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THEORETICAL ANALYSIS OF FLOW OF NANOFLUID OVER A STRETCHING SURFACE

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ABSTRACT

This is the first paper for the application of homotopy analysis method (HAM) in boundary layer flow of nanofluids on stretching sheet. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis. The system of nonlinear equations is solved using HAM. An analytical solution is presented which depends on the Prandtl number Pr, Lewis number Le, Brownian motion parameter Nb and thermophoresis parameter Nt. The variation of the reduced Nusselt and reduced Sherwood numbers with Nb and Nt is also presented in tabular forms. The effects of various parameters on the stream function, temperature distribution and volume fraction of nanofluid are also discussed.

Keywords: *Mathematical modeling, Boundary layer, Nanofluid, Stretching sheet, Brownian motion, Thermophoresis, Homotopy analysis method.*

1. INTRODUCTION

The nanofluid past a stretching sheet is an important problem in many engineering processes with applications in industries such as extrusion, melt-spinning, the hot rolling, wire drawing, glass-fiber production, manufacturing of plastic and rubber sheets, cooling of a large metallic plate in a bath, which may be an electrolyte, etc [1]. Nanofluid flows are also very useful in many applications in heat transfer, fuel cells, pharmaceutical process and hybrid-power engine, domestic refrigerator, chiller, heat exchange in grinding machine and boiler flow gas temperature reduction. Various flow of a nanofluid past a stretching sheet work has been done, as is evident from Table 1.

Crane [2] discussed two-dimensional incompressible boundary layer flow of a Newtonian fluid caused by the stretching of an elastic flat sheet. Crane obtained an exact solution of this two-dimensional Navier–Stokes equations. Nanofluid is first utilized by Choi [3]. Some numerical and experimental studies on nanofluids include thermal conductivity [4] and convective heat transfer [4–9]. Buongiorno[10] and Kakaç and Pramuanjaroenkij[11] made a comprehensive survey of convective transport in nanofluids.

Khan and Pop [16] have used the model of Kuznetsoz and Nield[17] to study the fundamental work on the boundary layer flow of nanofluid over a stretching sheet. Makinde and Aziz [18] extended the work of Khan and Pop [16] for convective boundary conditions. Ma *et al.* [19] have developed a hybrid approach that combines the Lattice Boltzmann model for fluid with a Brownian dynamics model for the nanoparticles.

Wang [20] discussed the partial slip effects on the planer stretching flow. of late, Noghrehabadi *et al.*[21]investigated the development of the slip effects on the boundary layer flow and heat transfer over a stretching sheet. Surana *et al.* [22] have used K-version finite element method to solve the viscoelastic flow through the parallel plates.

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Makinde and Aziz [18] conducted a numerical study of boundary layer flow of a nanofluid past a stretching sheet with convective boundary condition. Mustafa *et al.* [23] investigated stagnation point flow of a nanofluid towards a stretching sheet. Khan and Pop [16]. studied the boundary layer flow of a nanofluid past a stretching sheet with a constant surface temperature. Recently, Noghrehabadi *et al.* [21] investigated the effect of partial slip condition on the flow and heat transfer of nanofluids past stretching sheet, with prescribed constant wall temperature. This problem is solved by using Runge-Kutta-Fehlberg scheme with shooting method. Bidin and Nazar [24], Ishak[25] and Nadeem *et al.*, [26] numerically examined the flow and heat transfer over an exponentially stretching surface with thermal radiation. Elbashbeshy [27] numerically examined the flow and heat transfer over an exponentially stretching surface considering wall mass suction.

The heat and mass transfer problem with laminar flow of the nanofluids over a stretching surface has been studied [17].To the best of our knowledge, no rigorous analytical expressions for the stream function, temperature distribution and volume fraction of nanofluid for all values of parameters have been previously reported. In this manuscript, the approximate analytical expressions for the stream function, temperature distribution and volume fraction of nanofluid using homotopy analysis method is presented for the first time.

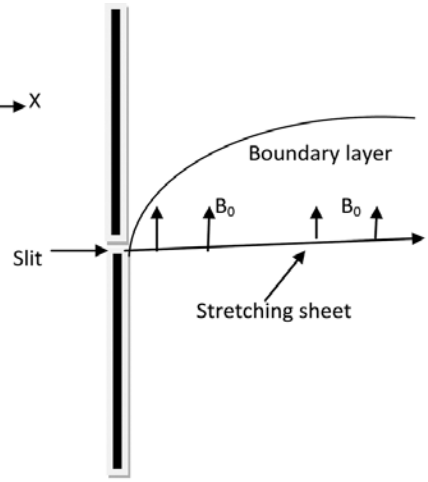

Figure-1: physical model and coordinate system [28].

2. MATHEMATICAL FORMULATION OF THE PROBLEM

The basic equation for steady two-dimensional boundary layer of a nanofluid past a stretching surface is given in ref [16]. The dimensionless, nonlinear coupled ordinary differential equation in boundary layer flow of nanofluid is given below [16]:

$$f''' + f\,f'' - f'^2 = 0 \tag{1}$$

$$\frac{1}{Pr}\theta'' + f\,\theta' + Nb\,\phi'\,\theta' + Nt\,\theta'^2 = 0 \tag{2}$$

$$\phi'' + Le\,\phi' + \frac{Nt}{Nb}\theta'' = 0 \tag{3}$$

where, f, θ *and* ϕ represents, the stream function, temperature distribution and volume fraction of nanofluid respectively. The boundary conditions are,

$$f(0) = 0,\quad f'(0) = 1,\quad f'(\infty) = 0,\quad \theta(0) = 1,\ \theta(\infty) = 0,\ \phi(0) = 1,\ \phi(\infty) = 0 \tag{4}$$

The four dimensionless parameters are defined by

$$Pr = \frac{\nu}{\alpha},\quad Le = \frac{\upsilon}{D_B},\quad Nb = \frac{(\rho c)_p\ D_B(\phi_w - \phi_\infty)}{(\rho c)_f\,\nu},\quad Nt = \frac{(\rho c)_p\ D_T(T_w - T_\infty)}{(\rho c)_f\ T_\infty \nu} \tag{5}$$

Here Pr, Le, Nb and Nt denote the Prandtl number, the Lewis number, the Brownian motion parameter and the thermophoresis parameter, respectively. It is important to note that this boundary value problem reduces to the classical problem of flow and heat and mass transfer due to a stretching surface in a viscous fluid when Nb and Nt are zero in Eqs. (2) and (3).The reduced Nusselt number and reduced Sherwood number are calculated by the following equations,

$$\text{Reduced Nusselt number} \quad \theta' = \mathrm{Re}_x^{-1/2}\ Nu = -\ (0)$$

$$\text{Reduced Sherwood number} = \mathrm{Re}_x^{-1/2}\ Sh = -\phi'(0) \tag{6}$$

where, $\mathrm{Re}_x = u_w(x)x/\nu$ is the local Reynolds number based on the stretching velocity $u_w(x)$.

