International Journal of Mathematical Archive-9(1), 2018, 213-218
Available online through www.ijma.info ISSN 2229 – 5046

ON GEOMETRIC INTERPRETATION OF THE SIMPLEX METHOD: A REVIEW

ASHU VIJ

Assistant Professor,
P. G. Department of Mathematics, D. A. V. College, Amritsar -143006, India.

(Received On: 13-12-17; Revised & Accepted On: 30-12-17)

ABSTRACT

The simplex method is the computation procedure to obtain the minimum (optimum) feasible solution of an optimization problem. This method starts with an initial basic feasible solution and a sequence of basic feasible solutions is obtained such that value of objective function at each member is less than the proceeding value. In this paper the geometric interpretation of the algebraic computations of the simplex procedure is reviewed. The basic concepts and other geometric properties used for this interpretation are also discussed.

Keywords: *The simplex method, feasible solution, optimization, geometric interpretation.*

1. INTRODUCTION

To solve the optimization problems involving the all relationships between the variables in linear form, the technique of Linear Programming is frequently used. It was the first method which was widely used for solving optimization problems using digital computation. The optimum solutions to different types of problems always attracted human throughout the ages. Some powerful techniques for optimization problems had been developed by the great mathematicians of 17th and 18th centuries. In recent times, a new type of optimization problems has come out of the different organizational structures. The General Linear Programming Problem is to find a vector $X = (x_1, x_2, ..., x_n)$ which optimizes the following linear function

$$c_1x_1 + c_2x_2 + ... + c_nx_n$$

subject to linear constraints

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_n \\ &x_i \geq 0,\ i = 1, 2, ..., n \end{aligned}$$

where a_{ij}, c_j and b_j are known constants and $m<n$.[1]

Definition 1: An n-dimensional Euclidean space E_n is defined as the set of vectors, which satisfies the following properties:

(a) The length of X defined by $\|X\| = \sqrt{X_1^2 + X_2^2 + ... + X_n^2}$ is always greater than or equal to zero. Further $\|X\| = 0$ iff $X = 0$.

(b) $\|\lambda X\| = |\lambda| \|X\|$, where λ is any scalar.

(c) $\|X + Y\| \leq \|X\| + \|Y\|$ for all X, Y in E_n.

(d) There always exists a set of n L.I. vectors in E_n and every set of (n+1) vectors in E_n is always L.D.

Corresponding Author: Ashu Vij
Assistant Professor, P. G. Department of Mathematics, D. A. V. College, Amritsar -143006, India.

Definition 2: A hyper plane in E_n is denoted by *H(A,K)* and is defined as the set of all vectors *X* of E_n such that $A - X = K$ where *A* is any non-zero vector and *K* is a real number. Further *A.X* is inner product of vector which is the sum of product of corresponding components of two vectors *A* and *X*.

For example, in two-dimensional Euclidean space E_2, one linear equation of the type $a_1x_1 + a_2x_2 = K$ represents a line which divides the whole plane E_2 into two half planes.

Similarly, in three dimensional Euclidean space E_3, one linear equation of the type $a_1x_1 + a_2x_2 + a_3x_3 = K$ represents a plane which divides the whole space E_3 into two half spaces.

In general, in E_n, the similar term is Hyper plane which divides whole Euclidean space E_n into two half spaces given by

$$H^+(\mathrm{X}, \mathrm{K}) = \{X : A.X \geq K\}$$

$$\text{and } H^-(\mathrm{X}, \mathrm{K}) = \{X : A.X \leq K\}$$

Definition 3: A subset S of Euclidean space E_n is called convex set if for any two vectors X_1 and X_2 in *S*, any convex combination $X = \lambda_1 X_1 + \lambda_2 X_2 \ \ where\ \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0$ is also contained in *S*. In other words, *S* is a convex set if the line segment joining any two points of *S* is also contained in *S*.

Definition 4: A point *X* in the convex set *S* is called an extreme point (corner point) of *S* if *X* cannot be expressed as convex combination of any other two different points of *S* i.e. if there does not exist two different points X_1 and X_2 other than *X* in S such that $X = \lambda_1 X_1 + \lambda_2 X_2$, $\lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 > 0$.

