International Journal of Mathematical Archive-9(1), 2018, 191-207
Available online through www.ijma.info ISSN 2229 – 5046

APPROXIMATE CONTROLLABILITY OF SECOND-ORDER NEUTRAL IMPULSIVE EVOLUTION DIFFERENTIAL INCLUSIONS OF SOBOLEV-TYPE WITH NONLOCAL CONDITIONS

M. TAMIL SELVAN*[1] & R. MURUGESU[2]

[1,2]Department of Mathematics,
SRMV College of Arts and Science, Coimbatore - 641 020, Tamil Nadu, India.

(Received On: 06-10-17; Revised & Accepted On: 05-01-18)

ABSTRACT

In this paper, we consider a class of second-order neutral impulsive evolution differential inclusions of Sobolev- type in Hilbert spaces. This paper deals with the approximate controllability for a class of second-order control systems. We establish a set of sufficient conditions for the approximate controllability for a class of second- order neutral impulsive evolution differential inclusions of Sobolev-type in Hilbert spaces by using the Bohnenblust- Karlin's fixed point theorem. Finally, an example is given to illustrate our main results.

Keywords: *Approximate controllability, Second- order differential inclusions of Sobolev-type, Cosine function of operators, Impulsive systems, Evolution equations, Nonlocal conditions.*

2010 Mathematics Subject Classification: *26A33, 34B10, 34K09, 47H10, 93B05.*

1. INTRODUCTION

In this paper, we establish a set of sufficient conditions for the approximate controllability for a class of second- order neutral impulsive evolution differential inclusions with nonlocal conditions in Hilbert spaces of the form

$$\frac{d}{dt}\left[(Ex(t))' - g(t,x(t))\right] \in A(t)x(t) + F(t,x(t)) + Bu(t), \quad t \in I = [0,b], \tag{1.1}$$

$$x(0) = x_0 + p(x),\ x'(0) = y_0 + q(x), \tag{1.2}$$

$$\Delta x(t_i) = I_i(x(t_i)), \tag{1.3}$$

$$\Delta x'(t_i) = J_i(x(t_i)), \quad i = 1,2,\ldots,n. \tag{1.4}$$

In this equation, $A(t): D(A(t)) \subseteq X \to X$ and $E: D(E) \subseteq X \to X$ are closed linear operators on a Hilbert space X and the control function $u(\cdot) \in L^2(I,U)$, a Hilbert space of admissible control functions. Further, B is a bounded linear operator from U to X, $g: I \times X \to X$, $p,q: PC(I,X) \to X$ and $F: I \times X \to 2^X \setminus \{\emptyset\}$ is a nonempty, bounded, closed and convex multivalued map. Similarly, the impulsive moments $\{t_i\}$ are given that $0 = t_1 < t_2 < \cdots < t_n = b$ are fixed numbers and the functions $I_i(\cdot): X \to X$, $J_i(\cdot): X \to X$ defined later in the preliminaries section. The symbol $\Delta\xi(t)$ represents the jump of the function $\xi(\cdot)$ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

Various evolutionary processes from fields such as physics, population dynamics, aeronautics, economics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. The duration of the changes made here are often negligible compared to the total period of time but such changes can be reasonably well approximated in the form of impulses. These process tend to more suitably modeled by impulsive differential equations. For more details on this theory, we refer the monographs of Samoilenko and Perestyuk [32] , Lakshmikantham *et al.* [23], Bainov *et al.* [7] and the references cited therein. The literature on this type of problrms is vast, and different topics on the existence and qualitative properties of solutions are considered. Concerning the general motivations, relevant developments and the current status of the theory, we refer the reader to papers [2, 16, 34, 35, 38, 42, 43].

Corresponding Author: M. Tamil selvan*[1]

There exists an extensive literature of differential equations with nonlocal conditions. Since it is demonstrated that the nonlocal problems have better effects in applications than the classical ones, differential equations with nonlocal problems have been studied extensively in the literature. The result concerning the existence and uniqueness of mild solutions to abstract Cauchy problems with nonlocal initial conditions was first formulated and proved by Byszewski, see [10, 11]. Since the appearance of these papers, several papers have addressed the issue of existence and uniqueness of nonlocal differential equations. Existence and controllability results of nonlinear differential equations and fractional differential equations with nonlocal conditions has been studied by several authors for different kinds of problems [3, 13, 16, 26]. Sobolev-type evolution equations often arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics, and shear in second order fluids. For Sobolev- type differential equations, readers may refer [1, 5, 8, 14]. It is helpful to study the controllability of the second- order abstract differential system directly rather than to convert that into first- order system since the conversion does not give the desired results due to the behavior of the semi group generated by the linear part of the converted first order system.

Since the introduction of controllability concept by Kalman in 1960, it has been one of the fundamental concepts of the mathematical control theory. This concept led to some very important conclusions regarding the linear and nonlinear dynamical systems. Many scientific and engineering problems are nonlinear in nature and so it is necessary to study the controllability results in infinite dimensional spaces. Two basic concepts of controllability can be distinguished which are given as exact and approximate controllability. Exact controllability enables to drive the system to arbitrary final state with a set of admissible control function while approximate controllability means that system can be steered to arbitrary small neighborhood of final state. In other words, approximate controllability gives the possibility of steering the system to states which form the dense subspace in the state space. Various authors extended the controllability concept to infinite dimensional systems and established sufficient conditions for the controllability of nonlinear systems in abstract spaces, see the references [2, 18, 26-28, 31, 39-41, 43, 45]. In the work [26], Vijayakumar *et al.* established a set of sufficient conditions for the approximate controllability for a class of second- order evolution differential inclusions in Hilbert spaces by using Bohnenblust- Karlin's fixed point theorem. In this paper, we extended the work to a class of second-order evolution impulsive neutral differential inclusions of Sobolev-type with nonlocal conditions in Hilbert spaces.

This paper is organized as follows. In Section 2, some necessary definitions and preliminary results to be used in the article are given. In Section 3, a set of sufficient conditions are established for the approximate controllability for a class of Sobolev type second-order neutral impulsive evolution differential inclusions with nonlocal conditions in Hilbert spaces. An example is presented in Section 4 to illustrate the theory of the obtained results.

2. PRELIMINARIES

In this section, we introduce few results, notations and lemmas which are required to establish our main results. The notations which are mentioned here will be used throughout the article without any further references. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Hilbert spaces and $L(Y, X)$ be the Hilbert space of bounded linear operators from Y into X equipped with its natural topology; in particular, when $Y = X$ we use the notation $L(X)$. The resolvent set of a linear operator A is given by $\rho(A)$. Throughout this paper, $B_r(x, X)$ will denote the closed ball with center at x and radius $r > 0$ in a Hilbert space X. We denote by PC, the Hilbert space endowed with supnorm given by $\|x\|_{PC} \equiv \sup_{t\in I} \|x(t)\|$, for $x \in PC$.

The operators $A(t) : D(A(t)) \subseteq X \to X$ and $E : D(E) \subseteq X \to X$ satisfy the following hypothesis.

(S_1) $A(t)$ and E are linear operators, and $A(t)$ is closed;

(S_2) $D(E) \subset D(A(t))$ and E is bijective;

(S_3) $E^{-1} : X \to D(E)$ is compact..

The hypothesis (S_1) - (S_3) and the closed graph theorem imply that the boundedness of the linear operator $AE^{-1} : X \to X$.

In recent times there has been an increasing interest in studying the abstract non-autonomous second order initial value problem

$$x''(t) = A(t)x(t) + f(t), \ 0 \le s, t \le a \tag{2.1}$$

$$x(s) = x_0, \ x'(s) = x_1 \tag{2.2}$$

where $A(t) : D(A(t)) \subseteq X \to X, t \in [0, a]$ is a closely densely defined operator and $f : [0, a] \to X$ is an appropriate function. Equations of this type have been considered in many papers. The reader is referred to [14, 25, 30] and the references mentioned in these works. In the most of works, the existence of solutions to the problem (2.1) – (2.2) is related to the existence of an evolution operator $S(t,s)$ for the homogenous equation

$$x''(t) = A(t)x(t), \quad 0 \le s,t \le a \tag{2.3}$$

Let us assume that the domain of $A(t)$ is a subspace D dense in X and independent of t, and for each $x \in D$ the function $t \to A(t)x$ is continuous.

Following Kozak [22], in this work, we will use the following concept of evolution operator.

