(\hat{V}_\lambda, \hat{W}_\lambda)$. In particular $L_E(V) \cong \lim_{\leftarrow \lambda} L_{E_\lambda}(\hat{V}_\lambda)$ and we conclude that $L_E(V)$ is a locally C^* -algebra. The connecting maps of the inverse system $\{L_{E_\lambda}(V_\lambda, W_\lambda)\}_{\lambda \in \Lambda}$ will be denoted by $(\pi_{\lambda_1 \lambda_2})_*$, $\lambda_1, \lambda_2 \in \Lambda$, $|\cdot|_{\lambda_1} \geq |\cdot|_{\lambda_2}$, where

$$(\pi_{\lambda_1 \lambda_2})_* : L_{E_{\lambda_1}}(V_{\lambda_1}, W_{\lambda_1}) \rightarrow L_{E_{\lambda_2}}(V_{\lambda_2}, W_{\lambda_2})$$

$$(\pi_{\lambda_1 \lambda_2})_*(T)(x + \xi_{\lambda_2}^V) = \sigma_{\lambda_1 \lambda_2}^W(T(x + \xi_{\lambda_1}^V)).$$

So $\{L_{E_\lambda}(V_\lambda, W_\lambda); (\pi_{\lambda_1 \lambda_2})_*\}_{\|\lambda_1\| \geq \|\lambda_2\|}$ is an inverse system of normed spaces and $\{L_{\widehat{E}_\lambda}(\widehat{V}_\lambda, \widehat{W}_\lambda); (\pi_{\lambda_1 \lambda_2})_*\}_{\|\lambda_1\| \geq \|\lambda_2\|}$ is an inverse system of Banach spaces. Also,

$$L_E(V, W) \cong \lim_{\leftarrow \lambda} L_{\widehat{E}_\lambda}(\widehat{V}_\lambda, \widehat{W}_\lambda).$$

So $L_E(V, W)$ is a complete locally convex space. On the other hand by [2] each $T \in B_{\widehat{E}_\lambda}(\widehat{V}_\lambda)$ belongs to $L_{\widehat{E}_\lambda}(\widehat{V}_\lambda)$.

From this we obtain that each $T \in B_E(V)$ belongs to $L_E(V)$.

Definition: 3.4 Let W be a Hilbert module over an l.m.c. H^* - algebra, E . Let $v, w \in W$ be basic vectors and let the operator $F_{v,w} : W \rightarrow W$ be defined with $F_{v,w}(x) = [x | w]v$. The linear span of the set $\{F_{v,w} : v, w \in W\}$ is denoted by $F_E(W)$ and an operator T belonging to $F_E(W)$ is called a generalized finite rank operator. Observe that $F_E(W) \subseteq B_E(W)$ and $F_{v,w}^* = F_{w,v}$, $TF_{v,w} = F_{Tv,w}$, $F_{v,w}T = F_{v,T^*w}$, for each $v, w \in W$, for each $T \in B_E(W)$. Therefore $F_E(W)$ is a selfadjoint two-sided ideal in $B_E(W)$.

Definition: 3.5 An operator $T \in B_E(W)$ is said to be a generalized compact operator if there exists a sequence of generalized finite rank operators $\{F_n\}$ such that $\lim_n F_n = T$. The set of all generalized compact operators is denoted by $K_E(W)$. By definition $K_E(W) = \overline{F_E(W)}$ is a closed two-sided ideal in $B_E(W)$. Moreover, $K_E(W)$ may be identified with $\lim_{\leftarrow \lambda} K_{\widehat{E}_\lambda}(\widehat{W}_\lambda)$.

We terminate with a result about unitary operators in Hilbert H^* - modules.

Definition: 3.6 Let E be a proper H^* - algebra. We say that Hilbert E - modules V and W are unitary equivalent if there is a unitary element U in $L_E(V, W)$, namely, $UU^* = id_W$ and $U^*U = id_V$.

If $U \in L_E(V, W)$ is unitary then it is clear that U is a surjective E - linear map and also that U is isometric,

since $\|U(x)\|^2 = tr[U(x) | U(x)] = tr[U^*U(x) | x] = tr[x | x] = \|x\|^2$. Our next result will be the converse assertion, that if $U : E \rightarrow F$ is an isometric, surjective E - linear map then U is unitary. For this we need the following lemma.

Lemma: 3.7 Let E be a proper H^* - algebra and $a \in E$. If $\|ac\| = \|bc\|$ for each $c \in E$ then $a^*a = b^*b$.

Proof: We have $\|ac\|^2 = \|bc\|^2$, so that $\langle ac, ac \rangle = \langle bc, bc \rangle$ and $\langle a^*ac, c \rangle = \langle b^*bc, c \rangle$.

Hence $\langle (a^*a - b^*b)c, c \rangle = 0$ for each $c \in E$. From this and by Lemma 3.1 of [1] we have

$$\sup_{\|c\|=1} \| (a^*a - b^*b)c \| = \sup_{\|c\|=1} | \langle (a^*a - b^*b)c, c \rangle | = 0.$$

Thus $\| (a^*a - b^*b)c \| = 0$, where $\|c\| = 1$, so that $(a^*a - b^*b)c = 0$ for arbitrary $c \in E$. Therefore

$$(a^*a - b^*b)E = 0, \text{ so that } a^*a - b^*b = 0. \text{ Q.E.D.}$$

Proposition: 3.8 With E, V, W as before, let U be an E - linear map from V to W . The following conditions are equivalent:

- (i) U is an isometric surjective E - linear map;
- (ii) U is a unitary element of $L_E(V, W)$.

Proof. Suppose that (i) holds. For x in V , $[U(x)|U(x)] = b^*b$ and $[x|x] = c^*c$ for some b, c in E . For each a in E , we have

$$\begin{aligned} \|ab^*\|^2 &= \text{tr}(a[U(x)|U(x)]a^*) \\ &= \text{tr}([U(ax)|U(ax)]) \\ &= \|U(ax)\|^2 \\ &= \|ax\|^2 \\ &= \text{tr}([ax|ax]) \\ &= \text{tr}(a[x|x]a^*) \\ &= \text{tr}(a(c^*c)a) \\ &= \|ac^*\|^2. \end{aligned}$$

Thus $\|ba^*\| = \|ca^*\|$ for each a in E . By previous lemma $b^*b = c^*c$. This implies that $[U(x)|U(x)] = [x|x]$ for each x in E and by polarization identity $[U(x)|U(y)] = [x|y]$ for each x, y in E .

Now let $x \in V$ and $z \in W$. Since U is surjective, there is a $y \in V$ such that

$$U(y) = z. \text{ We have } [U(x)|z] = [U(x)|U(y)] = [x|y] = [x|U^{-1}(z)].$$

Hence $U^* = U^{-1}$. This implies that U satisfies (ii); and the implication (ii) \Rightarrow (i) is obvious as already discussed. Q.E.D.

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