3. THE APPROXIMATE ANALYTICAL EXPRESSION OF STREAM FUNCTION, TEMPERATURE AND NANOPARTICLE VOLUME FRACTION USING HAM

The homotopy analysis method is a semi-analytical technique to solve nonlinear problems. This method was introduced by Liao [12]. Unlike other analytic techniques, this method is independent of small/large physical parameters. The HAM provides a simple way to guarantee the convergence of solution series in minimum number of iterations. The basic concept of this method is given in Appendix-A. By solving the nonlinear equations (1) to (4) using the HAM, the approximate analytical expression of the stream function, temperature distribution and volume fraction of nanofluid can be obtained as follows:

$$f(\eta) = 1 - e^{-\eta} \tag{8}$$

$$\begin{aligned}\theta(\eta) = {} & A_1\, e^{-\Pr\eta} + A_2\, e^{(-2\Pr\eta)} + A_3\, e^{(-3\Pr\eta)} + A_4 e^{(-(1+\Pr)\eta)} + A_5 e^{-(Le+\Pr)\eta} \\ & + A_6 e^{-(1+\Pr+Le)\eta} + A_7 e^{-(Le+2\Pr)\eta} + A_8\, e^{-(2Le+\Pr)\eta} + A_9 e^{-(1+2\Pr)\eta} \\ & + A_{10} e^{-(2+\Pr)\eta}\end{aligned} \tag{9}$$

$$\begin{aligned}\varphi(\eta) = {} & B_1 e^{-Le\eta} + B_2 e^{-(1+Le)\eta} + B_3 e^{-(2+Le)\eta} + B_4 e^{-(\Pr+Le)\eta} + B_5 e^{-\Pr\eta} \\ & + B_6 e^{-2\Pr\eta} + B_7 e^{-(1+\Pr)\eta}\end{aligned} \tag{10}$$

where the constant A_i, and B_i are given by (C26) and (C30).The temperature distribution function using two iterations become

$$\begin{aligned}\theta(\eta) = \theta_0 + \theta_1 = {} & e^{(-\Pr\eta)} - h\left(\frac{\Pr^2}{1+\Pr} + \frac{\Pr^2 Nb}{Le+\Pr} + \frac{\Pr Nt}{2}\right)e^{(-\Pr\eta)} + h\left(\frac{\Pr^2}{1+\Pr}e^{(-(1+\Pr)\eta)}\right. \\ & \left. + \frac{\Pr^2 Nb}{Le+\Pr}e^{(-(\Pr+Le)\eta)} + \frac{\Pr Nt}{2}e^{(-2\Pr\eta)}\right)\end{aligned} \tag{11}$$

The derivative of temperature distribution becomes

$$\begin{aligned}\theta'(\eta) = {} & -\Pr\, e^{(-\Pr\eta)} + h\left(\frac{\Pr^2}{1+\Pr} + \frac{\Pr^2 Nb}{Le+\Pr} + \frac{\Pr Nt}{2}\right)\Pr\, e^{(-\Pr\eta)} + \\ & h\left(\frac{\Pr^2(-1-\Pr)\ e^{(-(1+\Pr)\eta)}}{1+\Pr} + \frac{\Pr^2 Nb(-\Pr-Le)\, e^{(-(\Pr+Le)\eta)}}{Le+\Pr} - \Pr^2\ Nt\, e^{-2\Pr\eta}\right)\end{aligned} \tag{12}$$

When $\eta = 0$ the above equation becomes,

$$\theta'(0) = -\frac{\Pr}{2}\left[\frac{h\Pr^3 Nt + (2 + 2h\ Le\ Nb + h\,Nt + hNtLe + 2h)\Pr^2 + (2 + 2Le + h\,Nt\,Le + 2h\,Le\,Nb + 2hLe)\Pr + 2Le}{(1+\Pr)(Le+\Pr)}\right] \tag{13}$$

Also, when Nb = 0 and Nt = 0,

$$\theta'(0) = -\frac{1}{2}\left[\frac{(1+\Pr+h\Pr)\Pr}{(1+\Pr)}\right] \tag{14}$$

The volume fraction of nanofluid using two iterations become

$$\phi(\eta) = e^{(-\mathrm{Le}\,\eta)} - h\left(\frac{\mathrm{Le}^2}{\mathrm{Le}+1} + \frac{\mathrm{Pr\,Nt}}{\mathrm{Nb}(\mathrm{Pr}-\mathrm{Le})}\right)e^{(-\mathrm{Le}\,\eta)} + h\left(\frac{\mathrm{Le}^2\, e^{(-(\mathrm{Le}+1)\eta)}}{\mathrm{Le}+1} + \frac{\mathrm{Pr\,Nt}\; e^{(-\mathrm{Pr}\,\eta)}}{\mathrm{Nb}(\mathrm{Pr}-\mathrm{Le})}\right) \tag{15}$$

The derivative of the equation becomes

$$\phi'(\eta) = -\mathrm{Le}\; e^{(-\mathrm{Le}\,\eta)} + h\left(\frac{\mathrm{Le}^2}{\mathrm{Le}+1} + \frac{\mathrm{Pr\,Nt}}{\mathrm{Nb}(\mathrm{Pr}-\mathrm{Le})}\right)\mathrm{Le}\; e^{(-\mathrm{Le}\,\eta)} + h\left(\frac{\mathrm{Le}^2\,(-1-\mathrm{Le})e^{(-(\mathrm{Le}+1)\eta)}}{\mathrm{Le}+1} - \frac{\mathrm{Pr}^2\;\mathrm{Nt}\; e^{(-\mathrm{Pr}\,\eta)}}{\mathrm{Nb}(\mathrm{Pr}-\mathrm{Le})}\right) \tag{16}$$

When $\eta = 0$ the above equation becomes,

$$\phi'(0) = -\frac{\mathrm{Le}^2\;\mathrm{Nb} + \mathrm{Le}^3 h\;\mathrm{Nb} + \mathrm{Le\,Nb} + h\,\mathrm{Le\,Pr\,Nt} + h\,\mathrm{Pr\,Nt}}{\mathrm{Nb}\;(\mathrm{Le}+1)} \tag{17}$$

This boundary value problem reduces to the classical problem of flow and heat and mass transfer due to a stretching surface in a viscous fluid when Nb and Nt are zero in Eqs. (2) and (3). The boundary value problem for ϕ then becomes ill-posed and is of no physical significance.

4. NUMERICAL SIMULATION

The nonlinear differential eqns (1-4) are also solved by numerical methods. The function bvp4c in Matlab software, which is a function of solving two-point boundary value problems (BVPs) for ordinary differential equations is used to solve this system. This Matlab program is given in appendix E. The numerical solution is compared with approximate analytical solution obtained by Homotopy analysis method for $\mathrm{Pr} \le 10$ and $\mathrm{Le} \le 10$.

5. RESULTS AND DISCUSSION

Equations (8) to (10) are the new approximate analytical expression of the stream function, temperature distribution and volume fraction of nanofluid interms of the parameters Prandtl number, the Lewis number, the Brownian motion parameter and the thermophores, respectively.

