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CYCLIC COUPLED FIXED POINT THEOREM IN COMPLEX VALUED METRIC SPACES

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ABSTRACT

In this paper we prove a strong cyclic coupled fixed point theorem for Kannan type contractions in complex valued metric spaces.

MSC: *47H10; 54H25.*

Keywords: *fixed point; contractive mapping; complex valued metric space.*

INTRODUCTION

There are many generalizations of metric spaces such as partial metric spaces, generalized metric spaces, cone metric spaces, and quasi metric spaces. Recently, Azam *et al.* [1] obtained the generalization of Banach's contraction principle by introducing the concept of complex valued metric space and established some common fixed point theorems for mappings involving rational expressions which are not meaningful in cone metric spaces. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \precsim \mathrm{Re}(z_2), \mathrm{Im}(z_1) \precsim \mathrm{Im}(z_2)$.

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

$$\text{C1. } \mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2). \quad \text{C2. } \mathrm{Re}(z_1) < Re(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2).$$
$$\text{C3. } \mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) < Im(z_2). \quad \text{C4. } \mathrm{Re}(z_1) < Re(z_2), \mathrm{Im}(z_1) < Im(z_2).$$

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Definition 1.1: (Azam *et al.* [2011]) let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d: X \times X \to C$, satisfies the following conditions:

1. $0 \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preccurlyeq d(x, z) + d(z, y)$, for all $x, y, z \in X$. Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 1.2: (Azam *et al.* [2011]) Let (X, d) be a complex valued metric space. (i) A point $x \in X$ is called interior point of set $A \subseteq X$, whenever there exist $0 \prec r \in \mathbb{C}$ such that $B(x, r) := \{y \in X | d(x, y) \prec r\} \subseteq A$ where B(x, r) is an open Ball. (ii) A point $x \in X$ is called a limit of A whenever for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - \{x\}) \neq \emptyset$. (iii) A subset $A \subseteq X$ is called open whenever each element A is an interior point of A. (iv) A sub set $A \subseteq X$ is called closed whenever each limit point of A belongs to A. (v) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x, r) \mid x \in X$ and $0 \prec r\}$.

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Definition 1.3: (Azam *et al.* [2011]) Let (X, d) be a complex valued metric space,$\{x_n\}$ be a sequence in X and $x \in X$. (i) If for every$c \in C$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$, we denote this by $\lim_{n\to\infty} x_n = x$ (or) $\{x_n\} \to x$ as $n \to \infty$. (ii) If for every $c \in C$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence. (iii) If for every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued metric space.

Example 1.4: Let $X = \mathbb{C}$. Define the mapping $d: X \times X \to \mathbb{C}$ by $d(z_1, z_2) = 3i \mid z_1 - z_2 \mid$ for all $z_1, z_2 \in X$, then (X, d) is a complex valued metric space.

Lemma 1.5: (Azam *et al.* [2011]) Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\mid d(x_n, x) \mid \to 0$, as $n \to \infty$.

Lemma 1.6: (Azam *et al.* [2011]) Let (X, d) be a complex valued metric space and, let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\mid d(x_n, x_{n+m}) \mid \to 0$, as $n \to \infty$, where $m \in \mathbb{N}$.

In this paper, we establish a strong coupled fixed point result by using cyclic coupled Kannan type contractions in complex valued metric spaces. A mapping $f: X \to X$, where (X, d) is a complex valued metric space, is called a Kannan type mapping [2], [3] if

$$d(fx, fy) \leq k[d(x, fx) + d(y, fy)], \text{ for some } 0 < k < \frac{1}{2} \tag{1}$$

.

Let A and B be two nonempty subsets of a set X. A mapping $f: X \to X$, is cyclic (with respect to A and B) if $f(A) \subseteq B$ and $f(B) \subseteq A$. The fixed point theory of cyclic contractive mappings has a recent origin. Kirk *et al.* [4] in 2003 initiated this line of research. Cyclic contractive mappings are mappings of which the contraction condition is only satisfied between any two point's x and y with $x \in A$ and $y \in B$. The above notion of cyclic mapping is extended to the cases of mappings from $X \times X$ to X in the following definition.

