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AN APPLICATION OF HYPERGEOMETRIC FUNCTION ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT

The purpose of the present paper is to investigate some interesting criteria for hypergeometric functions belonging to classes $T(\lambda,\alpha)$ and $C(\lambda,\alpha)$. We also studied an integral operator related to hypergeometric functions relevant connection of the results presented herewith various well known results are briefly indicated.

Keywords: *Analytic, Univalent, Hypergeometric function, Integral Operator*

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1. INTRODUCTION

Let T be the class consisting of functions of the form:

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \quad (a_n\geq 0) \tag{1.1}$$

which are analytic and univalent in the open unit disc $U=\{z:|z|<1\}$. Let $T(\lambda,\alpha)$ be the subclass of T consisting of functions which satisfy the condition

$$\Re\left\{\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\}>\alpha \tag{1.2}$$

for some $\alpha(0\leq\alpha<1)$, $\lambda(0\leq\lambda<1)$ and for all $z\in U$.

Also let $C(\lambda,\alpha)$ denote the subclass of T consisting of function which satisfy the condition

$$\Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right\}>\alpha \tag{1.3}$$

for some $\alpha(0\leq\alpha<1)$, $\lambda(0\leq\lambda<1)$ and for all $z\in U$ also from (1.2) and (1.3) we have

$$f(z)\in C(\lambda,\alpha)\Leftrightarrow zf'(z)\in T(\lambda,\alpha). \tag{1.4}$$

We note that $T(0,\alpha)=T^*(\alpha)$, the class of starlike functions of order $\alpha(0\leq\alpha<1)$ and $C(0,\alpha)=C(\alpha)$, the class of convex functions of order $\alpha(0\leq\alpha<1)$, (see Mostafa [5], Silverman [7]).

Let $F(a,b;c;z)$ be the Gaussian Hypergeometric function defined by

$$F(a,b;c,z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n \tag{1.5}$$

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where $c \neq 0, -1, -2, ...$ and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1 & n = 0 \\ a(a+1).....(a+n-1), & n \in N \end{cases}$$

We note that $F(a,b;c;1)$ converges for $\Re(c-a-b) > 0$ and is related to the Gamma function by

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Silverman [8] and Mosftafa [5] (see also [2]-[4], [6]) gives necessary and sufficient conditions for $zF(a,b;c;z)$ to be in $T^*(\alpha)$ and $C(\alpha)$. Now presented here $I(a,b;c;z)$ by the $z\{(1-\mu)F(a,b;c,z) + \mu[zF(a,b;c,z)]'\}$ where $\mu(0 \le \mu < 1)$.

2. MAIN RESULTS

To establish our main result, we need the following lemma due to Altinats and Owa [1].

Lemma 2.1: A function *f(z)* defined by (1.1) is in the class $T(\lambda,\alpha)$ if and only if

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)a_n \le 1-\alpha. \tag{2.1}$$

Lemma 2.2: A function *f(z)* defined by (1.1) is in $C(\lambda,\alpha)$, if and only if

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)a_n \le 1-\alpha. \tag{2.2}$$

Theorem 2.1: If *a, b>-1* and *ab<0, c>a+b+1* then *zI(a, b; c; z)* where
$I(a,b;c;z) = (1-\mu)F(a,b;c;z) + \mu[zF(a,b;c;z)]'$ is in $T(\lambda,\alpha)$, if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\Big[\{(1-\lambda\alpha)(1-\mu)-\alpha\mu(1-\lambda)\}\left(\frac{a+bc-a-b-1}{ab}\right) + \mu(1-\lambda\alpha)\left\{5+\frac{(a+1)(b+1)}{c-a-b-2}\right\} - \alpha(1-\lambda)(1-\mu)(c-a-b-1)\Big] \le (1-\alpha)\left|\frac{c}{ab}\right| + \alpha(1-\lambda)(1-\mu)\frac{c}{ab}. \tag{2.3}$$

Proof: We have $zI(a,b;c;z) = z + \frac{ab}{c}\sum_{n=2}^{\infty}(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n$

$$= z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n, \tag{2.4}$$

According to Lemma 2.1 we must show that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \le \frac{c}{ab}(1-\alpha). \tag{2.5}$$