Fig.2 presents the stream function *f* and velocity f' versus boundary layer co-ordinate η. From the Fig.2, it is observed that stream function is always increasing function whereas the velocity profile f' is always a decreasing function. Also the stream function and velocity profile reaches the steady state value when $\eta = 4$. In Fig.3 the temperature distribution function is compared with simulation results for various values of parameters. A good agreement between analytical and numerical results is noted.

The effect of the parameters Pr, Le, Nb and Nt on the temperature distribution are shown in Fig.4.From the Fig it is observed that temperature decreases as Prandtl number Pr and Lewis number Le increases. It is also noted that temperature distribution has no significant effect due to Thermophoresis parameter Nt and Brownian motion parameter Nb. Analytical expressions of nanoparticle volume fraction is compared with simulation results in Fig.5for various values of the parameters. Satisfactory agreement is noted between the analytical and simulation results.

The influence of the parameters Le, Pr, Nb and Nt on nanoparticle volume fraction is shown in the Fig.6. The Lewis number is the ratio of thermal diffusion to mass diffusion which inturn becomes inversely proportional to mass diffusion, given thermal diffusion is constant. An increasing Lewis number is equivalent to decreasing mass diffusion or nanoparticle volume fraction. The nanoparticle volume fraction decreases with increase in Lewis number Le or decrease in the Prandl number Pr.

Fig.7 depicts the effect of parameters Nb or Nt on the reduced nusselt number $-\theta'(0)$ and reduced Sherwood number $-\phi'(0)$. From the Figs, it is inferred that reduced Sherwood number decreases when Nb increases. Fig.8. presents the comparison of analytical expression of stream function *f*, temperature distribution θ and nanoparticle volume fraction ϕ with numerical results.

The analytical solution represented by (13) to (16) contains the auxiliary parameter h. The region where the distribution of θ and ϕ versus h is a horizontal line is known as the convergence region for the corresponding function. The h curves of θ (Pr =1, Le = 5, Nt = Nb = 0.1) and ϕ (Pr =1, Le = 5, Nt = Nb = 0.1) are plotted in Fig.9. This figure clearly indicates that the valid region of h is about −0.77. Similarly we can find the value of the convergence-control parameter h for different values of the parameters.

Our analytical results for reduced Nusselt number $-\theta'(0)$ is compared with previous results khan.*et al.* [16], wang. *et al.* [13] and Golra and Sidawi *et al.*[14] in Table.2. Also, the variation of reduced Nusselt number and Sherwood number for various values of Nb and Nt with Pr=10 and Le=10 is presented in the Table.2. From the Table it is observed that the Nur is a decreasing function whereas Shr is increasing function for all values of parameters. From the table it is also inferred that our results give satisfactory agreement with previous results.

6. CONCLUSIONS

Two dimensional steady state boundary layer flow of nanofluid past a stretching sheet is discussed**.** HAM is used to find the analytical expression of the stream function, temperature distribution and volume fraction of nanofluid. It is demonstrated that the obtained results are in good agreement with the numerical results. In this study the effect of the parameters Prantl number, Lewis number, Brownian motion and Thermophoresis parameter on the temperature distribution and nanoparticle volume fraction is demonstrated. The Prandtl number Pr and the Lewis number Le have reverse effect on both the temperature distribution and the volume fraction. The behavior of Brownian motion and Thermophoresis parameter on the fluid temperature and concentration profile is insignificant.

NOMENCLATURE

D_B	Brownian diffusion coefficient	Re_x	Local Reynolds number
D_T	Thermophoretic diffusion coefficient	Sh	Local Sherwood number
$f(\eta)$	Dimensionless stream function	T	Fluid temperature
Pr	Prandtl number	T_w	Temperature at the stretching surface
Le	Lewis number	T_∞	Ambient temperature
Nb	Brownian motion parameter	u, v	Velocity components along x- and y-axes
Nt	Thermophoresis parameter	u_w	Velocity of the stretching sheet
Nu	Nusselt number	x, y	Cartesian coordinates (x-axis is aligned along the stretching surface and y-axis is normal to it)

GREEK SYMBOLS

α	Thermal diffusivity
$\phi(\eta)$	Rescaled nanoparticle volume fraction
η	Similarity variable
$\theta(\eta)$	Dimensionless temperature
υ	Kinematic viscosity of the fluid
$(\rho c)_f$	Heat capacity of the fluid
$(\rho c)_\rho$	Effective heat capacity of the nanoparticle material

Table-1: Contribution of the various flow of a nanofluid past a stretching sheet

Author	Reference	Flow	Analytical / numerical
Khan et al.	International Journal of Heat And Mass Transfer, 53 (2010) 2477-2483.	Laminar fluid flow	Implicit finite-difference method
Crane et. al	Journal ofApplied Mathematics And Physics (Zamp), 21, 645-647.	Boundary layer flow	Shooting method
Mania et.al	Applied Nano Science, (2014) 4, 761-767.	Laminar flow	Variational finite element method
Mahata et al.	Journal of Applied Fluid Mechanics, 9 (2016), 4, 1977-1989.	Boundary layer flow	Spectral relaxation method

Besthapu et al.	Journal ofApplied Mathematics And Physics, 2015, 3, 1580-1593.	Mixed convection magneto hydrodynamic flow	Implicit finite difference method
Mohamed et al.	Journal of Mechanical Engineering, 63 (2017), 2, 119-128.	MHD flow	Transform method
Abu baker et al.	Indian Journal of Science And Technology, 9(31), Aug (2016), 0974-5645.	Nanofluid flow	Shooting technique
Abel et al.	International Journal of Physics And Mathematical Sciences, 5 (2015), 4, 25-35.	Laminar 2d boundary layer flow	Similarity transformation
Nield et al.	International Journal of Heat And Mass Transfer, 52(2009), 5792-5795.	Convective boundary layer flow	Porus medium the darcy model
Kuznetsov et al.	International Journal of Thermal Sciences, 77(2014), 126-129.	Boundary layer flow of a nanofluid	A revised model
Takhar et al.	Acta Mechanica 146(2001), 59-71.	Laminar boundary layer flow	Implicit finite difference method
Ferdows et al.	Acta University. Sapientiae, Mathematica, 9, 1 (2017), 140-161.	MHD boundary layer flow	Runge-kutta sixth order iteration
Ibrahim et al.	Computers & Fluids, 75 (2013), 1–10.	MHD boundary layer flow and heat transfer of ananofluid	Fourth order runge–kutta method, similarity transformation
Lavanya et al.	Asian Journal of Science And Technology, 7,(2016), 4, 2815-2824.	Laminar, two dimensional, steady/unsteady, free/mixed convection flow	Runge-kutta fourth order , shooting technique
Mansur et al.	Journal of Applied Mathematics, 2014, Article Id 907152, 7 Pages.	(MHD) boundary layer flow of a nanofluid past a stretching	Shooting method

Table-2: Comparison of various results for the reduced Nusselt number $-\theta'(0)$ when Nb=Nt=0 for Le=0.