Definition 2.1: A family S of bounded linear operators $S(t,s):[0,a]\times[0,a] \to L(X)$ is called an evolution operator for (2.3) if the following conditions are satisfied:

(Z1) For each $x \in X$, the mapping $[0,a]\times[0,a] \ni (t,s) \to S(t,s)x \in X$ is of class C^1 and

(i) *for each* $t \in [0,a]$, $S(t,t) = 0$,

(ii) *for all* $t,s \in [0,a]$, *and for each* $x \in X$,

$$\frac{\partial}{\partial t} S(t,s)x\,|_{t=s} = x, \qquad \frac{\partial}{\partial s} S(t,s)x\,|_{t=s} = -x.$$

(Z2) For all $t,s \in [0,a]$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, the mapping $[0,a]\times[0,a] \ni (t,s) \to S(t,s)x \in X$ is of class C^2 and

(i) $\dfrac{\partial}{\partial t^2} S(t,s)x = A(t)S(t,s)x,$

(ii) $\dfrac{\partial}{\partial s^2} S(t,s)x = S(t,s)A(s)x,$

(iii) $\dfrac{\partial}{\partial s}\dfrac{\partial}{\partial t} S(t,s)x\,|_{t=s} = 0.$

(Z3) For all $t,s \in [0,a]$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, then $\dfrac{\partial}{\partial t^2}\dfrac{\partial}{\partial s} S(t,s)x$, $\dfrac{\partial}{\partial s^2}\dfrac{\partial}{\partial t} S(t,s)x$ and

(i) $\dfrac{\partial}{\partial t^2}\dfrac{\partial}{\partial s} S(t,s)x = A(t)\dfrac{\partial}{\partial s} S(t,s)x,$

(ii) $\dfrac{\partial}{\partial s^2}\dfrac{\partial}{\partial t} S(t,s)x = \dfrac{\partial}{\partial s} S(t,s)A(s)x,$

and the mapping $[0,a]\times[0,a] \ni (t,s) \to A(t)\dfrac{\partial}{\partial s} S(t,s)x$ is continuous.

Throughout the work, we assume that there exists an evolution operator $S(t,s)$ associated to the operator $A(t)$. To abbreviate the text, we introduce the operator $C(t,s) = -\dfrac{\partial S(t,s)}{\partial s}$. In addition, we set N and $\tilde{N}$ for positive constants such that $\sup_{0\le t,s\le a}\|S(t,s)\| \le N$ and $\sup_{0\le t,s\le a}\|C(t,s)\| \le \tilde{N}$. Furthermore, we denote by N_1 a positive constant such that

$$\|S(t+h,s) - S(t,s)\| \le N_1|h|, \tag{2.4}$$

for all $s,t,t+h \in [0,a]$. Assuming that $f:[0,a] \to X$ is an integrable function, the mild solution $x:[0,a] \to X$ of the problem (2.1) – (2.2) is given by

$$x(t) = C(t,s)x_0 + S(t,s)x_1 + \int_0^t S(t,\tau)f(\tau)d\tau.$$

In the literature several techniques have been discussed to establish the existence of the evolution operator $S(t,s)$. In particular, a much studied situation is that $A(t)$ is the perturbation of an operator A that generates a cosine operator function. For this reason, below we briefly review some essential properties of the theory of cosine functions. Let $A: D(A) \subseteq X \to X$ be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C_0(t))_{t\in R}$ on a Hilbert space X. We denote by $(S_0(t))_{t\in R}$ the sine function associated with $(C_0(t))_{t\in R}$ which is defined by

$$S_0(t)x = \int_0^t C_0(s)xds, \quad x \in X, \ t \in R.$$

We refer the reader to [15, 36, 37] for the necessary concepts about cosine functions. Next we only mention a few results and notations about this matter needed to establish our results. The function $S_0(\cdot)$ satisfies

$$S_0(t+s) = S_0(t)C_0(t)S_0(s), \quad s,t \in R. \tag{2.5}$$

The notation $[D(A)]$ stands for the domain of the operator A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|, x \in D(A)$. Moreover, in this paper the notation E stands for the space formed by the vectors $x \in X$ for which $C_0(\cdot)x$ is a function of class C^1. It was proved by Kisynski [21] that the space E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0 \le t \le 1} \|AS_0(t)x\|, \qquad x \in X,$$

is a Hilbert space. The operator valued function

$$G(t) = \begin{bmatrix} C_0(t) & S_0(t) \\ AS_0(t) & C_0(t) \end{bmatrix}$$

is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $A = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from that $AS_0(t)x : E \to X$ is abounded linear operator and that $AS_0(t)x \to 0, t \to 0$, for each $x \in E$. Furthermore, if $x:[0,\infty) \to X$ is a locally integrable function, then $z(t) = \int_0^t S_0(t-s)ds$ defines a $E-$valued continuous function.

The existence of solutions for the second- order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad 0 \le t \le a, \tag{2.6}$$

$$x(s) = x_0, \ x'(s) = x_{1,} \tag{2.7}$$

where $h:[0,a] \to X$, is an integrable function, has been discussed in [36]. Similarly, the existence of solutions of the semi linear second order Cauchy problem it has been treated in [37]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C_0(t-s)x_0 + S_0(t-s)x_1 + \int_s^t S_0(t-\tau)h(\tau)d\tau, \quad 0 \le t \le a, \tag{2.8}$$

is called the mild solution of (2.6) - (2.7), and that when $x_0 \in E, x(\cdot)$ is continuously differentiable and

$$x'(t) = AS_0(t-s)x_0 + C_0(t-s)x_1 + \int_s^t C_0(t-\tau)h(\tau)d\tau, \quad 0 \le t \le a,$$

In addition, if $x_0 \in D(A), x_1 \in E$ and h is a continuously differentiable function, then the function $x(\cdot)$ is a solution of the initial value problem (2.6) - (2.7).

Assume now that $A(t) = A + \tilde{B}(t)$ where $\tilde{B}(\cdot): R \to L(E,X)$ is a map such that the function $t \to \tilde{B}(t)x$ is continuously differentiable in X for each $x \in X$. It has been established by Serizawa [33] that for each $(x_0, x_1) \in D(A) \times E$, the non autonomous abstract Cauchy problem

$$x''(t) = (A + \tilde{B}(t))x(t), \ t \in R, \tag{2.9}$$

$$x(0) = x_0, \ x'(0) = x_{1,} \tag{2.10}$$

has a unique solution $x(\cdot)$ such that the function $t \to x(t)$ is continuously differentiable in E. It is clear that the same argument allows us to conclude that equation (2.9) with the initial condition (2.7) has a unique solution $x(\cdot, s)$ such that the function $t \to x(t)$ is continuously differentiable in E . It follows from (2.8) that

$$x(t,s) = C_0(t-s)x_0 + S_0(t-s)x_1 + \int_s^t S_0(t-\tau)\tilde{B}(\tau)x(\tau,s)d\tau. \tag{2.11}$$

In particular, for $x_0 = 0$, we have

$$x(t,s) = S_0(t-s)x_1 + \int_s^t S_0(t-\tau)\tilde{B}(\tau)x(\tau,s)d\tau. \tag{2.12}$$

Consequently,

$$\|x(t,s)\|_E \le \|S_0(t-s)\|_{L(X,E)}\|x_1\| + \int_s^t \|S_0(t-s)\|_{L(X,E)}\|\tilde{B}(\tau)\|_{L(X,E)}\|x(s,\tau)\|_E d\tau$$

and applying the Gronwall-Bellman lemma, we infer that

$$\|x(t,s)\|_E \le \tilde{M}\|x_1\|,\ s,t \in [0,a].$$

We define the operator $S(t,s)x_1 = x(t,s)$. It follows from the previous estimate that $S(t,s):(E,\|\cdot\|) \to (E,\|\cdot\|_E)$ is a bounded linear map on E. Since $(E,\|\cdot\|)$ is dense in X and $(E,\|\cdot\|)$ is complete, we can extend $S(t,s)$ to X. We keep the notation $S(t,s)$ for this extension. Combining this assertion with (2.12), we obtain that

$$S(t,s)x = S_0(t-s)x + \int_s^t S_0(t-\tau)\tilde{B}(\tau)S(\tau,s)xd\tau,\quad x \in X. \tag{2.13}$$

In similar way, from (2.11) we have

$$C(t,s)x_0 = C_0(t-s)x_0 + \int_s^t S_0(t-\tau)\tilde{B}(\tau)C(\tau,s)x_0 d\tau.$$

for $x_0 \in D(A)$..

Assume that the following condition holds:

(S_4) The operator $S_0(t)\tilde{B}(\tau)$ has an extension $H_0(t,\tau) \in L(X)$ for all $0 \le \tau, t \le a$, and the operator valued map $H_0 : [0,a]\times[0,a] \to L(X)$ is strongly continuous.

In this case, proceeding as above, we can consider $C(t,s)$ defined on X, and

$$C(t,s)x = C_0(t-s)x + \int_s^t H_0(t-\tau,\tau)C(\tau,s)xd\tau,\ x \in X. \tag{2.14}$$

We denote

$$R(t,s)x = \int_s^t H_0(t-\tau,\tau)C(\tau,s)xd\tau,\ x \in X.$$

Theorem 2.2: [18, Theorem 1.2]. Under the preceding conditions, and assuming that condition S_4 holds, then $S(\cdot,\cdot)$ is an evolution operator for (2.9) - (2.10). Moreover, if $S_0(t)$ is compact for all $t \in R$, then $S(t,s)$ is also compact for all $s \le t$.