Definition 5: The convex hull of a set *S* is set of all convex combinations of sets of points of *S*. It is denoted by C(S).

Definition 6: A convex polyhedron is defined as the convex hull of a set having finite number of points.

Definition 7: Any subset *S* of is called a cone if for X in *S*, αX is also in *S* for every nonnegative number α.

Definition 8: A simplex is an n-dimensional convex polyhedron which has exactly (n+1) vertices.

For example, in one dimension, a simplex is a line segment with its two end points and its vertices. In two dimensions, a simplex is a triangle having three vertices. In three dimensions, a simplex is a tetrahedron having four vertices.

The general linear programming problem is to determine a subset of points of the convex set *S* defined by a set of m linear constraints in E_n for which the linear objective function is minimized (or maximized).

Definition 9: An edge of a convex polyhedron is defined as the line segment joining two adjacent extreme points of convex set *S* of feasible solutions of LPP.

2. GEOMETRICAL PROPERTY OF LINEAR FUNCTIONAL (OBJECTIVE FUNCTION)

The linear functional $c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is always perpendicular to the vector through origin to the point $c = (\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_n)$. Further, if the linear functional is transformed to $z = d_1x_1 + d_2x_2 + \ldots + d_nx_n$ then *z* represents the distance of the linear functional from the origin where $d_j = \dfrac{c_j}{\sqrt{c_1^2 + c_2^2 + \ldots + c_n^2}}$ (2)

Proof: Let $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Perpendicular distance of origin from any point on the linear functional

$$= \frac{|z - c_1(0) - c_2(0) - \ldots - c_n(0)|}{\sqrt{c_1^2 + c_2^2 + \ldots + c_n^2}}$$

So perpendicular distance of origin to the point $c = (\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_n)$ on the linear functional

$$= \frac{|c_1c_1 + c_2c_2 + \ldots + c_nc_n|}{\sqrt{c_1^2 + c_2^2 + \ldots + c_n^2}}$$

$$= \sqrt{c_1^2 + c_2^2 + \ldots + c_n^2} \qquad (1)$$

Also distance between origin and point $(c_1, c_2, ..., c_n)$

$$= \sqrt{(c_1 - 0)^2 + (c_2 - 0)^2 + ... + (c_n - 0)^2} = \sqrt{c_1^2 + c_2^2 + ... + c_n^2} \tag{2}$$

From (1) and (2) it immediately follows that the linear functional is always perpendicular to the vector through origin to the point $(c_1, c_2, ..., c_n)$.

Further, if the linear functional is transformed to $z = dx_1 + dx_2 + \cdots + d_n x_n$ (3)

Then, distance of linear functional from origin

$$= \frac{|z - d_1(0) - d_2(0) - ... - d_n(0)|}{\sqrt{d_1^2 + d_2^2 + ... + d_n^2}}$$

Now $\sqrt{d_1^2 + d_2^2 + ... + d_n^2} = \sqrt{\frac{c_1^2}{c_1^2 + c_2^2 + ... + c_n^2} + ... + \frac{c_n^2}{c_1^2 + c_2^2 + ... + c_n^2}} = 1$

This immediately follows that Z given by (3) represents the distance of the linear functional from the origin. In other words, if the coefficients of the linear form and the equations are normalized then Z is the distance of linear form from the origin [4].

3. GEOMETRICAL PROPERTY OF SLACK VARIABLE

Let x_{n+i} be the i^{th} slack variable for the system $AX \le b,\ X \ge 0$

Show that for any nonnegative solution,

$$x_{n+1} = h_i \sqrt{a_{i1}{}^2 + a_{i2}{}^2 + ... + a_{in}{}^2}$$

where h_i is the perpendicular distance from the solution to the i^{th} hyperplane [3].