Pr	Our Results Eqn(14) Nb = Nt = 0	Khan[16]	Wang[13]	Golraand Sidawi[14]
0.07	0.0662	0.0663	0.0656	0.0656
0.20	0.1696	0.1691	0.1691	0.1691
0.70	0.4592	0.4539	0.4539	0.5349
2.00	0.9114	0.9113	0.9114	0.9114
7.00	1.8769	1.8954	1.8954	1.8905
20.00	3.3968	3.3539	3.3539	3.3539
70.00	6.4746	6.4621	6.4622	6.4622

Table-3.(a): Comparison of reduced Nusselt number $Nur = -\theta'(0)$ for various values of Nb and Nt when Pr =10 and Le=10 with previous results. Eqn (13) is used for our results.

Nt	Nb=0.1		Nb=0.2		Nb=0.3		Nb=0.4		Nb=0.5	
	Khan[16]	Our results Eq.(14)	Khan[16]	Our Results Eq.(14)	Khan[16]	Our Results Eq.(14)	Khan[16]	Our Results Eq.(14)	Khan[16]	Our Results Eq.(14)
0.1	0.9524	0.9524	0.5056	0.5051	0.2522	0.2522	0.1194	0.1194	0.0543	0.0543
0.2	0.6932	0.6936	0.3654	0.3651	0.1816	0.1816	0.0859	0.0859	0.0390	0.0390
0.3	0.5201	0.5200	0.2731	0.2731	0.1355	0.1355	0.0641	0.0641	0.0291	0.0291
0.4	0.4026	0.4023	0.2110	0.2118	0.1046	0.1046	0.0495	0.0495	0.0225	0.0225
0.5	0.3211	0.3215	0.1681	0.1689	0.0833	0.0833	0.0394	0.0394	0.0179	0.0179

Table- 3(b): Comparison of reduced Sherwood number $\mathrm{Shr} = -\phi'(0)$ for various values of Nb and Nb when Pr =10 and Le=10 with previous results. Eq.(17) is used for our results.

Nt	Nb=0.1		Nb=0.2		Nb=0.3		Nb=0.4		Nb=0.5	
	Khan[16]	Our Results Eq.(17)	Khan[16]	Our Results Eq.(17)	Khan[16]	Our Results Eq.(17)	Khan[16]	Our Results Eq.(17)	Khan[16]	Our Results Eq.(17)
0.1	2.1294	2.1293	2.3819	2.3820	2.4100	2.4100	2.3997	2.3997	2.3836	2.3836
0.2	2.2740	2.2745	2.5152.	2.5152	2.5150.	2.5150	2.4807	2.4807	2.4468	2.4468
0.3	2.5286	2.5288	2.6555	2.6554	2.6088	2.6088	2.5486	2.5486	2.4984	2.4984
0.4	2.7952	2.7956	2.7818	2.7817	2.6876	2.6878	2.6038	2.6038	2.5399	2.5399
0.5	3.0351	3.034	2.8883	2.8883	2.7519	2.7518	2.6483	2.6483	2.5731	2.5731

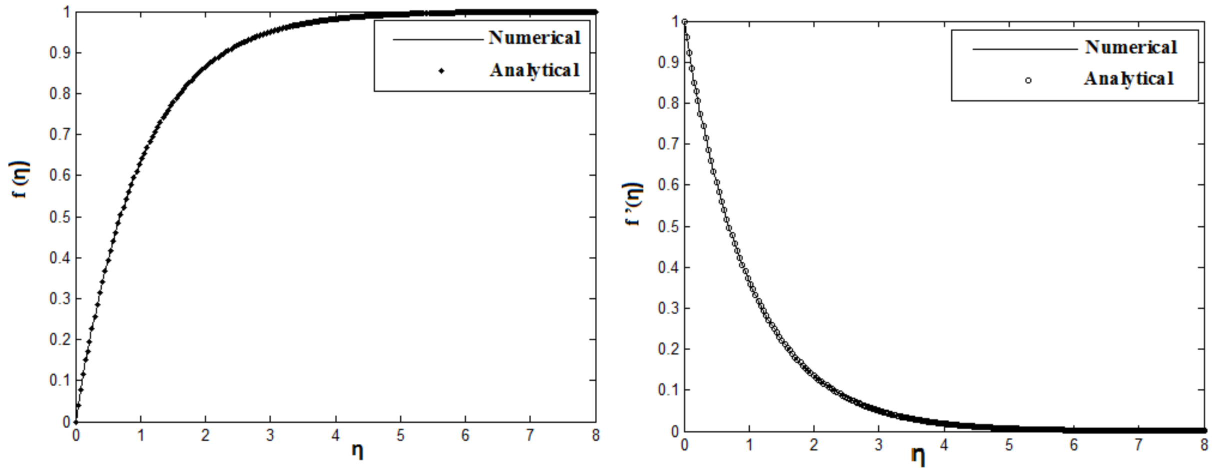

Figure-2: Comparison of analytical expression of stream function *f* (Eqn. (8)) and the velocity profile *f* ’with the numerical results.

