International Journal of Mathematical Archive-9(2), 2018, 44-47
IJMA Available online through www.ijma.info ISSN 2229 – 5046

COMMON FIXED POINT FOR THE WEAKLY COMPATIBLE MAPPINGS USING F-CONTRACTION ON A COMPLETE METRIC SPACES

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(Received On: 16-12-17; Revised & Accepted On: 18-01-18)

ABSTRACT

In 2012 Wardowski [1] defined the F-contraction. The main aim of this paper is to find the common fixed point for two mappings satisfying contraction conditions as similar to that of F-contraction.

Keywords: *Weakly Compatible mappings, F- contraction.*

INTRODUCTION

In1998, Jungck and Rhoades [4] introduced the notion of weakly compatible mappings. In 2012, Wardowski [1] defined the *F*-contraction The main purpose of this paper is to present fixed point results for pair of map satisfying a new contractive condition as similar to that of F-contraction by using the concept of weakly compatible maps in a complete metric space.

Definition 1.1(a): [1] Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be an F-contraction if there exists $\tau>0$ such that for all $x,y\in X$ with

$$d(Tx,Ty)>0 \quad \Rightarrow \tau+F\left(d\left(Tx,Ty\right)\right)<F\left(d\left(x,y\right)\right),$$

Where *F* satisfies following conditions:
(F1) *F* is strictly increasing.
(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}F(\alpha_n)=-\infty.$
(F3) There exist $k\in(0,1)$ such that $\lim_{\alpha\to0^+}\alpha^kF(\alpha)=0.$

Definition 1.1 (b): [4] Two self-mappings *f* and *g* on a metric space (X,d) are said to be weakly compatible if the commute at their coincidence points, that is, if $f(x)=g(x)$, for some $x\in X$, then $fg(x)=gf(x)$.

Wardowski [1] stated and proved the Banach contraction principle as follows.

Theorem 1.2[1]: Let (X,d) be a complete metric space and $T:X\to X$ be an F- contraction. Then T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T^nx\}_{n=1}^{\infty}$ converges to $x^*\in X$.

By replacing the conditions (F2) and (F3) by the following conditions

(F $2'$) $\inf F=-\infty$.

(F $3'$) F is continuous on $(0,\infty)$.

The family of functions satisfying the conditions (F1), (F $2'$) and (F $3'$) is denoted by $\Im$.

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Piri, H., Kumam, P. [2] have proved the following result.

Theorem 1.3[2]: Let *T* be a self-mapping of a complete metric space *X* into itself. Suppose $F \in \Im$ and there exists $\tau > 0$ for all $x, y \in X$ with $d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) < F(d(x, y))$

Then *T* has unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.

We denote the family of functions satisfying the conditions (F1), (F2) and (F$3'$) by Φ.

Recently Secelean [3] proved the following lemma.

Lemma 1.4: Let $F : R_+ \to R$ be an increasing mapping and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following conditions hold:

(a) If $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} \alpha_n = 0$.

(b) If $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

By proving the above lemma 1.4, Secelean proved that the condition (F$2'$) and (F2) are equivalent.

MAIN RESULT

Theorem 1.5: Let (X, d) be a complete metric space and f and g be two continuous self-mappings on *X* where *f* is one to one function and f and g weakly compatible mappings, $f(X)$ is closed subset of *X* and $f(X) \subseteq g(X)$.

Suppose $F \in \Phi$ and there exist $\tau > 0$ such that

$$\forall x, y \in X,\ d(f(x), f(y)) > 0 \Rightarrow \tau + F(d(f(x), f(y))) \leq F(d(g(x), g(y))). \tag{1}$$

Then *f* and *g* has a unique common fixed point.

Proof: let $x_0 \in X$ be an arbitrary point in *X*. Then we consider the following sequence

$f(x_0) = g(x_1)$, $f(x_2) = g(x_3)$, that is $f(x_n) = g(x_{n+1}) = y_n$.

Now since *f* is one to one, it follows that $f(x) = f(y)$ if $x = y$ then since $x_n \neq x_{n+1} \Rightarrow f(x_n) \neq f(x_{n+1})$ i.e., $d(f(x_n), f(x_{n+1})) > 0$ so from (1) it follows that

$$\tau + F(d(f(x_n), f(x_{n+1}))) \leq F(d(g(x_n), g(x_{n+1})))$$

$$\begin{aligned}
\Rightarrow F(d(f(x_n), f(x_{n+1}))) &\leq F(d(g(x_n), g(x_{n+1}))) - \tau. \\
&= F(d(f(x_{n-1}), f(x_n))) - \tau. \\
&\leq F(d(g(x_{n-1}), g(x_n))) - 2\tau. \\
&= F(d(f(x_{n-2}), f(x_{n-1}))) - 2\tau. \\
&\leq F(d(g(x_{n-2}), g(x_{n-1}))) - 3\tau. \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&= F(d(f(x_1), f(x_2))) - n\tau.
\end{aligned}$$