On the other hand, it is well known that, except in the case $\dim(X) < \infty$, the cosine function $C(t)$ cannot be compact for all $t \in R$. By contrast, for the cosine functions that arise in specific applications, the sine function $S(t)$ is very often a compact operator for all $t \to R$..

Concerning the impulsive conditions in system (1.1)-(1.4), it is convenient to introduce some additional concepts and notations.

A function $u:[\sigma,\tau] \to X$ is said to be a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $PC([\sigma,\tau];X)$ the space of normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, for fixed $a > 0$ and $t_i \in (0,a), i = 1,2,\dots,n,$, we introduce the space PC formed by all normalized piecewise continuous functions $u:[0,a] \to X$ such that u is continuous at $t = t_i, i = 1,2,\dots,n.$ It is clear that PC endowed with the norm $\|u\|_{PC} = \sup_{s\in I}\|u(s)\|$ is a Hilbert space. In what follows, we set $t_0 = 0, t_{n+1} = a,$, and for $u \in PC$, we denote by $\tilde{u}_i \in C([t_i,t_{i+1}];X), i = 0,1,2,\dots n,$, the function given by

$$\tilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t, t_{i+1}], \\ u(t_i^+, & \text{for } t = t_i. \end{cases}$$

Moreover, for a set $B \subseteq PC$, we represented by $\tilde{B}_i, i = 0,1,2,...n,$ the sets $\tilde{B}_i = \{\tilde{u}_i : u \in B\}.$

Lemma 2.3: A set $B \subseteq PC$ is relatively compact in PC if and only if, the set $\tilde{B}_i$, is relatively compact in the space $C([t_i, t_{i+1}]; X)$ for all $i = 0,1,2,...n.$

Throughout this paper, $B_r(x, X)$ will denote the closed ball with center at x and radius $r > 0$ in a Hilbert space X. Some of our results are proved using the next well-known results.

We also introduce some basic definitions and results of multivalued maps. For more details on multivalued maps, see the books of Deimling [12] and Hu and Papageorgious [20].

A multivalued map $G : X \to 2^X \setminus \{\emptyset\}$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(C) = \cup_{x \in C} G(x)$ is bounded in X for any bounded set C of X, i.e., $\sup_{x \in C} \{\sup\{\|y\| : y \in G(x)\}\} < \infty.$

Definition 2.4: G is called upper semicontinuous (u.s.c. for short) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set C of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq C.$.

Definition 2.5: G is called completely continuous if $G(C)$ is relatively compact for every bounded subset C of X.

If the multivalued map G is completely continuous with nonempty values, then G is u.s.c., if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$. G has a fixed point if there is a $x \in X$ such that $x \in G(x).$

Definition 2.6: A function $x \in PC$ is said to be a mild solution of system (1.1)-(1.4) if $x(0) + p(x) = x_0$, $x'(0) + q(x) = y_0$ and there exists $f \in L^1(I, X)$ such that $f(t) \in F(t, x(t))$ on $t \in I$ and the integral equation

$$x(t) = E^{-1}C(t,0)[Ex_0 + Ep(x)] + E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0)] + \int_0^t E^{-1}C(t,s)g(s, x(s))ds + \int_0^t E^{-1}S(t,s)f(s)ds$$
$$+ \int_0^t E^{-1}S(t,s)Bu(s)ds + \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x(t_i)) + \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x(t_i)), \qquad t \in I.$$

is satisfied.

In order to address the problem, it is convenient at this point to introduce two relevant operators and basic assumptions on these operators:

$$\Gamma_0^b = \int_0^b E^{-1}S(b,s)BB^*E^{-1}S^*(b,s)ds : X \to X,$$
$$R(\alpha, \Gamma_0^b) = (\alpha I + \Gamma_0^b)^{-1} : X \to X$$

where B^* denotes the adjoint of B and $S^*(t)$ is the adjoint of $S(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

To investigate the approximate controllability of system (1.1)-(1.2), we impose the following condition:
(H_0) $\alpha R(\alpha, \Gamma_0^b) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.

In view of [28], Hypothesis (H_0) holds if and only if the linear system

$$(Ex(t))'' = Ax(t) + (Bu)(t), \quad t \in [0,b], \tag{2.15}$$

$$x(0) = x_0, \quad x'(0) = y_0, \tag{2.16}$$

is approximately controllable on $[0,b]$.

Some of our results are proved using the next well-known results.

Lemma 2.7: [24, Lasota and Opial] Let I be a compact real interval, $BCC(X)$ be the set of all nonempty, bounded, closed and convex subset of X and F be a multivalued map satisfying $F: I \times X \to BCC(X)$ is measurable to t for each fixed $x \in X$, u.s.c. to x for each $t \in I$, and for each $x \in C$ the set

$$S_{F,x} = \{ f \in L'(I,X) : f(t) \in F(t,x(t)),\ t \in I \}$$

is nonempty. Let F be a linear continuous from $L'(I,X)$ to $C,$ then the operator

$$F \circ S_F : C \to BCC(X), \quad x \to (F \circ S_F)(x) = F(S_{F,x}),$$

is a closed graph operator in $C \times C.$

Lemma 2.8: [9, Bohnenblust and Karlin] Let D be a nonempty subset of X , which is bounded, closed and convex. Suppose $G: D \to 2^X \setminus \{\emptyset\}$ is u.s.c. with closed, convex values and such that $G(D) \subseteq D$ and $G(D)$ is compact. Then G has a fixed point.

3. APPROXIMATE CONTROLLABILITY RESULTS

In this section, we need the following hypotheses to establish a set of sufficient conditions for the approximate controllability for a class of second-order evolution differential inclusions of the form (1.1)-(1.4) in Hilbert spaces by using Bohnenblust-Karlin's fixed point theorem.

(H_1) $S_0(t), t > 0$ is compact.

(H_2) For each positive number r and $x \in PC$ with $\|x\|_{PC} \le r,$ there exists $L_{f,r}(\cdot) \in L^1(I, R^+)$ such that $\sup\{\|f\| : f(t) \in F(t,x(t))\} \le L_{f,r}(t),$ for a.e. $t \in I.$

(H_3) The function $s \to L_{f,r}(s) \in L^1([0,t], R^+)$ and there exists a $\gamma > 0$ such that

$$\lim_{r \to \infty} \frac{\int_0^t L_{f,r}(s)ds}{r} = \gamma < +\infty.$$

(H_4) The function $g: I \times X \to X$ satisfies the following conditions:

(i) The function $g(t,\cdot): X \to X$ is continuous a.e. $t \in I,$

(ii) The function $g(t,\cdot): I \to X$ is strongly measurable for each $x \in X,$

(iii) There exists positive constants $L_g > 0, \tilde{L}_g > 0,$ such that

$$\|g(t,x(t)) - g(t,y(t))\| \le L_g \|x - y\| \qquad \text{for a.e } x, y \in X,\ t \in I,$$

$$\|g(t,x(t))\| \le L_g \|x\| + \tilde{L}_g$$

where $\tilde{L}_g = \max_{t \in I} \|g(t,0)\|.$

(H_5) The function $p, q: PC(I,X) \to X$ are continuous and there exists constants $L_p > 0, L_q > 0$ such that

$$\|p(x) - p(y)\| \le L_p \|x - y\|, \quad \text{for } x, y \in PC,$$

$$\|q(x) - q(y)\| \le L_q \|x - y\|, \quad \text{for } x, y \in PC.$$

(H_6) (i) There are positive constants L_{I_i}, L_{J_i} such that

$$\|I_i(\psi_1) - I_i(\psi_2)\| \le L_{I_i} \|\psi_1 - \psi_2\|,$$

$$\|J_i(\psi_1) - J_i(\psi_2)\| \le L_{J_i} \|\psi_1 - \psi_2\|, \quad \psi_1, \psi_2 \in X, \quad i = 1,2,...n..$$

(ii) The maps $I_i, J_i : X \to X, i = 1,2,...n$ are completely continuous and there exists continuous non-decreasing functions $M_i, \tilde{M}_i : [0,\infty) \to (0,\infty), i = 1,2,...n,$ such that

$$\|I_i(\psi)\| \le M_i(\|\psi\|),$$
$$\|J_i(\psi)\| \le \tilde{M}_i(\|\psi\|), \quad \psi \in X.$$

It will be shown that the system (1.1-1.4) is approximately controllable, if for all $\alpha > 0,$ there exists a continuous function $x(\cdot)$ such that

$$x(t) = E^{-1}C(t,0)[Ex_0 + Ep(x)] + E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] + \int_0^t E^{-1}C(t,s)g(s, x(s))ds$$
$$+\int_0^t E^{-1}S(t,s)f(s)ds + \int_0^t E^{-1}S(t,s)Bu(s, x)ds + \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x(t_i)) + \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x(t_i)), \quad f \in S_{F,x}$$

$$u(t,x) = B^* E^{-1} S^*(b,t)R(\alpha, \Gamma_0^b)p(x(\cdot)),$$

$$p(x(\cdot)) = x_b - E^{-1}C(b,0)[Ex_0 + Ep(x)] - E^{-1}S(b,0)[Ey_0 + Eq(x) - g(0, x(0))] - \int_0^b E^{-1}C(b,s)g(s, x(s))ds$$
$$-\int_0^b E^{-1}S(b,s)f(s)ds - \sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x(t_i)) - \sum_{0<t_i<b} E^{-1}S(b,t_i)J_i(x(t_i)).$$

Theorem 3.1: Suppose that the hypotheses (H_0)-(H_6) are satisfied. Assume also

$$\left[1+\frac{1}{\alpha}\tilde{N}^2 M_B{}^2 \tilde{G}^2 b\right]\left(\tilde{G}N(GL_p + bL_g) + \tilde{G}\tilde{N}(GL_q + L_g + \gamma) + \tilde{G}\sum_{i=1}^n (NL_{I_i} + \tilde{N}L_{J_i})\right) < 1, \tag{3.1}$$

where $M_B = \|B\|,$ then the system (1.1) – (1.4) has a solution on I.