Definition 1.7: Let A and B be two nonempty subsets of a given set X. We call any function $F: X \times X \to X$ such that $F(x, y) \in B$ if $x \in A$ and $y \in B$ and $F(x, y) \in A$ if $x \in B$ and $y \in A$ a cyclic mapping with respect to A and B.

Definition 1.8 [5]: An element $(x, y) \in X \times X$, where X is any nonempty set, is called a coupled fixed point of the mapping $F: X \times X \to X$ *if* $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.9: We call the coupled fixed point in the above definition to be strong coupled fixed point if $x = y$, that is, if $(x, x) = x$. Combining the above concepts we define a cyclic coupled Kannan type contraction.

Definition 1.10 [cyclic coupled Kannan type contraction]: Let A and B be two nonempty subsets of a complex valued metric space (X, d). We call a mapping $F: X \times X \to X$ a cyclic coupled Kannan type contraction with respect to A and B if F is cyclic with respect to A and B satisfying, for some $k \in (0, 1/2)$, the inequality

$$d\big(F(x, y), F(u, v)\big) \precsim k\left[d\big(x, F(x, y)\big) + d\big(u, F(u, v)\big)\right] \tag{2}$$

where $x, v \in A, y, u \in B$.

Theorem 1.11: Let A and B be two nonempty closed subsets of a complete complex valued metric space (X, d). Let $F: X \times X \to X$ be a cyclic coupled Kannan type contraction with respect to A and B and $A \cap B \neq \phi$. Then F has a strong coupled fixed point in $A \cap B$.

Proof: Let $x_0 \in A$ and $y_0 \in B$ be any two elements and let the sequences $\{x_n\}$ *and* $\{y_n\}$ be defined as

$$x_{n+1} = F(y_n, x_n), y_{n+1} = F(x_n, y_n) \quad \forall n \geq 0 \tag{3}$$

Then, for all $n \geq 0, x_n \in A$ and $y_n \in B$. by (2),

We have

$$d(x_1, y_1) = d\big(x_1, F(x_1, y_1)\big) = d\big(F(y_0, x_0), F(x_1, y_1)\big) \tag{4}$$

$$\precsim k\big[d\big(y_0, F(y_0, x_0)\big) + d\big(x_1, F(x_1, y_1)\big)\big] = k[d(y_0, x_1) + d(x_1, y_2)]$$

which implies that

$$d(x_1, y_2) \precsim \nabla d(y_0, x_1) \tag{5}$$

Where $0 < \nabla = \frac{k}{1-k} < 1$ (6)

$$d(y_1, x_2) = d\big(y_1, F(y_1, x_1)\big) = d\big(F(x_0, y_0), F(y_1, x_1)\big) \tag{7}$$

$$\precsim k\big[d\big(x_0, F(x_0, y_0)\big) + d\big(y_1, F(y_1, x_1)\big)\big] = k[d(x_0, y_1) + d(y_1, x_2)],$$

which implies that

$$d(y_1, x_2) \precsim \nabla d(x_0, y_1) \tag{8}$$

where ∇ is the same as in (6)

Again, by (2), we have

$$d(x_2, y_3) = d\big(x_2, F(x_2, y_2)\big) = d\big(F(y_1, x_1), F(x_2, y_2)\big) \tag{9}$$

$$\precsim k\big[d\big(y_1, F(y_1, x_1)\big) + d\big(x_2, F(x_2, y_2)\big)\big] = k[d(y_1, x_2) + d(x_2, y_3)],$$

$$d(x_2, y_3) = \nabla\, d(y_1, x_2) \tag{10}$$

which, by (8), implies that $d(x_2, y_3) \precsim \nabla^2 d(x_0, y_1)$ (11)
where ∇ is the same as in (6). Similarly, by (2), we have

$$d(y_2, x_3) = d\big(y_2, F(y_2, x_2)\big) = d\big(F(x_1, y_1), F(y_2, x_2)\big) \tag{12}$$

$$\precsim k\big[d\big(x_1, F(x_1, y_1)\big) + d\big(y_2, F(y_2, x_2)\big)\big] = k[d(x_1, y_2) + d(y_2, x_3)],$$

$$d(y_2, x_3) = \nabla\, d(x_1, y_2) \tag{13}$$

which implies that $d(y_2, x_3) \precsim \nabla^2 d(y_0, x_1)$ (14)
where ∇ is the same as in (6).

