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CONNECTEDNESS AND COMPACTNESS VIA REGULAR*-OPEN SETS

S. PIOUS MISSIER*[1], M. ANNALAKSHMI[2]

[1]Assosiate Professor,
PG & Research Department of Mathematics, V. O. Chidambaram College, Tuticorin, India.

[2]Research Scholars,
PG & Research Department of Mathematics, V. O. Chidambaram College, Tuticorin, India.

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ABSTRACT

In this paper, we introduce new concepts, namely regular-connectedness and regular*-compactness using regular*-open sets. We investigate their basic properties and also discuss their relationships with already existing concepts of connectedness and compactness.*

Key wards: *Regular*-connected, regular*- compact, regular*-open, regular*-closed, regular*-closure.*

Mathematical Subject Classification: *54D05, 54D30.*

1. INTRODUCTION

In 1974, Das[2] defined the concept of semi-connectedness in topological spaces and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett [3] introduced and studied the concept of semi-compact spaces. Since then, Hanna and Dorsett [6], Ganster[5] investigated the properties of semi-compact spaces. S.PasunkiliPandian[12] introduced semi*-pre-compact spaces and investigated their properties. S.PiousMissier and A.Robert[16] introduced and studied semi*-α connectedness and semi*-α compactness in topological spaces. The authors [13], have defined regular*-open sets and regular*-closed sets and investigated their properties.

In this paper, we introduced the concept of regular*-connected spaces and investigate their basic properties. We also discuss their relationship with already existing concepts namely connectedness, pre-connectedness and regular-connectedness. Further we define regular*-compact spaces and investigate their properties.

3. PRELIMINARIES

Throughout this paper X, Y will always denote topological spaces. If A is a subset of the space X, *Cl(A)* and *Int(A)* dente the closure and the interior of A respectively.

Definition 2.1: [7] A subset A of a topological space (X, τ) is called
- (i) **generalized closed** (briefly g-closed) if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open.
- (ii) **generalized open** (briefly g-open) if X\A is g-closed in X.

Definition 2.2: [6] Let A be a subset of X. The**generalized closure** of A is defined as the intersection of all g-closed sets containing A, and is denoted by *Cl*(A).*

Definition 2.3: [13] A subset A of a topological space (X, τ) is
- (i) **Regular*-open** (resp. pre-open, regular open) if $A=Int(Cl^*(A))$ (resp. $A\subseteq Int(Cl(A))$, $A=Int(Cl(A))$).
- (ii) **Reguler*-closed** (resp. pre-closed, regular closed) if $A=Cl(Int^*(A))$ (resp. $Cl(Int(A))\subseteq A$, $A=Cl(Int(A))$).
- (iii) **regular*-regular** if it is both regular*-open and regular*-closed.

Corresponding Author: S. Pious Missier*[1],
[1]Assosiate professor, PG & Research Department of Mathematics,
V. O. Chidambaram College, Tuticorin, India.

Definition 2.4: Let A be a subset of X. Then the **regular*-closure** of A is defined as the intersection of all regular*-closed sets containing A and is denoted by *r*Cl(A).*

Theorem 2.5: (i) Every regular-open set is regular*-open. (ii) Every regular*-open set is open. (iii) Every regular*-open set is pre-open.

Definition 2.6: If A is subset of X, then **the regular*-frontier** of Ais defined by *r*Fr(A)=r*Cl(A)\r*Int(A).*

Theorem 2.7: Let A is a subset of X. Then A is regular*-regular, then *r*Fr(A)=*ϕ.

Theorem 2.8: If A is a subset of X, then
- (i) *r*Cl(X\A)=X\r*Int(A)*
- (ii) *r*Int(X\A)=X\r*Cl(A).*
- (iii) A is regular*-closed then *r*Cl(A)= A.*

Definition 2.9: A topological space X is said to be connected (resp. semi-connected, semi*-connected, pre-connected, regular-connected) if X cannot be expressed as the union of two disjoint nonempty open (resp. semi-open, semi*-open, pre-open, regular-open) sets in X.

Definition 2.10: A subset A of a topological space (X, τ) is called clopen if it is both open and closed in X.

Theorem 2.11: A topological space X is connected if and only if the only clopen subsets of X are ϕ and X.