Proof: If x_{n+i} is the i^{th} slack variable for the system $AX \le b,\ X \ge 0$, then i^{th} constraint can be written as

$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + x_{n+i} = b_i \tag{1}$$

$$h_i = \frac{|b_i - a_{i1}x_1 - a_{i2}x_2 - ... - a_{in}x_n|}{\sqrt{a_{i1}{}^2 + a_{i2}{}^2 + ... + a_{in}{}^2}}$$

or $$|b_i - a_{i1}x_1 - a_{i2}x_2 - ... - a_{in}x_n| = h_i \sqrt{a_{i1}{}^2 + a_{i2}{}^2 + ... + a_{in}{}^2}$$

This immediately follows from non-negativity of x_{n+i} and (1) that

$$x_{n+i} = h_i \sqrt{a_{i1}{}^2 + a_{i2}{}^2 + ... + a_{in}{}^2}$$

4. GEOMETRICAL PROPERTIES OF SET OF FEASIBLE SOLUTIONS TO LPP

(i) The set *S* of the feasible solutions of linear programming problem is always convex.
(ii) The boundary of *S* consists of sections of some of the corresponding hyper plane [4].
(iii) The set *S* which is subset of Euclidean space E_n is either void, a convex polyhedron or a convex region which is unbounded in some direction.
(iv) If the set of solutions *S* to a LPP is bounded then S is convex combination of extreme points of *S*. We can always bound S by putting additional constraint $\sum_{j=1}^{n} x_j \le M$ to any LPP, where M is very large number.
(v) Further, if in the optimal solution $\sum_{j=1}^{n} x_j = M$ then original problem has unbounded optimum [5].
(vi) The set of all feasible solutions S to the constraints $AX \le b,\ X \ge 0$ can be written as convex combination of extreme points of S and a non-negative linear combination of the extreme points of solution space

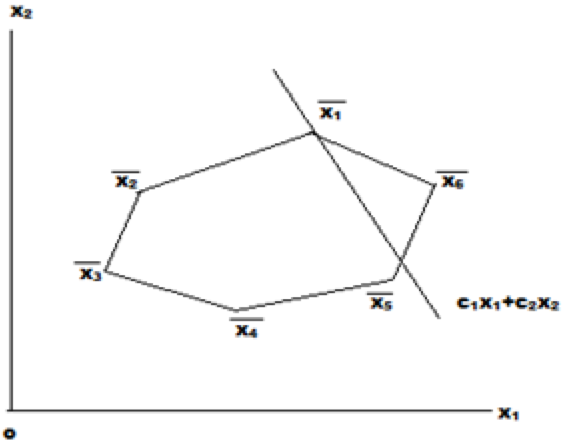
$$AX \le 0,\ \sum_{j=1}^{n} x_j = 1,\ X \ge 0$$ (6).

5. GEOMETRICAL INTERPRETATION OF SIMPLEX METHOD

The simplex procedure starts with an initial basic feasible solution. As BFS corresponds to an extreme point or corner point of set *S* of feasible solutions of LPP, so we can interpret the simplex method as follows:

The simplex method starts with a corner point of convex region consisting of all feasible solutions. As one iteration of simplex method is complicated, a new adjacent or neighboring corner point is obtained such that the value of linear functional is smaller than the preceding value. As an example, consider an arbitrary two dimensional convex polyhedron.

Let the starting point be $\overline{X_1}$. After one iteration of the simplex method, the linear functional would advance the linear form parallel to itself through point $\overline{X_2}$. In next iteration, this linear functional would move parallel to itself through $\overline{X_3}$ and then finally through $\overline{X_4}$ when the minimum feasible solution is attained. (See the figure).