Also, by (2), we have

$$d(x_3, y_4) = d\big(x_3, F(x_3, y_3)\big) = d\big(F(y_2, x_2), F(x_3, y_3)\big) \tag{15}$$

$$\precsim k\big[d\big(y_2, F(y_2, x_2)\big) + d\big(x_3, F(x_3, y_3)\big)\big] = k[d(y_2, x_3) + d(x_3, y_4)],$$

Implies that $d(x_3, y_4) \precsim \nabla\, d(y_2, x_3) = \nabla^3 d(y_0, x_1)$ ... (16). where ∇ is the same as in (6).

Similarly, by (2), we have

$$d(y_3, x_4) = d\big(y_3, F(y_3, x_3)\big) = d\big(F(x_2, y_2), F(y_3, x_3)\big) \tag{17}$$

$$\precsim k\big[d\big(x_2, F(x_2, y_2)\big) + d\big(y_3, F(y_3, x_3)\big)\big] = k[d(x_2, y_3) + d(y_3, x_4)],$$

$$d(y_3, x_4) = \nabla\, d(x_2, y_3) = \nabla^3 d(x_0, y_1) \tag{18}$$

Where ∇ is the same as in (6). Let m be any integer. We assume

$$d(x_n, y_{n+1}) = d(x_n, F(x_n, y_n)) \precsim \nabla^{\mathrm{n}}\, d(y_0, x_1) \tag{19}$$

$$d(y_n, x_{n+1}) = d(y_n, F(y_n, x_n)) \precsim \nabla^{\mathrm{n}}\, d(x_0, y_1) \tag{20}$$

for all $n \precsim m$ where n is odd and

$$d(x_n, y_{n+1}) = d(x_n, F(x_n, y_n)) \precsim \nabla^{\mathrm{n}}\, d(x_0, y_1) \tag{21}$$

$$d(y_n, x_{n+1}) = d(y_n, F(y_n, x_n)) \precsim \nabla^{\mathrm{n}}\, d(y_0, x_1) \tag{22}$$

for all $n \precsim m$ where n is even. Let m be even. Then, by (2) and (3), we have

$$d(x_{m+1}, y_{m+2}) = d\big(x_{m+1}, F(x_{m+1}, y_{m+1})\big) = d\big(F(y_m, x_m), F(x_{m+1}, y_{m+1})\big) \tag{23}$$

$$\precsim k[d(y_m, x_{m+1}) + d(x_{m+1}, y_{m+2})]$$

or by (22) we have $d(x_{m+1}, y_{m+2}) \precsim \frac{\mathrm{k}}{1-\mathrm{k}}[\nabla^{\mathrm{m}} d(y_0,\ x_1)] = \nabla^{\mathrm{m}+1}\, d(y_0, x_1)$ (24)

Similarly, by (2) and (3), we have

$$d(y_{m+1}, x_{m+2}) = d\big(y_{m+1}, F(y_{m+1}, x_{m+1})\big) = d\big(F(x_m, y_m), F(y_{m+1}, x_{m+1})\big) \tag{25}$$

$$\precsim k[d(x_m, y_{m+1}) + d(y_{m+1}, x_{m+2})]$$

or by (21) we have $d(y_{m+1}, x_{m+2}) \precsim \frac{\mathrm{k}}{1-\mathrm{k}}[\nabla^{\mathrm{m}} d(x_0, y_1)] = \nabla^{\mathrm{m}+1}\, d(x_0, y_1)$ (26)