Definition 2.12: A collection **B** of open (resp. semi-open) sets in X is called an open (resp. semi-open) cover of A⊆X if A⊆$\bigcup\{U_\alpha : U_\alpha \in \boldsymbol{B}\}$holds.

Definition 2.13: A space X is said to be compact (resp. semi-compact) if every open (resp. semi-open) cover of X has a finite sub cover.

Definition 2.14: A function *f:X→Y* is said to be
- (i) **regular*-continuous** if $f^{-1}(V)$ is regular*-open in X for every open set V in Y.
- (ii) **contra-regular*-continuous** if $f^{-1}(V)$ is regular*-closed in X for every open set V in Y.
- (iii) **regular*-irresolute** if $f^{-1}(V)$ is regular*-open in X for every regular*-open set V in Y.
- (iv) **contra-regular*-irresolute** if $f^{-1}(V)$ is regular*-closed in X for every regular*-open set V in Y.
- (v) **strongly regular*-irresolute** if $f^{-1}(V)$ is open in X for every regular*-open set V in Y.
- (vi) **regular*-open** if *f(U)* is regular*-open in Y for every open set U in X.
- (vii) **regular*-closed** if *f(F)* is regular*-closed in Y for every closed set F in X.
- (viii) **pre-regular*-open** if *f(U)* is regular*-open in Y for every regular*-open set U in X.
- (ix) **pre-regular*-closed** if *f(F)* is regular*-closed in Y for every regular*-closed set F in X.
- (x) **regular*-totally continuous** if $f^{-1}(V)$ is clopen in X for every regular*-open set V in Y.
- (xi) **totallyregular*-continuous** if $f^{-1}(V)$ is regular*-regular in X for every open set V in Y.

Theorem 2.15: Let $f : X\to Y$ be a function. Then
- (i) *f* is regular*-continuous if and only if f^{-1}(F) is regular*-closed in X for every closed set F in Y.
- (ii) *f* is regular*-irresolute if and only if f^{-1}(F) is regular*-closed in X for every regular*-closed set F in Y.
- (iii) *f* is contra-regular*-continuous if and only if f^{-1}(F) is regular*-open in X for every closed set F in Y.
- (iv) *f* is contra-regular*-irresolute if and only if f^{-1}(F) is regular*-open in X for every regular*-closed set F in Y.
- (v) *f* is regular*-totally continuous if and only if f^{-1}(F) is clopen in X for every regular*-closed set F in Y.

3. REGULAR*-CONNECTED SPACES

Definition 3.1: A topological space X is said to be **regular*-connected**if X cannot be expressed as the union of two disjoint non-empty regular*-open sets in X.

Theorem 3.2:
- (i) Every regular*-connected space is regular-connected.
- (ii) Every connected space is regular*-connected.
- (iii) Every pre-connected space is regular*-connected.

Proof: Follows from Theorem 2.5 and Definitions.

Remark 3.3: It can be seen that the converse of each of the statement in Theorem 3.2 is not true.

Definition 3.4: The sets A and B in a topological space X are said to be regular*-separated if $A\cap r^*Cl(B)=r^*Cl(A)\cap B=\phi$.

Theorem 3.5: For a topological space X, the following statements are equivalent:
- (i) X is regular*-connected.
- (ii) X cannot be expressed as the union of two disjoint non-empty regular*-closed sets in X.
- (iii) The only regular*-regular subsets of X are ϕ and X itself.
- (iv) Every regular*-continuous function of X into a discrete space Y is constant.

Proof:
(i)⇒(ii): Let X be a regular*-connected space. Suppose X=A∪B, where A and B are disjoint non-empty regular*-closed sets. Then A=X\B and B=X\A are disjoint non-empty regular*-open sets in X. This is a contradiction to X is regular*-connected. This proves (ii).

(ii)⇒(i): Assume that X cannot be expressed as the union of two disjoint non-empty regular*-closed sets in X. Suppose X=A∪B, where A and B are disjoint non-empty regular*-open sets. Then A=X\B and B=X\A are disjoint non-empty regular*-closed sets in X. This is a contradiction to (ii).