Note that the left side of (2.5) diverges if *c<a+b+1*. Now

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$= [(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)]\sum_{n=2}^{\infty}n\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$+\mu(1-\lambda\alpha)\sum_{n=2}^{\infty}n^2\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} - \alpha(1-\mu)(1-\lambda)\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$= \{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left\{\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + \sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}\right\}$$

$$+\mu(1-\lambda\alpha)\sum_{n=0}^{\infty}(n+2)^2\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} - \alpha(1-\mu)(1-\lambda)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left\{F(a+1,b+1;c+1;1)+\frac{c}{ab}F(a,b;c;1)\right\}$$

$$+\mu(1-\lambda\alpha)\left\{4F(a+1,b+1;c+1;1)+\frac{c}{a}\left(\frac{(a)_2(b)_2}{b(c-a-b-2)_2}+\frac{ab}{(c-a-b-1)}\right)F(a,b;c;1)\right\}$$

$$-\alpha(1-\mu)(1-\lambda)\frac{c}{ab}\{F(a,b;c;1)-1\}$$

$$=\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left\{\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+\frac{c}{a}\frac{\Gamma(c)\Gamma(c-a-b)}{b\Gamma(c-a)\Gamma(c-b)}\right\}$$

$$+\mu(1-\lambda\alpha)\left\{\frac{4\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+\frac{c(a+1)(b+1)}{(c-a-b-2)_2}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right.$$

$$\left.+\frac{c}{a}\frac{a\ b}{b(c-a-b-1)}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right\}-\alpha(1-\mu)(1-\lambda)\frac{c}{a}\left\{\frac{\Gamma(c)\Gamma(c-a-b)}{b\Gamma(c-a)\Gamma(c-b)}-1\right\}$$

$$=\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}$$

$$\left[\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left(\frac{ab+c-a-b-1}{ab}\right)+\mu(1-\lambda\alpha)\left\{5+\frac{(a+1)(b+1)}{(c-a-b-2)}\right\}\right]$$

$$-\alpha(1-\mu)(1-\lambda)(c-a-b-1)\le(1-\alpha)\left|\frac{c}{ab}\right|+\alpha(1-\mu)(1-\lambda)\frac{c}{ab}. \qquad (2.6)$$

Remark 1: If we put $\mu=0$ in (2.6) of Theorem 2.1 we obtain the result which is correct to the Mostafa [5].

Corollary 2.1: If $a,b>-1, ab<0$ and $c>a+b+1$ then $zI(a,b;c;z)$ is in $T(\alpha)$, if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[ab+(c-a-1)(1-ab)\right]\ge c.$$

Theorem 2.2: If $a,b>-1, ab<0$ and $\mathrm{c}>\mathrm{a}+\mathrm{b}+1$, then $zI(a,b;c;z)$ where $I(a,b;c;z)=(1-\mu)F(a,b;c;z)+\mu(zF(a,b;c;z))'$ is in $C(\lambda,\alpha)$ if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{\mu(1-\lambda\alpha)}{abc}\{(a)_3(b)_3(c+3)+(a)_2(b)_2(c+2)(c-a-b-2)+7a\ b(c-a-b-2)_2(c+1)\right.$$

$$\left.+(c+1)(c-a-b)_3\}+\{1-\lambda\alpha)(1-\mu)-\alpha\mu(1-\lambda)\}n\left(5+\frac{(a+1)(b+1)}{(c-a-b-2)}\right)-\alpha(1-\mu)(1-\lambda)(c-a-b-1)\right]$$

$$\le(1-\alpha)\left|\frac{c}{ab}\right|+\{(\alpha-\mu)(1-\alpha)-\lambda\alpha\}\frac{c}{ab}.$$

Proof: We have

$$zI(a,b;c;z)=z+\frac{ab}{c}\sum_{n=2}^{\infty}(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n$$

$$=z-\left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n$$

Now according to the Lemma 2.2, we have

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\le\left|\frac{c}{ab}\right|(1-\alpha)$$

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha n-\alpha+\lambda\alpha)(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$= \mu(1-\lambda\alpha)\sum_{n=0}^{\infty}(n+2)^3\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + \{(1-\lambda\alpha)(1-\mu)-\alpha\mu(1-\lambda)\}$$

$$\sum_{n=0}^{\infty}(n+2)^2\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} - \alpha(1-\mu)(1-\lambda)\sum_{n=0}^{\infty}\frac{(n+2)(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

By solving the above calculation we get the result.

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{\mu(1-\lambda\alpha)}{abc}\{(a)_3(b)_3(c+3)+(a)_2(b)_2(c+2)(c-a-b-2)\right.$$

$$+7a\ (b-a-b-2)_2(c+1)+(c+1)(c-a-b)_3\}+\{1-\lambda\alpha)(1-\mu)-\alpha\mu(1-\lambda)\}$$

$$\left.\left(5+\frac{(1+1)(b+1)}{(c-a-b-2)}\right)-\alpha(1-\mu)(1-\lambda)(c-a-b-1)\right]$$

$$\le (1-\alpha)\left|\frac{c}{ab}\right| + (\alpha-\mu)(1-\alpha)-\lambda\alpha)\frac{c}{ab}.$$