6. GEOMETRIC INTERPRETATION OF THE SIMPLEX METHOD IN TERMS OF COLUMN VECTORS.

We shall explain another geometric interpretation of Simplex method [7] with the help of following numerical example [4]. Consider the LPP:

Minimize $5x_1 + 3x_2 + 4x_3 + 6x_4$ subject to

$$x_1 + 3x_2 + 5x_3 + 6x_4 = 3,$$
$$6x_1 + 4x_2 + 2x_3 + x_4 = 2,$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Let the 3-dimensional vectors P_1', P_2', P_3', P_4' be obtained by augmenting the cost coefficients c_1, c_2, c_3, c_4 respectively to 2-dimensional vectors P_1, P_2, P_3, P_4 of the coefficient matrix of given linear equations as follows: $P_1' = (1\ \ 6\ \ 5)^t, P_2' = (3\ \ 4\ \ 3)^t, P_3 = (5\ \ 2\ \ 4)^t, P_4 = (6\ \ 1\ \ 6)^t$. Let B be the line in 3-dimensional space consisting of all points whose first two elements are 3 and 2 i.e. R.H.S of linear constraints and third element takes on all the values. The objective is to find the lowest point of B which is also in C, the convex cone in 3-dimensional space determined by P_1', P_2', P_3', P_4'. (See figure 1)

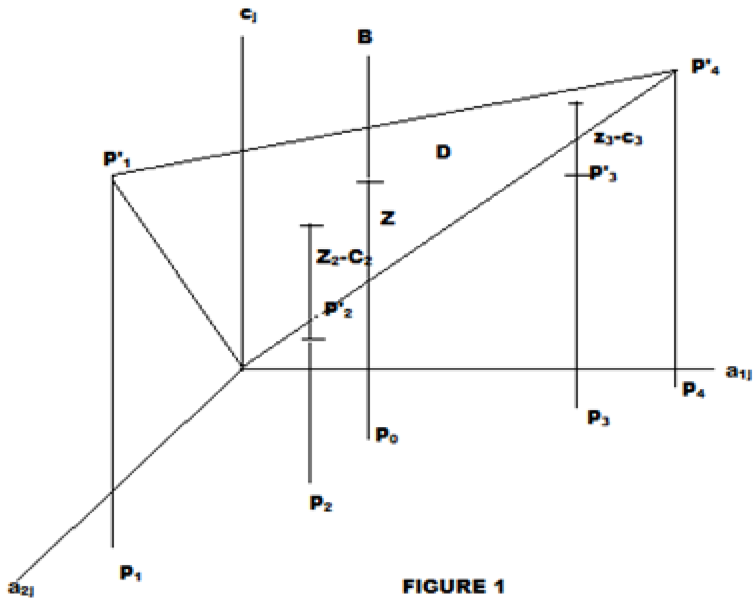


FIGURE 1

Obviously $P_1^{'}$ and $P_4^{'}$ are L.I. We take P_1 and P_4 as the basis and determine two dimensional cone determined by $P_1^{'}$ and $P_4^{'}$.

The point $P_2^{'}$ is below the hyper plane containing D (cone generated by $P_1^{'}$ and $P_4^{'}$) by a distance $z_2 - c_2$ and point $P_3^{'}$ is below the hyper plane containing D by a distance $z_3 - c_3$.

Now P_2 is introduced in basis as $z_2 - c_2 > z_3 - c_3$. Also P_0 is in cone generated by P_2 and P_4 and not the cone generated P_1 and P_4 by i.e. P_1 is eliminated from basis. The new basis is now P_2 and P_4 . (See figure 2)