(i)⇒(iii): Suppose X is a regular*-connected space. Let A be non-empty proper subset of X that is regular*-regular. Then X\A is a non-empty regular*-open and X = A∪(X\A). This is a contradiction to X is regular*-connected.

(iii)⇒(i): Suppose X = A∪B where A and B are disjoint non-empty regular*-open sets. Then A = X\B is regular*-closed. Thus A is a non-empty proper subset that is regular*-regular. This is a contradiction to (iii).

(iii)⇒(iv): Let *f* be a regular*-continuous function of the regular*-connected space X into the discrete space Y. Then for each y∈Y, $f^{-1}(\{y\})$ is a regular*-regular set of X. Since X is regular*-connected, $f^{-1}(\{y\})$ = ϕ or X. If $f^{-1}(\{y\})$ = ϕ for all y∈Y, then *f* fails to be a function. Therefore $f^{-1}(\{y_0\})$ = X for a unique y_0∈Y. This implies $f(x)=\{y_0\}$ and hence *f* is a constant function.

(iv)⇒(iii): Let U be a regular*-regular set in X. Suppose U≠ ϕ, We claim that U = X. Otherwise, choose two fixed points y_1 and y_2 in Y. Define $f:X\to Y$ such that $f(x)=\begin{cases} y_1 & if x\in U \\ y_2 otherwise\end{cases}$ Then for any open set V in Y, $f^{-1}(V)$ equals U if V contains y_1 but not y_2, equals X\U if V contains y_2 but not y_1, equals X if V contains both y_1 and y_2 and equals ϕ otherwise. In all the cases $f^{-1}(V)$ is regular*-open in X. Hence *f* is a non-constant regular*-continuous function of X into Y. This is a contradiction to our assumption. This proves that the only regular*-regular subsets of X are ϕ and X.

Theorem 3.6: Let $f: X\to Y$ be a regular*-continuous bijection and X be regular*-connected. Then Y is connected.

Proof: Let $f: X\to Y$ be a regular*-continuous bijection and X be regular*-connected. Let V be a clopen subset of Y. By definition 2.14(i) $f^{-1}(V)$ is regular*-open and by Theorem 2.15(i) $f^{-1}(V)$ is regular*-closed and hence $f^{-1}(V)$ is regular*-regular in X. Since X is regular*-connected, by Theorem 3.5 $f^{-1}(V)$ = ϕ or X. Hence V = ϕ or Y. This proves that Y is connected.

Theorem 3.7: Let $f: X\to Y$ be a regular*-irresolute bijection. If X is regular*-connected, so is Y.

Proof: Let $f: X\to Y$ be a regular*-irresolute bijection and let X be regular*-connected. Let V be a subset of Y that is regular*-regular in Y. By Definition 2.14(iii) and by Theorem 2.15(ii), $f^{-1}(V)$ is regular*-regular in X. Since X is regular*-connected, $f^{-1}(V)$ = ϕ or X. Hence V = ϕ or Y. This proves that Y is regular*-connected.

Theorem 3.8: Let $f: X\to Y$ be a pre-regular*-open and pre-regular*-closed bijection. If Y is regular*-connected, so is X.

Proof: Let A be a subset of X that is regular*-regular in X. Since *f* is both pre-regular*-open and pre-regular*-closed, *f(A)* is regular*-regular in Y. Since Y is regular*-connected, *f(A)* = ϕ or Y. Hence A = ϕ or X. Therefore by Theorem 3.5, X is regular*-connected.

Theorem 3.9: If $f: X\to Y$ is a regular*-open and regular*-closed bijection and Y is regular*-connected, then X is connected.

Proof: Let A be a clopen subset of X. Since *f* is regular*-open, *f(A)* is regular*-open in Y. Since *f* is regular*-closed, *f(A)* is regular*-closed in Y. Hence *f(A)* is regular*-regular in Y. Since Y is regular*-connected, by Theorem 3.5, *f(A)* = ϕ or Y. Hence A = ϕ or X. Therefore by Theorem 2.11, X is connected.

Theorem 3.10: If there is a regular*-totally continuous function from a connected space X onto Y, then the only regular*-open sets in Y are ϕ and Y.