Proof: The main aim in this section is to find conditions for solvability of system (1.1)-(1.4) for $\alpha > 0$. We show that, using the control $u(t, x)$, the operator $\Upsilon : PC \to 2^{PC}$, defined by

$$\Upsilon(x) = \left\{ \begin{array}{l} \varphi \in PC : \varphi(t) = E^{-1}C(t,0)[Ex_0 + Ep(x)] + E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] + \int_0^t E^{-1}C(t,s)g(s, x(s))ds \\ \quad + \int_0^t E^{-1}S(t,s)f(s)ds + \int_0^t E^{-1}S(t,s)Bu(s,x)ds + \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x(t_i)) + \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x(t_i)) \end{array} \right\},$$

has a fixed point $x,$ which is a mild solution of system (1.1)-(1.4).

We now show that Υ satisfies all the conditions of Lemma 2.8. We subdivide the proof into five steps for our convenience.

Step-1: Υ is convex for each $x \in PC$.
In fact, if φ_1, φ_2 belong to $\Upsilon(x),$ then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in I,$ we have

$$\phi_i(t) = E^{-1}C(t,0)[Ex_0 + Ep(x)] + E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] + \int_0^t E^{-1}C(t,s)g(s, x(s))ds$$
$$+\int_0^t E^{-1}S(t,s)f_i(s)ds + \int_0^t E^{-1}S(t,s)BB^*E^{-1}S(b,t)R(\alpha, \Gamma_0^b)[x_b - E^{-1}C(b,0)[x_0 + Ep(x)]$$
$$-E^{-1}S(b,0)[Ey_0 + Eq(x) - g(0, x(0))] - \int_0^b E^{-1}C(b,s)g(s, x(s))ds - \int_0^b E^{-1}S(b,s)f_i(\eta)d\eta$$
$$-\sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x(t_i)) - \sum_{0<t_i<b} E^{-1}S(b,t_i)J_i(x(t_i))](s)ds + \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x(t_i))$$
$$+\sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x(t_i)).$$

Let $\lambda \in [0,1]$. Then for each $t \in I$, we get

$$\lambda\phi_1(t)+(1-\lambda)\phi_2(t)=E^{-1}C(t,0)[Ex_0+Ep(x)]+E^{-1}S(t,0)[Ey_0+Eq(x)-g(0,x(0))]+\int_0^t E^{-1}C(t,s)g(s,\ x(s))ds$$

$$+\int_0^t E^{-1}S(t,s)[\lambda f_1(s)+(1-\lambda)f_2(s)]d\ s+\int_0^t E^{-1}S(t,s)BB^*E^{-1}S(b,t)R(\alpha,\ \Gamma_0^b)[x_b-E^{-1}C(b,0)[x_0+Ep(x)]$$

$$-E^{-1}S(b,0)[Ey_0+Eq(x)-g(0,x(0))]-\int_0^b E^{-1}C(b,s)g(s,\ x(s))ds-\int_0^b E^{-1}S(b,s)[\lambda f_1(\eta)+(1-\lambda)f_2(\eta)]d\eta$$

$$-\sum_{0<t_i<b}E^{-1}C(b,t_i)I_i(x(t_i))-\sum_{0<t_i<b}E^{-1}S(b,t_i)J_i(x(t_i))](s)ds+\sum_{0<t_i<t}E^{-1}C(t,t_i)I_i(x(t_i))$$

$$+\sum_{0<t_i<t}E^{-1}S(t,t_i)J_i(x(t_i)).$$

It is easy to see that $S_{F,x}$ is convex since F has convex values. So, $\lambda f_1+(1-\lambda)f_2\in S_{F,x}$. Thus,

$$\lambda\varphi_1+(1-\lambda)\varphi_2\in \Upsilon(x).$$

Step-2: For each positive number $r>0$, let $B_r=\{x\in PC:\|x\|_{PC}\le r\}$. Obviously, B_r is a bounded, closed and convex set of PC. We claim that there exists a positive number r such that $\Upsilon(B_r)\subseteq B_r$.

If this is not true, then for each positive number r, there exists a function $x^r\in B_r$, but $\Upsilon(x^r)$ does not belong to B_r, i.e.,

$$\left\|\Upsilon(x^r)\right\|_{PC}\equiv\sup\left\{\left\|\varphi^r\right\|_{PC}:\varphi^r\in\Upsilon(x^r)\right\}>r$$

and

$$\varphi^r(t)=E^{-1}C(t,0)[Ex_0+Ep(x)]+E^{-1}S(t,0)[Ey_0+Eq(x)-g(0,x(0))]+\int_0^t E^{-1}C(t,s)g(s,x^r(s))ds$$

$$+\int_0^t E^{-1}S(t,s)f^r(s)ds+\int_0^t E^{-1}S(t,s)Bu^r(s,x)ds+\sum_{i=1}^n E^{-1}C(t,t_i)I_i(x^r(t_i))+\sum_{i=1}^n E^{-1}S(t,t_i)J_i(x^r(t_i)),$$

for some $f^r\in S_{F,x^r}$. Furthermore we take $G=\|E\|$ and $\tilde{G}=\left\|E^{-1}\right\|$ and by using (H[1])-(H[6]), we have

$$r<\left\|\Upsilon(x^r)(t)\right\|$$

$$\le\left\|E^{-1}C(t,0)[Ex_0+Ep(x)]\right\|+\left\|E^{-1}S(t,0)[Ey_0+Eq(x)-g(0,x(0))]\right\|+\int_0^t\left\|E^{-1}C(t,s)g(s,\ x^r(s))\right\|ds$$

$$+\int_0^t\left\|E^{-1}S(t,s)f^r(s)\right\|d\ s+\int_0^t\left\|E^{-1}S(t,s)Bu^r(s,\ x)\right\|d\ s+\sum_{i=1}^n\left\|E^{-1}C(t,t_i)I_i(x^r(t_i))\right\|+\sum_{i=1}^n\left\|E^{-1}S(t,t_i)J_i(x^r(t_i))\right\|$$

$$\le\left[\tilde{G}NG\left(\|x_0\|+L_pr+\|p(0)\|\right)+\tilde{G}\tilde{N}\left(G\|y_0\|+GL_qr+G\|q(0)\|+L_gr+\tilde{L}_g\right)+\tilde{G}N\int_0^t\left(L_gr+\tilde{L}_g\right)ds+\tilde{G}\tilde{N}\int_0^t L_{f,r}(s)ds\right]$$

$$+\frac{1}{\alpha}\tilde{N}^2M_B{}^2\tilde{G}^2b\left[\tilde{G}NG\left(\|x_0\|+L_pr+\|p(0)\|\right)+\tilde{G}\tilde{N}\left(G\|y_0\|+GL_qr+G\|q(0)\|+L_gr+\tilde{L}_g\right)+\tilde{G}N\int_0^t\left(L_gr+\tilde{L}_g\right)ds\right.$$

$$\left.+\tilde{G}\tilde{N}\int_0^t L_{f,r}(s)ds+\tilde{G}N\sum_{i=1}^n\left[\|I_i(x(t_i))-I_i(0)\|+\|I_i(0)\|\right]+\tilde{G}\tilde{N}\sum_{i=1}^n\left[\|J_i(x(t_i))-J_i(0)\|+\|J_i(0)\|\right]\right]$$

$$+\tilde{G}N\sum_{i=1}^n\left[\|I_i(x(t_i))-I_i(0)\|+\|I_i(0)\|\right]\quad+\tilde{G}\tilde{N}\sum_{i=1}^n\left[\|J_i(x(t_i))-J_i(0)\|+\|J_i(0)\|\right].$$

Dividing both sides of the above inequality by r and taking the limit as $r \to \infty$, using (H$_3$), we get

$$\left[1+\frac{1}{\alpha}\tilde{N}^2 M_B{}^2\tilde{G}^2 b\right]\left(\tilde{G}N(GL_p+bL_g)+\tilde{G}\tilde{N}(GL_q+L_g+\gamma)+\tilde{G}\sum_{i=1}^{n}(NL_{I_i}+\tilde{N}L_{J_i})\right)\geq 1,$$

This contradicts with the condition (3.1). Hence, for some $r>0,\ \Upsilon(B_r)\subseteq B_r$.