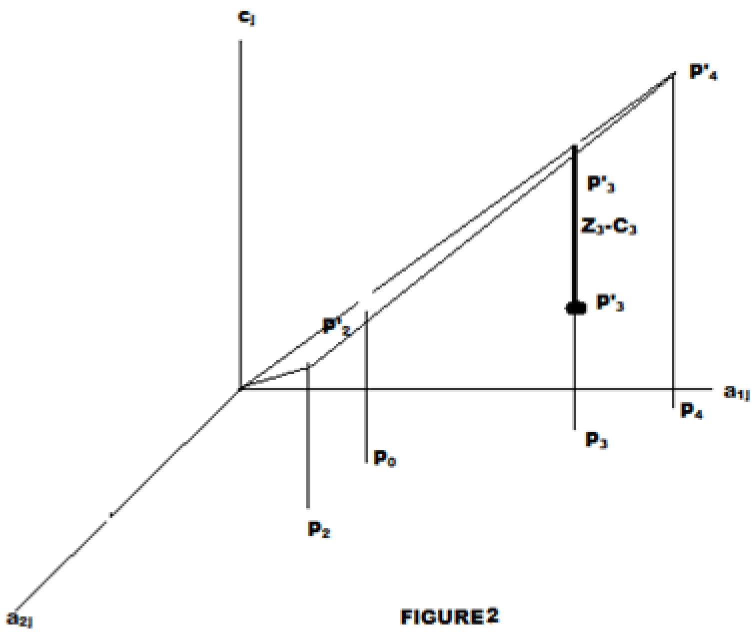


FIGURE 2

Using the similar procedure of obtaining $z_3 - c_3$, $P_3^{'}$ is below the hype rplane containing cone generating by $P_2^{'}$ and $P_4^{'}$ at a distance $z_3 - c_3$, vector P_3 is introduced in the basis. Now P_0 lies in the cone generated by P_2 and P_3 and not in the cone generated by P_3 and P_4 so P_4 is eliminated from the basis. Hence P_2 and P_3 vectors form basis and correspond to minimum feasible solution. The minimum value of *z* for this problem is the distance of point of intersection of line *B* with the two dimensional cone generated by $P_2^{'}$ and $P_3^{'}$.

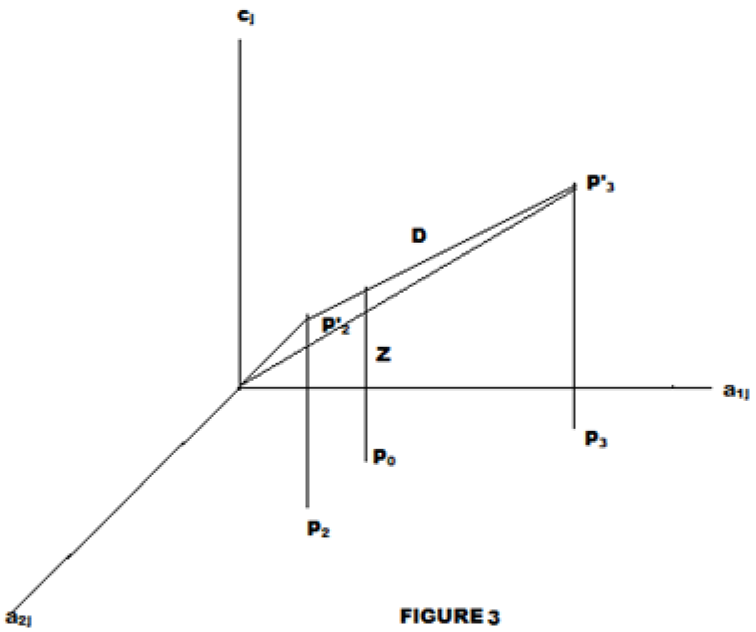


FIGURE 3

7. CONCLUSION

In simplex method, in each step or iteration, one moves from one extreme point to the other extreme point of convex polyhedron. The simplex procedure selects the new adjacent extreme point of the convex polyhedron by moving on the edge of the convex polyhedron such that the movement along the edge causes greatest decrease in the value of objective function per unit change in the value of new variable which is to be introduced.

REFERENCES

1. G.B. Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton N.J. 1963.
2. Linear Programmig, Hadley, G.; Addison- Wesley Publishing company, Inc, Reading Mass, 1961.
3. What is linear programming, Bassov, A.S., D.C. Heath and company, Boston, 1964.
4. Linear Programming: Methods and Applications, Gass Saul I, McGraw Hill Book Company, 4th edition.
5. Linear Programming, Kreko, B: Elsevier publishing company, Inc, NY, 1968.
6. Linear Inequalities and convex Polyhedral sets, Tucker, A.W.; Directorate of management Analysis
7. Hoffman,A., M. Mannos, D. Sokolowsky and N. Weigmann; Computational experience in solving linear programs, Journal of Society for Industrial and applied Mathematics, Vol. No.1,1953.

Source of support: Nil, Conflict of interest: None Declared.