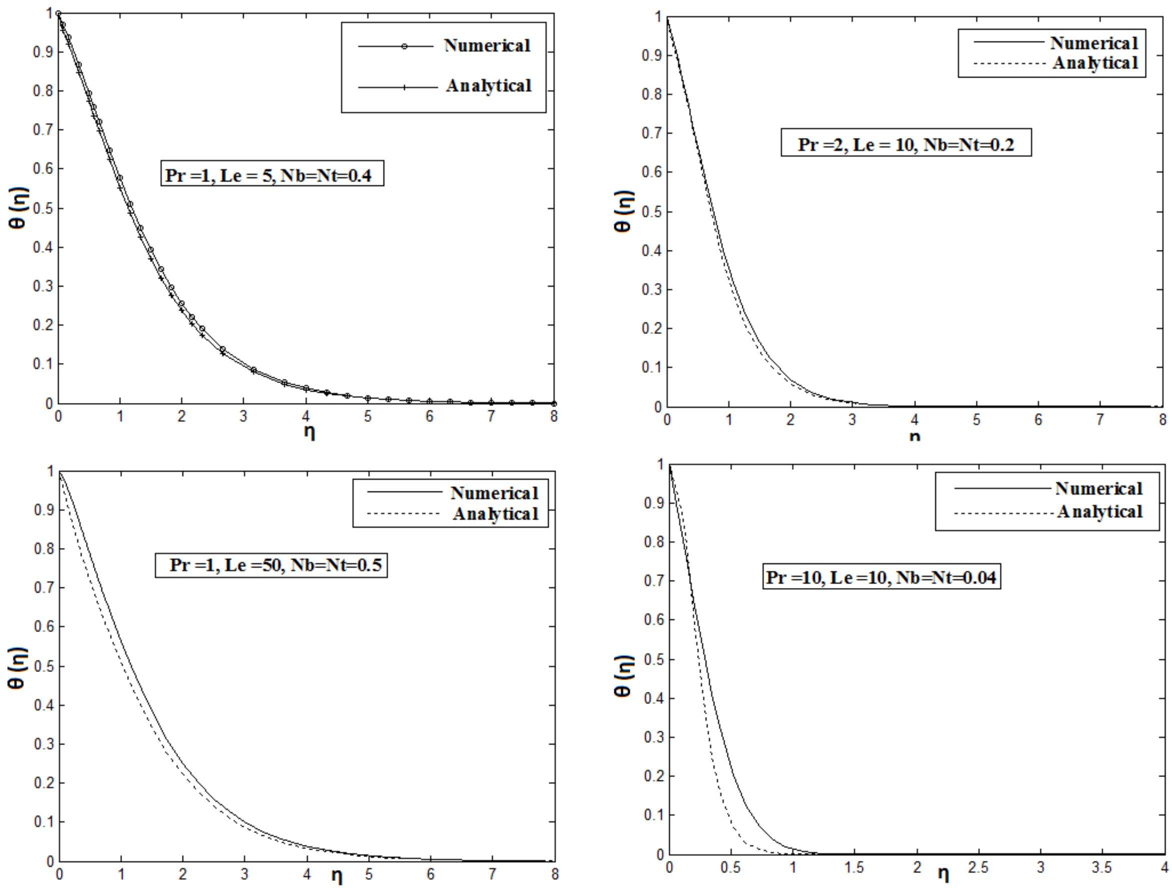

Figure-3: Comparison of analytical expression of temperature (Eqn. (9)) with numerical results.

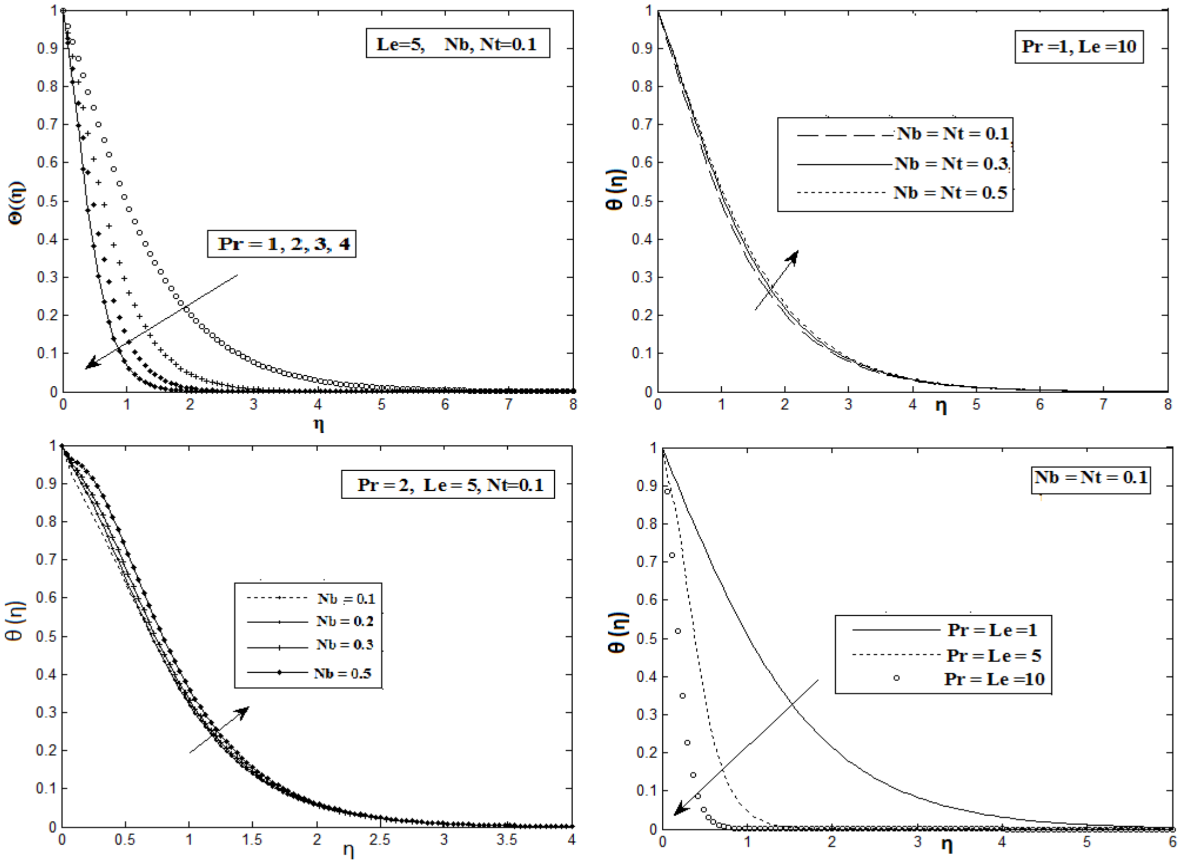

Figure-4: Effect of the various parameters on the temperature distribution θ.