Again, let m be odd. Then by (2) and (3), we have

$$d(x_{m+1}, y_{m+2}) = d\big(x_{m+1}, F(x_{m+1}, y_{m+1})\big) = d\big(F(y_m, x_m), F(x_{m+1}, y_{m+1})\big) \tag{27}$$

$$\precsim k[d(y_m, x_{m+1}) + d(x_{m+1}, y_{m+2})]$$

or by (20), we have $d(x_{m+1}, y_{m+2}) \precsim \frac{\mathrm{k}}{1-\mathrm{k}}[\nabla^{\mathrm{m}} d(x_0, y_1)] = \nabla^{\mathrm{m}+1} d(\ x_0, y_1)$ (28)

Similarly, by (2) and (3), we have

$$d(y_{m+1}, x_{m+2}) = d\big(y_{m+1}, F(y_{m+1}, x_{m+1})\big) = d\big(F(x_m, y_m), F(y_{m+1}, x_{m+1})\big) \tag{29}$$

$\precsim k[d(x_m, y_{m+1}) + d(y_{m+1}, x_{m+2})]$ Or by (19), we have

$$d(y_{m+1}, x_{m+2}) \precsim \frac{\mathrm{k}}{1-\mathrm{k}}[\nabla^{\mathrm{m}} d(y_0, x_1)] = \nabla^{\mathrm{m}+1}\, d(y_0, x_1) \tag{30}$$

Thus (19)–(22) hold for m + 1. But we have shown in (5), (8)–(18) that this is valid for m = 1, 2, 3. Then, by induction, we conclude that (19)–(22) are valid for all m. From the above we conclude that, for all odd integer n, we have

$$d(x_n, y_{n+1}) = d(x_n, F(x_n, y_n)) \precsim \nabla^n\, d(y_0, x_1)$$
$$d(y_n, x_{n+1}) = d(y_n, F(y_n, x_n)) \precsim \nabla^n\, d(x_0, y_1) \quad (31)$$

And for all even integer n, we have

$$d(x_n, y_{n+1}) = d(x_n, F(x_n, y_n)) \precsim \nabla^n\, d(x_0, y_1)$$
$$d(y_n, x_{n+1}) = d(y_n, F(y_n, x_n)) \precsim \nabla^n\, d(y_0, x_1) \quad (32).$$

By (2), we have

$$d(x_1, y_1) = d\big(F(y_0, x_0), F(x_0, y_0)\big) \precsim k[d\big(y_0, F(y_0, x_0)\big) + d\big(x_0, F(x_0, y_0)\big)]$$
$$= \frac{\nabla}{\nabla+1}[d(y_0, x_1) + d(x_0, y_1)] \quad (33)$$

Again, by (2), we have

$$d(x_2, y_2) = d\big(F(y_1, x_1), F(x_1, y_1)\big) \precsim k[d\big(y_1, F(y_1, x_1)\big) + d\big(x_1, F(x_1, y_1)\big)]$$
$$= k[d(y_1, x_2) + d(x_1, y_2)] \quad (34)$$
$$\precsim k\begin{bmatrix} d(x_0, y_1) + d(y_0, x_1) \\ (by\ (5)\ and\ (8)) \end{bmatrix} = \frac{\nabla^2}{\nabla+1}[d(y_0, x_1) + d(x_0, y_1)]$$

Let, for some integer m, $d(x_m, y_m) = \frac{\mathrm{t}^{\mathrm{m}}}{t+1}[d(y_0, x_1) + d(x_0, y_1)]$ (35)

Let m be odd. Then, by (2) and (3), we have

$$d(x_{m+1}, y_{m+1}) = d\big(F(y_m, x_m), F(x_m, y_m)\big)$$
$$\precsim k[d(y_m, x_{m+1}) + d(x_m, y_{m+1})] \quad (36)$$
$$\precsim k[\nabla^{\mathrm{m}}[d(x_0, y_1) + d(y_0, x_1)]]\ (by\ (31))$$
$$= \frac{\nabla^{\mathrm{m}+1}}{\nabla + 1}[d(x_0, y_1) + d(y_0, x_1)] \quad (by\ (6)$$