Proof: Let *f* be a regular*-totally continuous function from a connected space X onto Y. Let V be any regular*-open set in Y. Since *f* is regular*-totally continuous, $f^{-1}(V)$ is clopen in X. Since X is connected, by Theorem 2.11,$f^{-1}(V)$ = ϕ or X. This implies V = ϕ or Y.

Theorem 3.11: If *f: X→Y* is a strongly regular*-continuous bijectinand Y is a space with atleast two points, then X is not regular*-connected.

Proof: Let y∈Y. Then $f^{-1}(\{y\})$ is a non-empty proper subset that is regular*-regular in X. Hence by Theorem 3.5, X is not regular*-connected.

Theorem 3.11: Let *f: X→Y* be a contra-regular*-continuous surjection and X be regular*-connected. Then Y is connected.

Proof: Let *f: X→Y* be a contra-regular*-continuous surjection and X be regular*-connected. Let V be a clopen subset of Y. By Definition 2.14(ii) and Theorem 2.15(iii), $f^{-1}(V)$ is regular*-regular in X. Since X is regular*-connected, $f^{-1}(V)$ = ϕ or X. Hence V = ϕ or Y. This proves that Y is connected.

4. REGULAR*-COMPACT SPACES

Definition 4.1: A collection $\mathcal{A}$ of regular*-open sets in X is called **a regular*-open cover** of a subset B of X if B ⊆ ∪ $\{U_\alpha : U_\alpha \in \mathcal{A}\}$holds.

Definition 4.2: A space X is said to be **regular*-compact** if every regular*-open cover of X has a finite subcover.

Definition 4.3: A subset B of X is said to be **regular*-compact relative to X** if for every regular*-open cover $\mathcal{A}$ of B, there is a finite sub collection of $\mathcal{A}$ that covers B.

Remark 4.4: Every finite topological space is regular*-compact.

Theorem 4.5:

(i) Every compact space is regular*-compact.
(ii) Every regular*-compact space is regular-compact.
(iii) Every pre-compact space is regular*-compact.

Proof: Follows from Theorem 2.5 and Definitions.

Theorem 4.6: Every regular*-closed subset of a regular*-compact space X is regular*-compact relative to X.

Proof: Let A be a regular*-closed subset of a regular*-compact space X. Let $\mathcal{B}$ be a regular*-open cover of A. Then ∪{X\A} is a regular*-open cover of X. Since X is regular*-compact, this cover contains a finite sub cover of X and hence contains a finite sub collection of $\mathcal{B}$ that covers A. This shows that A is regular*-compact relative to X.

Theorem 4.7: A space X is regular*-compact if and only if for every family of regular*-closed sets in X which has empty intersection has a finite sub family with empty intersection.

Proof: Suppose X is regular*-compact and $\{F_\alpha : \alpha \in \Delta\}$ is a family of regular*-closed sets in X such that ∩ $\{F_\alpha : \alpha \in \Delta\} = \phi$. Then ∪ $\{X \backslash F_\alpha : \alpha \in \Delta\}$ is a regular*-open cover for X. Since X is regular*-compact, this cover has a finite sub cover $\{X\backslash F_{\alpha 1}, X\backslash F_{\alpha 2}, \ldots\ldots, X\backslash F_{\alpha n}\}$. That is X =∪$\{F_{\alpha i}$: i=1,2,…..,n}. On taking the complements on both sides we get $\bigcap_{i=1}^{n} F_{\alpha i} = \phi$. Conversely, suppose that every family of regular*-closed sets in X which has empty intersection has a finite sub family with empty intersection. Let $\{U_\alpha : \alpha \in \Delta\}$ be a regular*-open cover for X. Then ∪ $\{U_\alpha : \alpha \in \Delta\} = X$. Taking the complements, we get ∩ $\{X \backslash U_\alpha : \alpha \in \Delta\} = \phi$. Since $X\backslash U_\alpha$ is regular*-closed for each $\alpha \in \Delta$, by the assumption, there is a finite sub family$\{X\backslash U_{\alpha 1}, X\backslash U_{\alpha 2}, \ldots\ldots, X\backslash U_{\alpha n}\}$ with empty intersection. That is $\bigcap_{i=1}^{n}(X \backslash U_{\alpha i}) = \phi$. Taking the complements on both sides, we get $\bigcup_{i=1}^{n} U_{\alpha i} = X$. Hence X is regular*-compact.