Taking limit $n \to \infty$ both sides and using condition (F2) we get,

$$\lim_{n\to\infty} F(d(f(x_n), f(x_{n+1}))) = -\infty.$$

$$\Rightarrow \lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0.$$

$$\Rightarrow \lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2}$$

Now we prove that the sequence $\{y_n\}$ is a Cauchy sequence. If not then there exist $\varepsilon > 0$ such that for a natural number *N*, $d(y_n, y_m) \geq \varepsilon$ and $d(y_{n-1}, y_m) < \varepsilon$ whenever $n > m > N$.

From (2), we have

$$\varepsilon \leq d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\Rightarrow \lim_{m\to\infty} d(y_n, y_m) = \varepsilon. \tag{3}$$

Now $d(f(x_n), f(x_m)) = d(y_n, y_m) > 0.$

So Now from (1) we have

$$\tau + F(d(f(x_n), f(x_m))) \leq F(d(g(x_n), g(x_m))).$$

$$F(d(f(x_n), f(x_m))) \leq F(d(f(x_{n-1}), f(x_{m-1}))) - \tau.$$

Since $\tau > 0$ is arbitrary

$$\Rightarrow F(d(f(x_n), f(x_m))) < F(d(f(x_{n-1}), f(x_{m-1}))).$$

Since *F* is strictly increasing we have,

$$d(f(x_n), f(x_m)) < d(f(x_{n-1}), f(x_{m-1})). \tag{4}$$

So from (2), (4)

$$\varepsilon \leq d(f(x_n), f(x_m)) < d(f(x_{n-1}), f(x_{m-1}))$$

$$\leq d(f(x_{n-1}), f(x_n)) + d(f(x_n), f(x_m)) + d(f(x_m), f(x_{m-1}))$$

$$\lim_{m\to\infty} d(f(x_{n-1}), f(x_{m-1})) = \varepsilon. \tag{5}$$

Since $d(f(x_n), f(x_m)) = d(y_n, y_m) > 0.$ from (1) we have

$$\tau + F(d(f(x_n), f(x_m))) \leq F(d(g(x_n), g(x_m))).$$

$$\tau + F(d(f(x_n), f(x_m))) \leq F(d(f(x_{n-1}), f(x_{m-1})))$$

So by condition($F3$`), (3), (5) we have,

$$\tau + F(\varepsilon) \leq F(\varepsilon).$$

Which is a contradiction. Therefore the sequence $\{y_n\}$ must be Cauchy sequence in *X*. Now since *X* is complete we have

$$\lim_{n\to\infty} y_n = z.$$

Where $z \in X$. That is we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n+1}) = z.$$

Since $f(X)$ is closed it follows that $z \in f(X)$. So there exist some $u \in X$ such that $z = f(u).$

Now, $d(z, g(u)) = \lim_{n\to\infty} d(z, g(x_{n+1})) = 0.$

$$\Rightarrow z = g(u). \text{ So } f(u) = z = g(u).$$

Therefore *z* is the coincident point of *f* and g. since *f* and *g* are weakly compatible mappings we have,

$$f(u) = g(u). \Rightarrow gf(u) = fg(u). \Rightarrow g(z) = f(z). \tag{6}$$

If $f(z) \neq z$. Then $d(f(z), z) > 0$,

$$\Rightarrow d(f(z), f(u)) > 0.$$

So from (1),

$$\tau + F(d(f(z), f(u)) \le F(d(g(z), g(u)).$$
$$\tau + F(d(f(z), f(u)) \le F(d(f(z), f(u))$$

Since $\tau > 0$, it follows that this is a contradiction. Hence $f(z) = z = g(z)$. so z is the common fixed point of f and g.

To prove the uniqueness. Let us assume that u is another fixed point of f and g Then $f(z) = z = g(z)$. and $f(u) = u = g(u)$.

If $u \ne z$ $d(u,z) > 0 \Rightarrow d(f(u), f(z)) > 0$. So from (1) we have

$$\tau + F(d(f(u), f(z))) \le F(d(g(u), g(z))).$$
$$\Rightarrow \tau + F(d(u,z)) \le F(d(u,z)).$$

Since $\tau > 0$, this is a contradiction. So $u = z$.

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Source of support: Nil, Conflict of interest: None Declared.