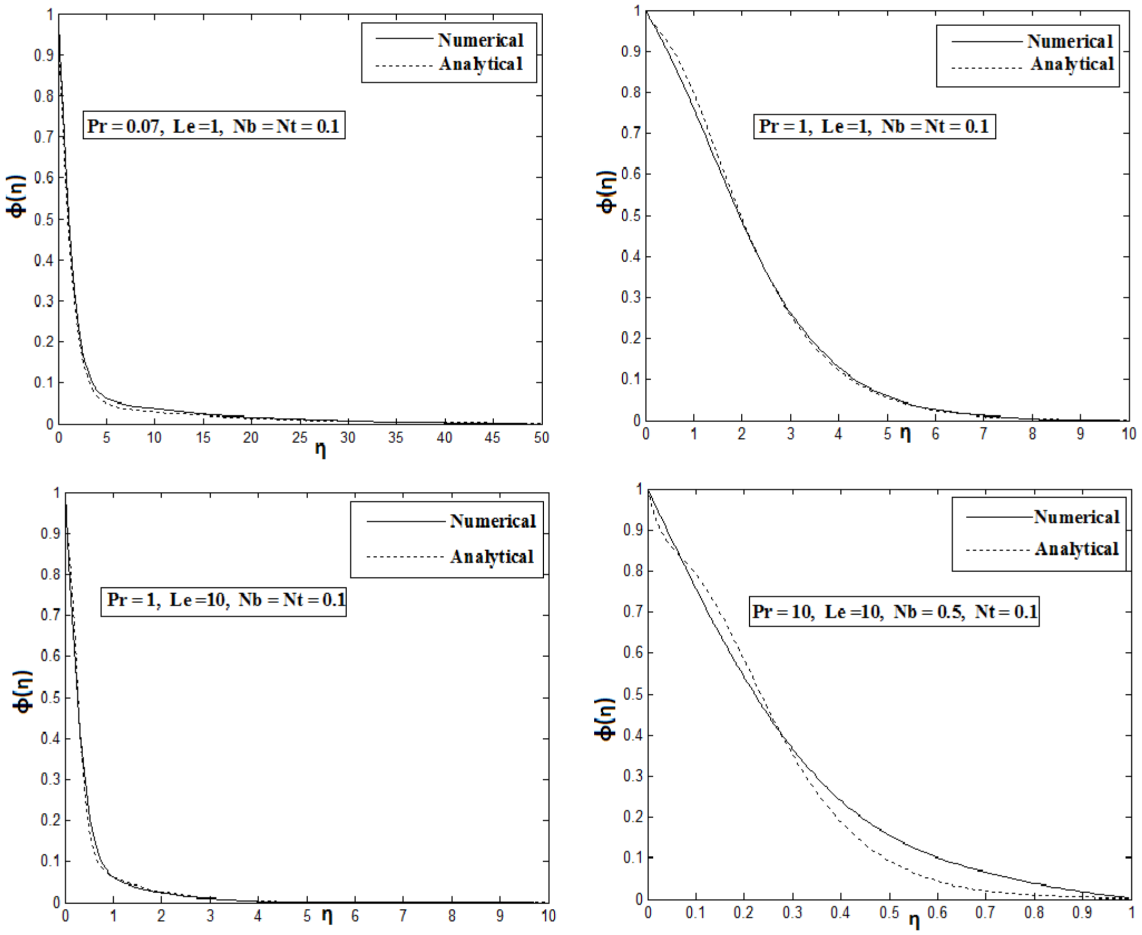

Figure-5: Comparison of analytical expression of volume fraction (Eqn (10)) with numerical results.

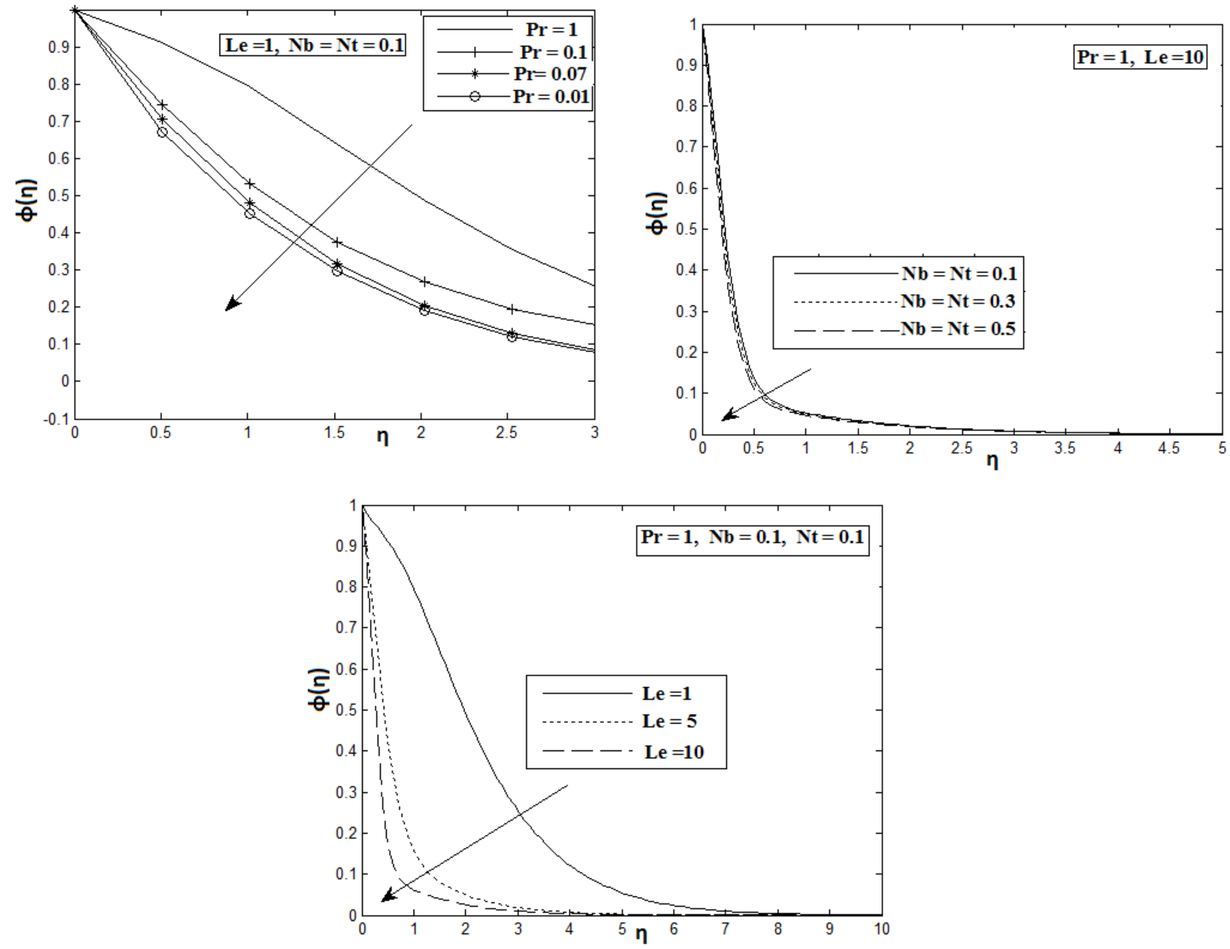

Figure-6: Effect of various parameters on nanoparticle volume fraction ϕ.

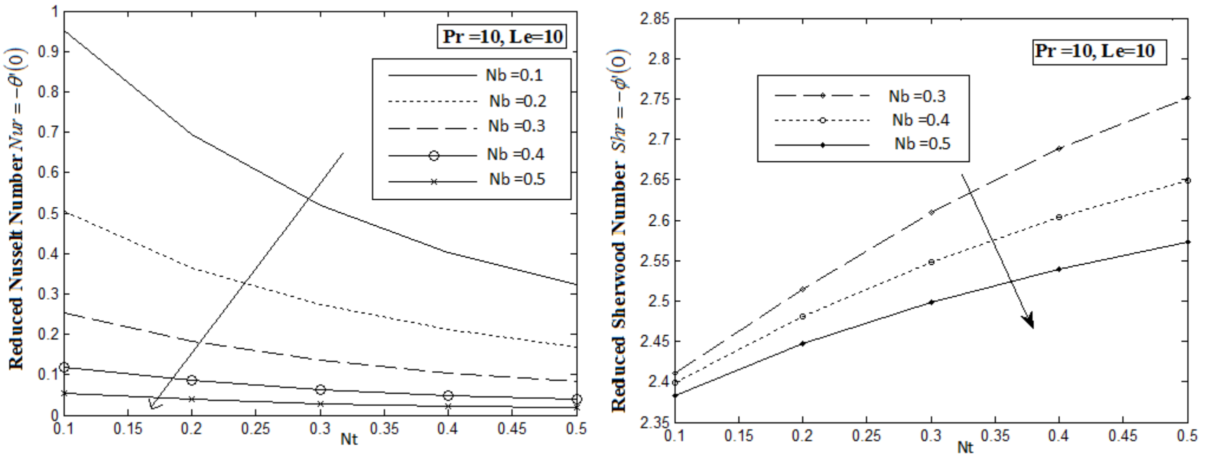

Figure-7: Effect of parameters on reduced Nusselt number $-\theta'(0)$ and reduced Sherwood number $-\phi'(0)$