Step-3: Υ sends bounded sets into equicontinuous sets of PC. For each $x\in B_r,\varphi\in\Upsilon(x)$, there exists a $f\in S_{F,x}$ such that

$$\phi(t)=E^{-1}C(t,0)[Ex_0+Ep(x)]+E^{-1}S(t,0)[Ey_0+Eq(x)-g(0,x(0))]+\int_0^t E^{-1}C(t,s)g(s,\ x(s))ds$$
$$+\int_0^t E^{-1}S(t,s)f(s)ds+\int_0^t E^{-1}S(t,s)Bu(s,\ x)ds+\sum_{0<t_i<t}E^{-1}C(t,t_i)I_i(x(t_i))+\sum_{0<t_i<t}E^{-1}S(t,t_i)J_i(x(t_i)),$$

Let $\varepsilon>0$ and $0<t_1<t_2\leq b$, then

$$\|\phi(t_1)-\phi(t_2)\|=\|E^{-1}\|\|C(t_1,0)-C(t_2,0)\|\|Ex_0+Ep(x)\|+\|E^{-1}\|\|S(t_1,0)-S(t_2,0)\|\|Ey_0+Eq(x)-g(0,x(0))\|$$
$$+\left\|\int_0^{t_1-\varepsilon}E^{-1}\left[C(t_1,s)-C(t_2,s)\right]g(s,\ x(s))ds\right\|+\left\|\int_{t_1-\varepsilon}^{t_1}E^{-1}\left[C(t_1,s)-C(t_2,s)\right]g(s,\ x(s))ds\right\|$$
$$+\left\|\int_{t_1}^{t_2}E^{-1}C(t_2,s)g(s,\ x(s))ds\right\|+\left\|\int_0^{t_1-\varepsilon}E^{-1}\left[S(t_1,s)-S(t_2,s)\right]f(s)ds\right\|$$
$$+\left\|\int_{t_1-\varepsilon}^{t_1}E^{-1}\left[S(t_1,s)-S(t_2,s)\right]f(s)ds\right\|+\left\|\int_{t_1}^{t_2}E^{-1}S(t_2,s)f(s)ds\right\|$$
$$+\left\|\int_0^{t_1-\varepsilon}E^{-1}\left[S(t_1,\eta)-S(t_2,\eta)\right]Bu(\eta,x)d\eta\right\|+\left\|\int_{t_1-\varepsilon}^{t_1}E^{-1}\left[S(t_1,\eta)-S(t_2,\eta)\right]Bu(\eta,x)d\eta\right\|$$
$$+\left\|\int_1^{t_2}E^{-1}S(t_2,\eta)B(u,\eta)d\eta\right\|+\left\|\sum_{0<t_i<t}E^{-1}\left[C(t_2,t_i)-C(t_1,t_i)\right]I_i(x(t_i))\right\|+\left\|\sum_{t_1<t_i<t_2}E^{-1}C(t_2,t_i)I_i(x(t_i))\right\|$$
$$+\left\|\sum_{0<t_i<t}E^{-1}\left[S(t_2,t_i)-S(t_1,t_i)\right]J_i(x(t_i))\right\|+\left\|\sum_{t_1<t_i<t_2}E^{-1}S(t_2,t_i)J_i(x(t_i))\right\|$$

$$\leq\tilde{G}\|C(t_1,0)-C(t_2,0)\|\left(G\|x_0\|+GL_pr+G\|p(0)\|\right)+\tilde{G}\|S(t_1,0)-S(t_2,0)\|\left(G\|y_0\|+GL_qr+G\|q_0\|+L_gr+\tilde{L}_g\right)$$
$$+\tilde{G}\int_0^{t_1-\varepsilon}\|C(t_1,s)-C(t_2,s)\|\left(L_gr+\tilde{L}_g\right)ds+\tilde{G}\int_{t_1-\varepsilon}^{t_1}\|C(t_1,s)-C(t_2,s)\|\ \left(L_gr+\tilde{L}_g\right)ds+\tilde{G}N\int_{t_1}^{t_2}\left(L_gr+\tilde{L}_g\right)ds$$
$$+\tilde{G}\int_0^{t_1-\varepsilon}\|S(t_1,s)-S(t_2,s)\|\ L_{f,r}(s)ds+\tilde{G}\int_{t_1-\varepsilon}^{t_1}\|S(t_1,s)-S(t_2,s)\|\ L_{f,r}(s)ds+\tilde{G}\tilde{N}\int_{t_1}^{t_2}L_{f,r}(s)ds$$
$$+\tilde{G}M_B\int_0^{t_1-\varepsilon}\|S(t_1,\eta)-S(t_2,\eta)\|\ \|u(\eta,x)\|d\eta+\tilde{G}M_B\int_{t_1-\varepsilon}^{t_1}\|S(t_1,\eta)-S(t_2,\eta)\|\ \|u(\eta,x)\|d\eta$$
$$+\tilde{G}\tilde{N}M_B\int_{t_1}^{t_2}\|u(\eta,x)\|d\eta+\tilde{G}\sum_{0<t_i<t}\|C(t_2,t_i)-C(t_1,t_i)\|\ M_i\|x_{t_i}\|+\tilde{G}N\sum_{t_1<t_i<t_2}M_i\|x_{t_i}\|$$
$$+\tilde{G}\sum_{0<t_i<t}\|S(t_2,t_i)-S(t_1,t_i)\|\ \tilde{M}_i\|x_{t_i}\|+\tilde{G}\tilde{N}\sum_{t_1<t_i<t_2}\tilde{M}_i\|x_{t_i}\|$$

The right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $(t_1 - t_2) \to 0$ and ε sufficiently small, since the compactness of the evolution operator $S(t,s)$ implies the continuity in the uniform operator topology. Thus $\Upsilon(x^r)$ sends B_r into equicontinuous family of functions.

Step-4: The set $\Pi(t) = \{\varphi(t) : \varphi \in \Upsilon(B_r)\}$ is relatively compact in X. Let $t \in (0,b]$ be fixed and ε a real number satisfying $0 < \varepsilon < t.$ for $x \in B_r$, we define

$$\phi_\varepsilon(t) = E^{-1}C(t,0)[Ex_0 + Ep(x)] + E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] + \int_0^{t-\varepsilon} E^{-1}C(t,s)g(s, x(s))ds$$

$$+ \int_0^{t-\varepsilon} E^{-1}S(t,s)f(s)ds + \int_0^{t-\varepsilon} E^{-1}S(t,s)Bu(s, x)ds + \sum_{0<t_i<t-\varepsilon} E^{-1}C(t,t_i)I_i(x(t_i)) + \sum_{0<t_i<t-\varepsilon} E^{-1}S(t,t_i)J_i(x(t_i))$$

Since $S(t)$ is a compact operator, the set $\Pi_\varepsilon(t) = \{\varphi_\varepsilon(t) : \varphi_\varepsilon \in \Upsilon(B_r)\}$ is relatively compact in X for each $\varepsilon, 0 < \varepsilon < t.$ Moreover, for each $0 < \varepsilon < t,$ we have

$$\left|\phi(t) - \phi_\varepsilon(t)\right| \le \tilde{G}\tilde{N}\int_{t-\varepsilon}^{t} \left(L_g r + \tilde{L}_g\right)ds + \tilde{G}\tilde{N}\int_{t-\varepsilon}^{t} L_{f,r}(s)ds + \tilde{G}\tilde{N}M_B\int_{t-\varepsilon}^{t} \|u(\eta, x)\| d\eta + \tilde{G}N\sum_{t-\varepsilon<t_i<t} M_i \left\|(x_{t_i})\right\|$$

$$+ \tilde{G}N\sum_{t-\varepsilon<t_i<t} M_i \left\|(x_{t_i})\right\|$$

Hence there exist relatively compact sets arbitrarily close to the set $\Pi(t) = \{\varphi(t) : \varphi \in \Upsilon(B_r)\}$ and the set $\Pi(t)$ is relatively compact in X for all $t \in (0,b].$ Since it is compact at $= 0,$ hence $\Pi(t)$ is relatively compact in X for all $t \in [0,b].$