3. AN INTEGRAL OPERATOR

In the theorem below, we obtain similar results in connection with a particular integral operator *k(a, b; c; z)* acting on *I(a, b; c; z)* as follows

$$k(a,b;c;z)=\int_0^z I(a,b;c;t)d\,t. \tag{3.1}$$

Theorem 3.1: Let *a, b>-1, ab<0* and *c>a+b+1*. Then *k(a, b; c; z)* defined in (3.1) is in $T(\lambda,\alpha)$, if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left\{\mu(1-\lambda\alpha)\frac{(c-a-b+1)}{ab}+\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left(\frac{c-a-b}{ab}\right)\right.$$

$$\left.+\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\left(\frac{c-a-b}{c}\right)\right\}-(1-\alpha\lambda+\mu-\lambda\mu)\frac{c}{ab}-\left(1+\frac{a}{c}\right)\!b\,\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\le(1-\alpha)\left|\frac{c}{ab}\right|. \tag{3.2}$$

Proof: Since

$$k(a,b;c;z)=z-\left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}(1+\mu-\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}z^n}{(c+1)_{n-2}(1)_n}. \tag{3.3}$$

By Lemma 2.1, we have

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}\le\left|\frac{c}{ab}\right|(1-\alpha).$$

Now

$$\sum_{n=2}^{\infty}n(n-\lambda\alpha-\alpha+\lambda\alpha)(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}$$

$$=\{\mu(1-\lambda\alpha)\}\sum_{n=2}^{\infty}\frac{n^2(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}+\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}$$

$$\sum_{n=2}^{\infty}\frac{n^2(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}+\alpha(1-\mu)\sum_{n=2}^{\infty}n\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}$$

$$=\mu(1-\lambda\alpha)\{F(a+1,b+1;c+1;1)+\frac{c}{ab}F(a,b;c;1)-1\}$$

$$+\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda\}\frac{c}{ab}\{F(a,b;c;1)-1\}+\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\left\{F(a,b;c;1)-1-\frac{ab}{c}\right\}$$

$$=\mu(1-\lambda\alpha)\left\{\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+\frac{\Gamma(c+1)\Gamma c\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right\}$$

$$+\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\left\{\frac{\Gamma c\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1-\frac{ab}{c}\right\}$$

$$=\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\{\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left(\frac{c-a-b}{ab}\right)$$

$$+\mu(1-\lambda\alpha)\left(\frac{(c-a-b+1)}{ab}\right)+\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\left(\frac{c-a-b}{c}\right)\Bigg\}$$

$$-\{1-\alpha\lambda+\mu-\lambda\mu\}\frac{c}{ab}-\left(1+\frac{ab}{c}\right)\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\le(1-\alpha)\left|\frac{c}{ab}\right|.$$

Corollary 3.2: Let *a,b>-1*, *ab<0* and *c>a+b+1* with constant $\mu=0$ then k(a,b;c;z) is in $T(\lambda,\alpha)$, if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left\{(1-\lambda\alpha)\frac{(c-a-b)}{ab}+\alpha(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\left(\frac{c-a-b}{c}\right)\right\}\backslash\backslash$$

$$-\alpha\left(a+\frac{ab}{c}\right)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\le(2-\alpha-\lambda\alpha)\left|\frac{c}{ab}\right|.$$

Proof: By the (3.2) of the above Theorem 3.1 if we put $\mu=0$ then the above result can be find easily.

Corollary 3.3: If *a,b>-1*, *ab<0* and *c>a+b+1*, then k(a, b; c; z) is in $T(\alpha)$,if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left\{1+\alpha\frac{(a)_2(b)_2}{(c)_2}\left(\frac{ab}{c}\right)\right\}-\alpha\left(1+\frac{ab}{c}\right)\frac{(a)_2(b)_2}{(c)_2}ab\ge(2-\alpha)\,|\,c\,|$$

Theorem 3.4: If *a, b>-1*, *ab<0* and *c>a+b+1*, then *k(a, b; c; z)* defined by (3.1) is in $C(\lambda,\alpha)$ iff

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[\mu(1-\lambda\alpha)\left\{\frac{(ab)}{c^2}(c-a-b)+3+\left(\frac{c-a-b}{c}\right)\right\}$$

$$+\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left(1+\frac{c-a-b}{ab}\right)+\alpha(1-\mu)(1-\lambda)\left(\frac{c-a-b}{ab}\right)\Bigg]$$

$$\le\left|\frac{c}{ab}\right|(1-\alpha)+\frac{c}{ab}\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)+\alpha(1-\mu)(1-\lambda)\}.$$

Proof: The proof of above theorem is much akin to that of Theorem 3.1, therefore we omit the details involved.

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