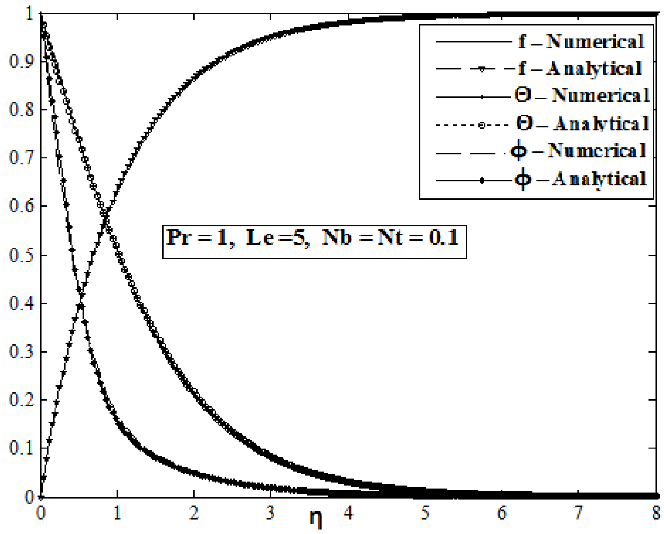

Figure-8: Comparison of analytical expression of stream function f, temperature distribution θ and nanoparticle volume fraction ϕ with numerical results.

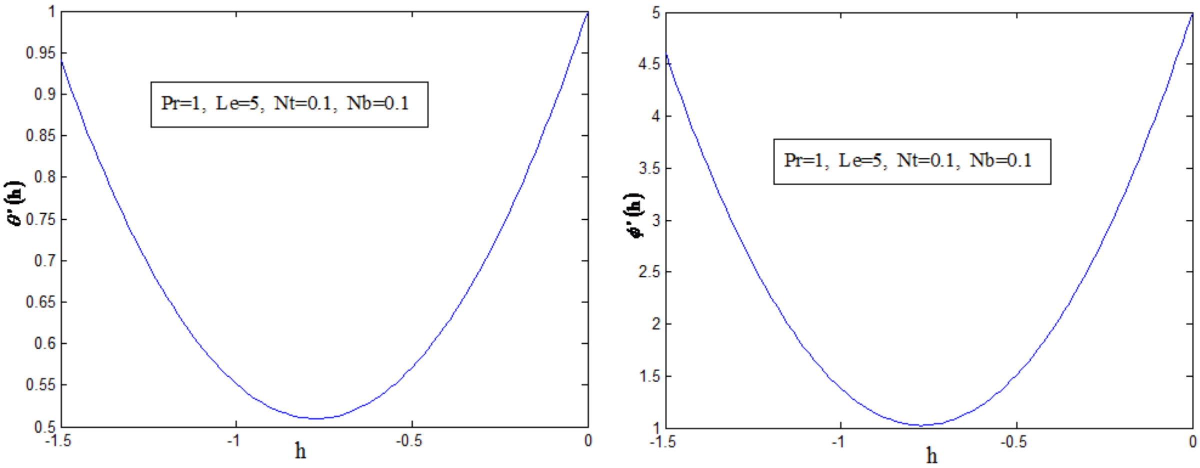

Figure-9: Validity Region of h curve of θ and ϕ

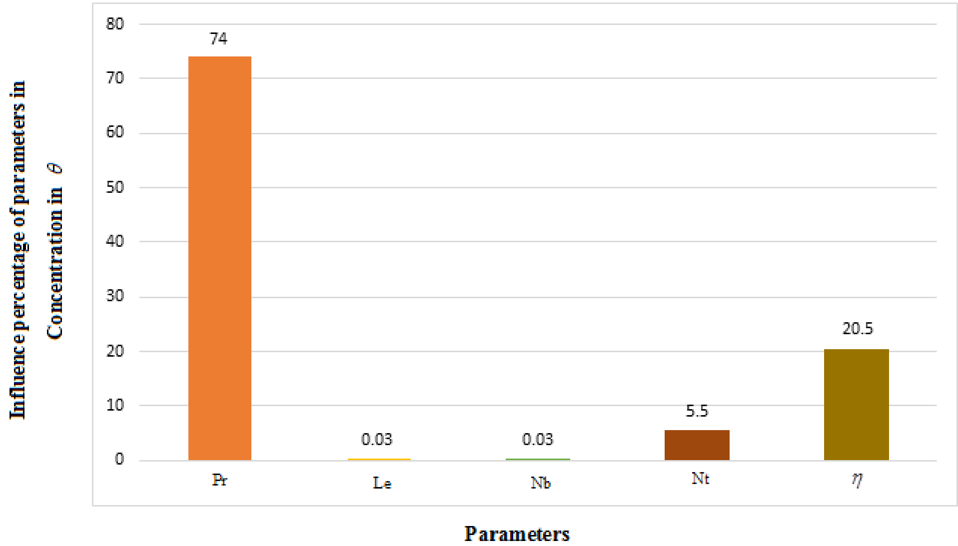

Figure-10: Sensitivity analysis for evaluating the influence of the concentration of temperature distribution in Eqn. (9)

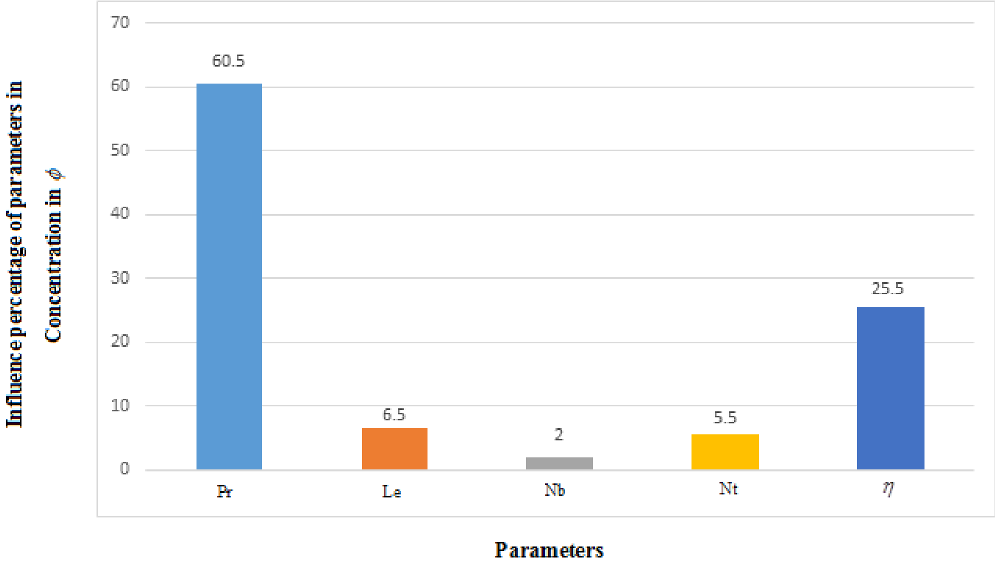

Figure-11: Sensitivity analysis for evaluating the influence of the concentration of nanoparticle volume fraction in Eqn.(10)