Theorem 4.8: Let *f: X→Y* be a regular*-irresolute bijection. If X is regular*-compact, then so is Y.

Proof: Let *f: X→Y* be a regular*-irresolute bijection and X is regular*-compact. Let $\{V_\alpha\}$ be a regular*-open cover for Y. Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by regular*-open sets. Since X is regular*-compact, $\{f^{-1}(V_\alpha)\}$ contains a finite sub cover namely, $\{f^{-1}(V_{\alpha 1}), f^{-1}(V_{\alpha 2}), \ldots\ldots, f^{-1}(V_{\alpha n})\}$. Then $\{V_{\alpha 1}, V_{\alpha 2}, \ldots\ldots, V_{\alpha n}\}$ is a finite sub cover for Y. Then Y is regular*-compact.

Theorem 4.9: Let $f: X\to Y$ be a regular*-continuous bijection and X be regular*-compact.Then Y is compact.

Proof: Let $f: X\to Y$ be a regular*-continuous bijection and X is regular*-compact. Let $\{V_\alpha\}$ be an open cover for Y. Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by regular*-open sets. Since X is regular*-compact, $\{f^{-1}(V_\alpha)\}$ contains a finite sub cover namely, $\{f^{-1}(V_{\alpha 1}), f^{-1}(V_{\alpha 2}), \ldots\ldots, f^{-1}(V_{\alpha n})\}$. Then $\{V_{\alpha 1}, V_{\alpha 2},\ldots\ldots,V_{\alpha n}\}$ is a finite sub cover for Y. Then Y is compact.

Theorem 4.10: Let $f: X\to Y$ be a pre-regular*-continuous injection. If Y is regular*-compact, then so is X.

Proof: Let $\{V_\alpha\}$ be a regular*-open cover for X.Then $\{f(V_\alpha)\}$ is a cover of Y by regular*-open sets. Since Y is regular*-compact, $\{f(V_\alpha)\}$ contains finite sub cover, namely $\{f(V_{\alpha 1}), f(V_{\alpha 2}),\ldots\ldots, f(V_{\alpha n})\}$. Then $\{V_{\alpha 1}, V_{\alpha 2},\ldots\ldots,V_{\alpha n}\}$ is a finite sub cover for X. Therefore X is regular*-compact.

Theorem 4.11: If $f: X\to Y$ be a regular*-open injection and Y is regular*-compact, then X is compact.

Proof: Let $\{V_\alpha\}$ be an open cover for X.Then $\{f(V_\alpha)\}$ is a cover of Y by regular*-open sets. Since Y is regular*-compact, $\{f(V_\alpha)\}$ contains finite sub cover, namely $\{f(V_{\alpha 1}), f(V_{\alpha 2}),\ldots\ldots, f(V_{\alpha n})\}$. Then $\{V_{\alpha 1}, V_{\alpha 2},\ldots\ldots,V_{\alpha n}\}$ is a finite sub cover for X. Therefore X is regular*-compact.

Theorem 4.12: Let $f: X\to Y$ be a contra-regular*-continuous function and Y be T_1. If X is regular*-compact, then the range of *f* is finite. Further if Y is infinite, *f* cannot be onto.

Proof: Since Y is T_1, for each $y \in Y$, {y} is closed in Y. Since *f* is contra-regular*-continuous, by Theorem $f^{-1}(\{y\})$ is regular*-open in X. Therefore $\{f^{-1}(\{y\}): y\in Y\}$ is a regular*-open cover for X. Since X is regular*-compact, There are $y_1, y_2,\ldots\ldots,y_n$ in Y such that $\{f^{-1}(\{y_i\}): i=1,2,\ldots\ldots,n\}$ is a cover of X by regular*-open sets. Therefore $\cup\{f^{-1}(\{y_i\}): i=1,2,\ldots\ldots,n\}$ = X, that is $f^{-1}(\{y_1, y_2,\ldots.., y_n\}) = X$. This implies $f(X) = \{y_1, y_2,\ldots..,y_n\}$. Thus the range of *f* is finite. If Y is infinite, $f(X) \neq Y$. Hence *f* cannot be onto.

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