Step-5: Υ has a closed graph. Let $x_n \to x_*$ as $n \to \infty, \varphi_n \in \Upsilon(x_n),$ and $\varphi_n \to \varphi_*$ as $n \to \infty.$ we will show that $\varphi_* \in \Upsilon(x_*).$ Since $\varphi_n \in \Upsilon(x_n),$ there exists a $f_n \in S_{F,x_n}$ such that

$$\phi_n(t) = E^{-1}C(t,0)[Ex_0 + Ep(x)] + E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] + \int_0^t E^{-1}C(t,s)g(s, x_n(s))ds$$

$$+ \int_0^t E^{-1}S(t,s)f_n(s)ds + \int_0^t E^{-1}S(t,s)BB^*E^{-1}S(b,t)R(\alpha, \Gamma_0^b)[x_b - E^{-1}C(b,0)[x_0 + Ep(x)]$$

$$- E^{-1}S(b,0)[Ey_0 + Eq(x) - g(0, x(0))] - \int_0^b E^{-1}C(b,s)g(s, x_n(s))ds - \int_0^b E^{-1}S(b,s)f_n(\eta)d\eta$$

$$- \sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x_n(t_i)) - \sum_{0<t_i<b} E^{-1}S(b,t_i)J_i(x_n(t_i))](s)ds + \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x_n(t_i))$$

$$+ \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x_n(t_i)).$$

We must prove that there exists a $f_* \in S_{F,x_*}$ such that

$$\phi_*(t) = E^{-1}C(t,0)[Ex_0 + Ep(x)] + E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] + \int_0^t E^{-1}C(t,s)g(s, x_*(s))ds$$

$$+ \int_0^t E^{-1}S(t,s)f_*(s)ds + \int_0^t E^{-1}S(t,s)BB^*E^{-1}S(b,t)R(\alpha, \Gamma_0^b)[x_b - E^{-1}C(b,0)[x_0 + Ep(x)]$$

$$- E^{-1}S(b,0)[Ey_0 + Eq(x) - g(0, x(0))] - \int_0^b E^{-1}C(b,s)g(s, x_*(s))ds - \int_0^b E^{-1}S(b,s)f_*(\eta)d\eta$$

$$- \sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x_*(t_i)) - \sum_{0<t_i<b} E^{-1}S(b,t_i)I_i(x_*(t_i))](s)ds + \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x_*(t_i))$$

$$+ \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x_*(t_i)).$$

Clearly we have

$$\Bigg\| \Bigg(\phi_n(t) - E^{-1}C(t,0)[Ex_0 + Ep(x)] - E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] + \int_0^t E^{-1}C(t,s)g(s,\ x_n(s))ds$$

$$- \int_0^t E^{-1}S(t,s)BB^* E^{-1}S(b,t)R(\alpha,\ \Gamma_0^b)[x_b - E^{-1}C(b,0)[x_0 + Ep(x)] - E^{-1}S(b,0)[Ey_0 + Eq(x) - g(0, x(0))]$$

$$-\int_0^b E^{-1}C(b,s)g(s,\ x_n(s))ds - \int_0^b E^{-1}S(b,s) f_n(\eta)d\eta \ - \sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x_n(t_i))$$

$$- \sum_{0<t_i<b} E^{-1}S(b,t_i)I_i(x_n(t_i))](s)ds - \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x_n(t_i)) - \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x_n(t_i)) \Bigg)$$

$$-\Bigg(\phi_*(t) - E^{-1}C(t,0)[Ex_0 + Ep(x)] - E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] - \int_0^t E^{-1}C(t,s)g(s,\ x_*(s))ds$$

$$- \int_0^t E^{-1}S(t,\eta)BB^* E^{-1}S(b,t)R(\alpha,\ \Gamma_0^b)[x_b - E^{-1}C(b,0)[x_0 + Ep(x)] - E^{-1}S(b,0)[Ey_0 + Eq(x) - g(0, x(0))]$$

$$-\int_0^b E^{-1}C(b,s)g(s,\ x_*(s))ds - \int_0^b E^{-1}S(b,s) f_*(\eta)d\eta \ - \sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x_*(t_i))$$

$$- \sum_{0<t_i<b} E^{-1}S(b,t_i)I_i(x_*(t_i))](s)ds - \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x_*(t_i)) - \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x_*(t_i)) \Bigg) \Bigg\|_{PC}$$

$$\to 0 \text{ as } n \to \infty.$$

Consider the operator $\tilde{W} : L^1(I, X) \to PC$,

$$(\tilde{W}f)(t) = \int_0^t E^{-1}S(t,s) f(s)ds - \int_0^t E^{-1}S(t,s)BB^* E^{-1}S^*(b,t)R(\alpha,\Gamma_0^b)\left(\int_0^b S(b,\eta) f(\eta)d\eta\right)(s)ds$$

We can see that the operator $\tilde{W}$ is linear and continuous. From Lemma 2.8 again, it follows that $\tilde{W} \circ S_F$ is a closed graph operator.

Moreover,

$$\Bigg(\phi_n(t) - E^{-1}C(t,0)[Ex_0 + Ep(x)] - E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] + \int_0^t E^{-1}C(t,s)g(s,\ x_n(s))ds$$

$$- \int_0^t E^{-1}S(t,s)BB^* E^{-1}S(b,t)R(\alpha,\ \Gamma_0^b)[x_b - E^{-1}C(b,0)[x_0 + Ep(x)] - E^{-1}S(b,0)[Ey_0 + Eq(x) - g(0, x(0))]$$

$$-\int_0^b E^{-1}C(b,s)g(s,\ x_n(s))ds - \int_0^b E^{-1}S(b,s) f_n(\eta)d\eta \ - \sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x_n(t_i))$$

$$- \sum_{0<t_i<b} E^{-1}S(b,t_i)J_i(x_n(t_i))](s)ds - \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x_n(t_i)) - \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x_n(t_i)) \Bigg) \in \tilde{W}(S_{F,x_n}).$$

In view of $x_n \to x_*$ as $n \to \infty$, it follows again from Lemma 2.8 that

$$\Bigg(\phi_*(t) - E^{-1}C(t,0)[Ex_0 + Ep(x)] - E^{-1}S(t,0)[Ey_0 + Eq(x) - g(0, x(0))] - \int_0^t E^{-1}C(t,s)g(s, x_*(s))ds$$

$$- \int_0^t E^{-1}S(t,\eta)BB^*E^{-1}S(b,t)R(\alpha, \Gamma_0^b)[x_b - E^{-1}C(b,0)[x_0 + Ep(x)] - E^{-1}S(b,0)[Ey_0 + Ey(x) - g(0, x(0))]$$

$$- \int_0^b E^{-1}C(b,s)g(s, x_*(s))d s - \int_0^b E^{-1}S(b,s) f_*(\eta)d\eta - \sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x_*(t_i))$$

$$- \sum_{0<t_i<b} E^{-1}S(b,t_i)I_i(x_*(t_i))](s)d s - \sum_{0<t_i<t} E^{-1}C(t,t_i)I_i(x_*(t_i)) - \sum_{0<t_i<t} E^{-1}S(t,t_i)J_i(x_*(t_i))\Bigg) \in \tilde{W}(S_{F,x_*}).$$

Therefore Y has a closed graph. As a consequence of Steps 1-5 together with the Arzela-Ascoli theorem, we conclude that Y is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 2.8, we can deduce that Y has a fixed point x which is a mild solution of system (1.1)-(1.4).

Definition 3.2: The control system (1.1) - (1.4) is said to be approximately controllable on I if $\overline{R(b, x_0)} = \{x_b(x_0; u) : u \in L^1(I, U)\}$ is called the reachable set of system (1.1) - (1.4) at terminal time b and its closure in X is denoted by $\overline{R(b, x_0)}$; Let $x_b(x_0, u)$ be the state value of (1.1) –(1.4) at terminal time b corresponding to the control u and the initial value $x_0 \in X$.

Frankly speaking, by using the control function u, from any given initial point $x_0 \in X$ we can move the system with the trajectory as close as possible to any other final point $x_b \in X$.

Theorem 3.3: Suppose that the assumptions (S_1)-(S_3) and (H_0)-(H_6) hold. Assume additionally that $g(t,\cdot)$ is continuous from the weak topology of X to the strong topology of X and (H_b) there exists $N \in L^1(I, [0,\infty))$ such that $\sup_{x\in X} \|F(t, x)\| + \sup_{x\in X} \|g(t, x)\| \le N(t)$ for a.e. $t \in I$, then the system (1.1)-(1.4) is approximately controllable on I.