Again, let m be even. Then, by (2) and (3), we have

$$d(x_{m+1}, y_{m+1}) = d\big(F(y_m, x_m), F(x_m, y_m)\big) \precsim k[d(y_m, x_{m+1}) + d(x_m, y_{m+1})] \quad (37)$$
$$\precsim k[\nabla^{\mathrm{m}}[d(y_0, x_1) + d(x_0, y_1)]]\ (by\ (31))$$
$$= \frac{\nabla^{\mathrm{m}+1}}{\nabla + 1}[d(y_0, x_1) + d(x_0, y_1)] \quad (by\ (6))$$

Therefore, (35) also holds if we replace m by m + 1. But, (33) and (34) imply that (35) is true for m = 1, 2. Then, by induction, for all n, it follows that

$$d(x_n, y_n) = \frac{\nabla^{\mathrm{n}}}{\nabla+1}[d(x_0, y_1) + d(y_0, x_1)] \quad (38)$$

Now, by (31) – (32) and (38), we have

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \precsim d(x_n, y_n) + d(y_n, x_{n+1}) + d(y_n, x_n) + d(x_n, y_{n+1})$$
$$= [\, d(x_n, y_n) + d(y_n, x_n)] + [\, d(y_n, x_{n+1}) + d(x_n, y_{n+1})\,]$$
$$\precsim \frac{2\nabla^{\mathrm{n}}}{\nabla + 1}[\, d(x_0, y_1) + d(y_0, x_1)] + \mathrm{t}^{\mathrm{n}}[\, d(x_0, y_1) + d(y_0, x_1)] \quad (39)$$

Since $0 < t < 1$, it follows that $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \precsim \mathbb{C}$. This implies that $\{x_n\}\, and\, \{y_n\}$ are Cauchy sequences and hence are convergent. Since A and B are closed subsets $\{x_n\} \subset A$ and $\{y_n\} \subset B$, it follows that

$$x_n \longrightarrow x \in A,\ y_n \longrightarrow y \in B\ as\ n \longrightarrow \infty \quad (40)$$

Again, from (38), $d\big(x_{n,}y_n\big) \longrightarrow 0\ as\ n \longrightarrow \infty$. Therefore, from (38), x = y (41)

Since $A \cap B \neq \emptyset$, then from the above it follows that $x \in A \cap B$. Now, by (2) and (3), we have

$$d(x, F(x, y)) \precsim d(x, x_{n+1}) + d\big(x_{n+1}, F(x, y)\big) = d(x, x_{n+1}) + d(F(y_n, x_n), F(x, y)) \quad (42)$$
$$\precsim d(x, x_{n+1}) + k[d(y_n, x_{n+1}) + d\big(x, F(x, y)\big)] \quad \text{or}$$
$$d\big(x, F(x, y)\big) \precsim \frac{1}{1-k} d(x, x_{n+1}) + \frac{k}{1-k} d(y_n, x_{n+1}) \quad (43)$$

Taking the limit as $n \longrightarrow \infty$ in the above inequality, using (40) and (41), we obtain $d\big(x, F(x, y)\big) = 0$. Again, in view of (41), we conclude that $x = F(x, x)$; that is, we have a strong coupled fixed point of F.
This completes the proof of the theorem.

Example 1.12: Let $X = \{0, 1, 2\}$

$d(0,1) = d(1,0) = i$, $d(1,2) = d(2,1) = 2i$

$d(0,0) = d(1,1) = d(2,2) = 0$

$d(2,0) = d(0,2) = 3i$

$A = \{0,1\}$, $B = \{1,2\}$

Define $F: X \times X \to X$, F is cyclic with respect to $A\ and\ B$

$$F(\alpha,\beta) = \begin{cases} 1, & (\alpha,\beta) \in A \times B \\ 1, & (\alpha,\beta) \in B \times A \\ 0, & otherwise \end{cases}$$

$$d\big(F(x,y), F(u,v)\big) = d(1,1) = 0$$

$$d\big(F(x, F(x,y)) + d(u, F(u,v)\big) = d(x,1) + d(u,1) = \{2i, i, 3i, 0, 4i\}$$

$Let\ k = 1/3$

Then $F(1,1) = 1$.

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