Proof: Let $\hat{x}^\alpha(\cdot)$ be a fixed point of Γ in B_r. By theorem 3.1, any fixed point of Γ is a mild solution of (1.1) – (1.4) under the control

$$\hat{u}^\alpha(t) = B^* E^{-1} S^*(b,t) R(\alpha, \Gamma_0^b) p(\hat{x}^\alpha)$$

and satisfies the following inequality

$$\hat{x}^\alpha(b) = x_b + \alpha R(\alpha, \Gamma_0^b) p(\hat{x}^\alpha). \tag{3.2}$$

By the condition (H_b),

$$\int_0^b \left\| f(s, \hat{x}^\alpha(s)) \right\|^2 ds + \int_0^b \left\| g(s, \hat{x}^\alpha(s)) \right\|^2 ds \le \int_0^b N^2(s) ds$$

Moreover by assumption on F and Dunford-Pettis Theorem, we have that the $\{f^\alpha(s)\}$ is weakly compact in $L^1(I, X)$, so there is a sub-sequence, still denoted by $\{f^\alpha(s)\}$ that converges weakly to say $f(s)$ in $L^1(I, X)$. Define

$$w = x_b - E^{-1}C(b,0)[Ex_0 + Ep(x)] - E^{-1}S(b,0)[Ey_0 + Eq(x) - g(0, x(0))] - \int_0^b E^{-1}C(b,s)g(s, x(s))ds$$

$$- \int_0^b E^{-1}S(b,s) f(s)d s - \sum_{0<t_i<b} E^{-1}C(b,t_i)I_i(x(t_i)) - \sum_{0<t_i<b} E^{-1}S(b,t_i)J_i(x(t_i)).$$

Now, we have

$$\left\| p(x^\alpha) - w \right\| = \left\| \int_0^b E^{-1}C(b,s)\left[g(s, \hat{x}^\alpha(s)) - g(s) \right] ds \right\| + \left\| \int_0^b E^{-1}S(b,s)\left[f(s, \hat{x}^\alpha(s)) - f(s) \right] ds \right\|$$

$$\le \sup_{t\in I} \left\| \int_0^t E^{-1}C(t,s)\left[g(s, \hat{x}^\alpha(s)) - g(s) \right] ds \right\| + \sup_{t\in I} \left\| \int_0^t E^{-1}S(t,s)\left[f(s, \hat{x}^\alpha(s)) - f(s) \right] ds \right\| \tag{3.3}$$

By using the infinite-dimensional version of the Ascoli- Arzela theorem, one can show that an operator $l(\cdot) \to \int_0^{\cdot} S(\cdot,s)l(s)ds : L^1(I,X) \to PC$ is compact. Therefore, we obtain that $\left\|p(x^{\alpha})-w\right\| \to 0$ as $\alpha \to 0^+$. Moreover, from (3.2) we get

$$\begin{aligned}\left\|\hat{x}^{\alpha}(b)-x_b\right\| &\le \left\|\alpha R(\alpha,\Gamma_0^b)(w)\right\| + \left\|\alpha R(\alpha,\Gamma_0^b)\right\|\left\|p(x^{\alpha})-w\right\| \\ &\le \left\|\alpha R(\alpha,\Gamma_0^b)(w)\right\| + \left\|p(x^{\alpha})-w\right\|\end{aligned}$$

It follows from the assumption (H_0) and the estimation (3.3) that $\left\|\hat{x}^{\alpha}(b)-x_b\right\| \to 0$ as $\alpha \to 0^+$. This proves the approximate controllability of system (1.1) – (1.4).

4. APPLICATION

In this section, we apply our abstract results to a concrete partial differential equation. In order to establish our results, we need to introduce the required technical tools. Following the equations (2.9) – (2.10), here we consider $A(t) = A + \tilde{B}(t)$ where A is the infinitesimal generator of a cosine function is $C(t)$ with associated sine function $S(t)$, and $\tilde{B}(t) : D(\tilde{B}(t)) \to X$ is a closed linear operator with $D \subseteq D(\tilde{B}(t))$ for all $t \in I$. We model this problem in the space $X = L^2(T,C)$, where the group T is defined as the quotient $R/2\pi Z$. We will use the identification between functions on T and 2π - periodic 2-integrable functions from R into C. Similarly, $H^2(T,C)$ denotes the Sobolev space of 2π - periodic functions $x : R \to C$ such that $x'' \in L^2(T,C)$.

We consider the operator $Ax(\xi) = x''(\xi)$ with domain $D(A) = H^2(T,C)$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function $C(t)$ on X. Moreover, A has discrete spectrum, the spectrum of A consists of eigenvalues $-n^2$ for $n \in Z$, with associated eigenvectors

$$w_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}, \quad n \in Z,$$

the set $\{w_n : n \in Z\}$ is an orthonormal basis of X. In particular, define $A(t) : D(A(t)) \subset X \to X$ by $A(t) : z_{\theta\theta}$ and $E : D(E) \subset X \to X$ by $Ez = z - z_{\theta\theta}$ where A and E can be written as

$$Ax = -\sum_{n=1}^{\infty} n^2 \langle z, w_n \rangle w_n \qquad \text{for } z \in D(A)$$

$$Ex = \sum_{n=1}^{\infty} (1+n^2) \langle z, w_n \rangle w_n \quad \text{for } z \in D(E)$$

The cosine function $C(t)$ is given by

$$C(t)x = \sum_{i=1}^{\infty} \cos(nt) \langle z, w_n \rangle w_n, \quad t \in R,$$

With associated sine function

$$S(t)x = \sum_{i=1}^{\infty} \frac{\sin(nt)}{n} \langle z, w_n \rangle w_n, \quad t \in R.$$

It is clear that $\left\|C(t)\right\| \le 1$ for all $t \in R$. Thus, $C(\cdot)$ is uniformly bounded on R.

Consider the second-order Cauchy problem with control

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} z(t,\theta) - z_{\theta\theta}(t,\theta) - G(t,z(t,\theta))\right] \in z_{\theta\theta}(t,\theta) + \frac{\partial^2}{\partial \theta^2} z(t,\theta) + b(t)\frac{\partial}{\partial t} z(t,\theta) + \mu(t,\theta) + Q(t,z(t,\theta)) \tag{4.1}$$

for $t \in I, 0 \le \tau \le \pi$, subject to the initial conditions

$$z(t,0) = z(t,\pi) = 0, \quad t \in I, \tag{4.2}$$

$$z(0,\tau) = z_0(\tau) + \sum_{i=1}^{n} \alpha_i z(t_i,\tau), \tag{4.3}$$

$$\frac{\partial}{\partial t}z(0,\tau)=z_1(\tau)+\sum_{i=1}^{n}\beta_i z(s_i,z),\quad 0\le\tau\le\pi, \tag{4.4}$$

$$\Delta z(t_i)(\tau)=\int_{-\infty}^{t_i}a_i(t_i-s)z(s,\tau)ds,\quad i=1,2,\ldots n, \tag{4.5}$$

$$\Delta z'(t_i)(\tau)=\int_{-\infty}^{t_i}\tilde{a}_i(t_i-s)z(s,\tau)ds,\quad i=1,2,\ldots n, \tag{4.6}$$

where $z_0, z_1\in X$, $a_i,\tilde{a}_i\in PC(R,R)$ and $0<t_i,s_i<\pi$, α_i,β_i are prefixed numbers, $b:[0,\infty)\to R$, $G,F:I\times X\times X\to X$ are continuous functions. We fix $\alpha>0$ and set $\beta=\sup_{0\le t\le a}|b(t)|$.

We take $\tilde{B}(t)x(\tau)=b(t)x'(\tau)$ defined on $H^1(T,C)$. It is easy to see that $A(t)=A+\tilde{B}(t)$ is a closed linear operator. Initially we will show that $A+\tilde{B}(t)$ generates an evolution operator. It is well known that the solution of the scalar initial value problem

$$q''(t)=-n^2q(t)+p(t),$$
$$q(s)=0,\qquad q'(s)=q_1,$$

is given by

$$q(t)=\frac{q_1}{n}\sin n(t-s)+\frac{1}{n}\int_s^t\sin n(t-\tau)p(\tau)d\tau.$$

Therefore, the solution of the scalar initial value problem

$$q''(t)=-n^2q(t)+in\ b(t)p(t), \tag{4.7}$$

$$q(s)=0,\ q'(s)=q_1, \tag{4.8}$$

satisfies the integral equation

$$q(t)=\frac{q_1}{n}\sin n(t-s)+i\int_s^t\sin n(t-\tau)b(\tau)q(\tau)d\tau.$$

Applying the Gronwall – Bellman lemma we can affirm that

$$|q(t)|\le\frac{|q_1|}{n}e^{\beta(t-s)} \tag{4.9}$$

for $s\le t$. We denote by $q_n(t,s)$ the solution of (4.7) – (4.8). We define

$$S(t,s)x=\sum_{n=1}^{\infty}q_n(t,s)\langle z,\ w_n\rangle w_n.$$

It follows from the estimate (4.9) that $S(t,s)x: X\to X$ is well-defined and satisfies the conditions of Definition 2.1. Put $z(t)=z(t,\cdot)$, that is $z(t)(\tau)=z(t,\tau), t\in I, \tau\in[0,\pi]$ and $u(t)=\mu(t,\cdot)$, here $\mu: I\times[0,\pi]\to[0,\pi]$ is continuous. Let us define $f: I\times X\to X$ and $g: I\times X\to X$ as

$$F(t,z)(\tau)=Q(t,z(\tau)),\qquad g(t,z)(\tau)=G(t,z(\tau)),\qquad z\in X,\qquad \tau\in[0,\pi],$$

$$p(x)(\tau)=\sum_{i=1}^{n}\alpha_i x(t_i,\tau),\qquad q(x)(\tau)=\sum_{i=1}^{n}\beta_i x(t_i,\tau),$$

$$I_i(x)(\tau)=\int_{-\infty}^{0}a_i(s)x(s,\tau)ds,\qquad J_i(x)(\tau)=\int_{-\infty}^{0}\tilde{a}_i(s)x(s,\tau)ds.$$

Furthermore for $z\in X$, we have,

$$E^{-1}z=\sum_{n=1}^{\infty}\frac{1}{1+n^2}\langle z,w_n\rangle w_n,$$

$$AE^{-1}z=\sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}\langle z,w_n\rangle w_n$$

and the bounded linear operator $B:U\to X$ by

$$Bu(t)(\tau)=\mu(t,\tau),\quad t\in I,\quad \tau\in[0,\pi].$$

Assume these functions satisfy the requirement of hypothesis. From the above choices of the functions and evolution operator $A(t)$ with $B = I$, the system (4.1) – (4.6) can be formulated as the system (1.1)-(1.2) in X. Since all the hypothesis of Theorem 3.3 are satisfied, approximate controllability of system (4.1) – (4.6) on I follows from Theorem 3.3.

REFERENCES

1. Agarwal, S and Bahuguna, D, Existence of solutions to Sobolev-type partial neutral diff erential equations, *Journal of Applied Mathematics and Stochastic Analysis*, 2006 (2006), 1-10. Article ID 16308.
2. Abada, N, Benchohra, M and Hammouche, H, Existence and controllability results for impulsive partial functional differential inclusions, Nonlinear Analysis, 69 (2008), 2892-2909.
3. Aizicovici, S and Gao, Y, Functional Differential Equations with nonlocal initial conditions, Journal of Applied Mathematics and Stochastic Analysis, 10(2) (1997), 145-156.
4. Bainov, D.D and Simeonov, P.S, Impulsive Differential Equations: Periodic Solutions and Applications, Longman Scientific and Technical Group, England, 1993.
5. Balachandran, K and Kiruthika, S, Existence of solutions of abstract fractional integrodiff erential equations of Sobolev type, Computers & Mathematics with Applications, 64 (2012), 3406-3413.
6. Benchohra, M, Henderson, J and Ntouyas, S.K, Impulsive neutral functional differential inclusions in Banach spaces, Applied Mathematics Letters, 15(8) (2002), 917-924.
7. Benchohra, M, Henderson, J and Ntouyas, S.K, Impulsive Differential Equations and Inclusions, Contemporary Mathematics and Its Applications, Hindawi Publishing Corporation, New York, 2006.
8. Brill, H, A semilinear Sobolev evolution equation in Banach space, Journal of Differential Equations, 24 (1977), 412-425.
9. Bohnenblust, H.F and Karlin, S, On a Theorem of Ville, in: Contributions to the Theory of Games, Vol. I, Princeton University Press, Princeton, NJ, 1950, pp. 155-160.
10. Byszewski, L, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, Journal of Mathematical Analysis and Applications, 162 (1991), 494-505.
11. Byszewski, L and Akca, H, On a mild solution of a semilinear functional differential evolution nonlocal problem, Journal of Applied Mathematics and Stochastic Analysis, 10(3) (1997), 265-271.
12. Deimling, K, Multivalued Differential Equations, De Gruyter, Berlin, 1992.
13. Ezzinbi, K, Fu, X and Hilal, K, Existence and regularity in the α -norm for some neutral partial differential equations with nonlocal conditions, Nonlinear Analysis, 67 (2007), 1613-1622.
14. Faraci, F and Iannizzotto, A, A multiplicity theorem for a perturbed second-order non-autonomous system, Proceedings of the Edinburgh Mathematical Society, 49 (2006), 267-275.
15. Fattorini, H.O, Second Order Linear Differential Equations in Banach Spaces, North-Holland Mathematical Studies, 108, Amsterdam: North-Holland, 1985.
16. Fu, X and Ezzinbi, K, Existence of solutions for neutral functional differential evolution equations with nonlocal conditions, Nonlinear Analysis, 54 (2003), 215-227.
17. Grans, A and Dugundji, J, Fixed Point Theory, Springer-Verlag, New York, 2003.
18. Henriquez, H.R, M. Eduardo Hernandez, Approximate controllability of second-order distributed implicit functional systems, Nonlinear Analysis, 70, 2009, 1023-1039.
19. Henriquez , H.R, Existence of solutions of non-autonomous second order functional diff erential equations with infinite delay, Nonlinear Analysis: TMA, 74 (2011), 3333-3352.
20. Hu, S and Papageorgiou, N.S, Handbook of Multivalued Analysis (Theory), Kluwer Academic Publishers, Dordrecht Boston, London, 1997.
21. Kisynski, J, On cosine operator functions and one parameter group of operators, Studia Mathematica, 49 (1972), 93-105.
22. Kozak, M, A fundamental solution of a second order differential equation in a Banach space, Universitatis Lagellonicae Acta Mathematica, 32 (1995), 275-289.
23. Lakshmikantham, V, Bainov, D and Simenov, P.S, Theory of impulsive differential equations, Series in Modern Applied Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.
24. Lasota, A and Opial, Z, An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations or noncompact acyclic-valued map, Bulletin L'Academie Polonaise des Science, Serie des Sciences Mathematiques, Astronomiqueset Physiques, 13 (1965), 781-786.
25. Lin, Y, Time-dependent perturbation theory for abstract evolution equations of second order, Studia Mathematica, 130 (1998), 263-274.
26. Mahmudov, N.I, Vijayakumar, V and Murugesu, R, Approximate controllability of second-order evolution differential inclusions in Hilbert spaces, Mediterranean Journal of Mathematics, 13 (5), (2016), 3433-3454.
27. Mahmudov, N.I, Murugesu, R, Ravichandran, C and Vijayakumar, V, Approximate controllability results for fractional semilinear integro-differential inclusions in Hilbert spaces, Results in Mathematics, 71(1), (2017), 45-61.

28. Mahmudov, N.I and Denker, A, On controllability of linear stochastic systems, Inter-national Journal of Control, 73, (2000), 144-151.
29. Martin, R.H, Nonlinear Operators and Differential Equations in Banach Spaces, Robert E Krieger Publishing Co, Florida, 1987.
30. Obrecht, E, Evolution operators for higher order abstract parabolic equations, Czechoslovak Mathematical Journal, 36 (1986), 210-222.
31. Sakthivel, R, Suganya, S, Anthoni, S.M, Approximate controllability of fractional stochastic evolution equations, Computers & Mathematics with Applications, 63(2012), 660-668.
32. Samoilenko, A.M and Perestyuk, N.A, Impulsive Differential Equations, World Scientific, Singapore, 1995.
33. Serizawa, H and Watanabe, M, Time-dependent perturbation for cosine families in Banach spaces, Houston Journal of Mathematics, 12 (1986), 579-586.
34. Sivasankaran, S, Mallika Arjunan, M and Vijayakumar, V, Existence of global solutions for second order impulsive abstract partial differential equations, Nonlinear Analysis: TMA, 74(17), (2011), 6747-6757.
35. Sivasankaran, S, Mallika Arjunan, M, Vijayakumar, V, Existence of global solutions for impulsive functional differential equations with nonlocal conditions, Journal of Nonlinear Sciences and Applications, 4(2) (2011), 102-114.
36. Travis, C.C and Webb, G.F, Compactness, regularity, and uniform continuity properties of strongly continuous cosine families, Houston Journal of Mathematics, 3(4) (1977), 555-567.
37. Travis, C.C and Webb, G.F, Cosine families and abstract nonlinear second order dif-ferential equations, Acta Mathematica Academiae Scientiarum Hungaricae, 32 (1978), 76-96.
38. Vijayakumar, V, Sivasankaran, S and Mallika Arjunan, M, Existence of global solutions for second order impulsive abstract functional integrodifferential equations, Dynamics of Continuous Discrete and Impulsive Systems: Mathematical Analysis, 18 (2011), 747-766.
39. Vijayakumar, V, Ravichandran, C and Murugesu, R, Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay, Nonlinear studies, 20(4) (2013), 511-530.
40. Vijayakumar, V, Approximate controllability results for abstract neutral integro-differential inclusions with infinite delay in Hilbert spaces, IMA Journal of Mathematical Control and Information, (2016), 1-18.
41. Vijayakumar, V, Approximate controllability results for analytic resolvent integro-di fferential inclusions in Hilbert spaces, International Journal of Control, (2017), 1-11.
42. Vijayakumar, V, Approximate controllability results for impulsive neutral differential inclusions of Sobolev-type with infinite delay, International Journal of Control, (2017), 1-20.
43. Vijayakumar, V, Murugesu, R, Poongodi, R and Dhanalakshmi, S, Controllability of second order impulsive nonlocal Cauchy problem via measure of noncompactness, Mediterranean Journal of Mathematics, 14(1), (2017), 29-51.
44. Wang, J, Feckan, M and Zhou, Y, Controllability of Sobolev-type fractional evolution systems, Dynamics of Partial Differential Equations, 11 (2014), 71-87.
45. Zhou, Y, Vijayakumar, V and Murugesu, R, Controllability for fractional evolution inclusions without compactness, Evolution Equations and Control Theory, 4(4) (2015), 507-524.

Source of support: Nil, Conflict of interest: None Declared.