APPENDIX A: Basic idea of Homotopy Analysis Method

In order to show the basic idea of HAM, we consider a linear or nonlinear equation in a general form:

$$N[u(t)] = 0; \tag{A1}$$

where N is a nonlinear operator $u(t)$ is an unknown function, t is independent variable. Let $u_0(t)$ denote an initial approximation of the solution of equation, h a nonzero auxiliary parameter, $H(t)$ a nonzero auxiliary function and L an auxiliary linear operator. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM, we first construct the so-called zero-th order deformation equation.

$$(1-p)L[\varphi(t;p)-u_0(t)] = phH(t)N[\varphi(t;p)]; \tag{A2}$$

where $p \in [0,1]$ is the embedding parameter, $\varphi(t;p)$ is an unknown function. It is obvious that when the embedding parameter $p=0$ and $p=1$, it holds

$$\varphi(t;0) = u_0(t); \quad \varphi(t;1) = u(t); \tag{A3}$$

respectively. Thus as p increases from 0 to 1, $\varphi(t;p)$ varies from the initial guesses $\varphi(t;0)$ to the equation $\varphi(t;1)$ of equation. Expanding $\varphi(t;p)$ in Taylors series with respect to p, we have

$$\varphi(t;p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)p^m, \tag{A4}$$

Where $$u_m(t) = \left[\frac{1}{m!}\frac{\partial^m \phi(t;p)}{\partial p^m}\right]_{p=0} \tag{A5}$$

The convergence of the series (A4) depends upon the auxiliary parameter h. If it is convergent at $p=1$, one has

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t); \tag{A6}$$

This must be one of the solutions of the original nonlinear equation. Define the vectors

$$\vec{u} = \{u_0(t), u_1(t), \ldots u_n(t)\} \tag{A7}$$

Differentiating the zero-order deformation Eq. (A1) m-times with respect to p and then dividing them by m! and finally setting $p=0$, we get the following m[th]-order deformation equation.

$$L[u_m(t) - \chi_m u_{m-1}(t)] = hH(t)\Re_m\left(\vec{u}_{m-1}\right) \tag{A8}$$

where $$\Re_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N\varphi(t;p)}{\partial p^{m-1}} \tag{A9}$$

and

$$\chi_m = \begin{cases} 0, m \le 1, \\ 1, \ m > 1 \end{cases}$$

Operating the inverse operation of L^{-1} on the both sides of Eq. (A8), we have

$$\boldsymbol{u_m}(\boldsymbol{t}) = \chi_m \boldsymbol{u_{m-1}}(\boldsymbol{t}) + \boldsymbol{h}\boldsymbol{L}^{-1}\left[\boldsymbol{H}(\boldsymbol{t})\Re_m \left(\overrightarrow{\boldsymbol{u}_{m-1}} \right) \right] \tag{A10}$$

In this way, it is easy to obtain $\boldsymbol{u}_1(\boldsymbol{t}); \boldsymbol{u}_2(\boldsymbol{t}); \ldots$ one after another; finally, we get an exact solution of the original equation.

$$u(t) = \sum_{m=0}^{+\infty} u_m(t) \tag{A11}$$

For the convergence of the above method we refer the reader to Liao [15]. If Eq. (A1) admits unique solution, then this method will produce the unique solution. If Eq. (A1) does not possess a unique solution, the HAM will give a solution among many other possible solutions.

APPENDIX B: Analytical solution of stream function f

Analytical solution of eqn (1)

$$f''' + f\ f'' - f'^2 = 0 \tag{B1}$$

The homotopy analysis method for the equations (B1) can be constructed as follows:

$$(1-p)\left(f''' - f'\right) = h\,p\left(f''' + f\ f'' - f'^2\right) \tag{B2}$$

The approximate solution of the equations (B2) is as follows:

$$f = f_0 + f_1 p + f_2 p^2 + ... \tag{B3}$$

where p is the embedding parameters and $p = [0, 1]$. Substituting (B3) in (B2) and equating the like coefficients of p on both sides we get,

$$\begin{aligned} & p^0: \ f_0''' - f_0' = 0 \\ & p^1: = f_1''' - f_1' - f_0''' + f_0' = h\left(f_0''' + f_0\ f_0'' - f_0'^2\right) \\ & p^2: \ f_2''' - f_2' - f_1''' + f_1' = h\left(f_1''' + f_0\ f_1'' + f_1\ f_0'' - 2\ f_0'\ f_1'\right) \end{aligned} \tag{B4}$$

With the boundary conditions,

$$\begin{aligned} & f_0(0) = 0, \quad f_0'(0) = 1, \quad f_0'(\infty) = 0 \\ & f_1(0) = 0, \quad f_1'(\infty) = 0, \quad f_2'(\infty) = 0 \\ & f_2(0) = 0, \quad f_2'(\infty) = 0, \quad f_2'(\infty) = 0 \end{aligned} \tag{B5}$$

Solving the eqns (B4) and using (B5) ,

$$\begin{aligned} & f_0(\eta) = 1 - exp(-\eta) \\ & f_1(0) = 0 \\ & f_2(0) = 0 \end{aligned} \tag{B6}$$

Substituting in (B3) and taking the limit as p tends to 1,

$$f(\eta) = 1 - \exp(-\eta) \tag{B7}$$

APPENDIX C: Analytical expression of temperature θ and nanoparticle volume fraction ϕ by solving the equations (2) and (3) using HAM

Analytical solution of (2) and (3) using HAM.

Substituting $f(\eta)=1-\exp(-\eta)$ in the eqn (2) and (3) we get,

$$\theta''+\Pr\theta'-\Pr\ e^{-\eta}\theta'+\Pr\ \mathrm{Nb}\ \theta'\phi'+\Pr\ \mathrm{Nt}\ \theta'^2=0 \quad (C1)$$

$$\phi''+\mathrm{Le}\ \phi'-\mathrm{Le}\ \phi' e^{-\eta}+\left(\frac{\mathrm{Nt}}{\mathrm{Nb}}\right)\theta''=0 \quad (C2)$$

The homotopy analysis method for the equations (C1) and (C2) can be written as follows:

$$(1-p)(\theta''+\Pr\ \theta')=h\,p\left(\theta''+\Pr\ \theta'-\Pr\ \exp(-\eta)\theta'+\Pr Nb\ \theta'\ \varphi'+\Pr\ Nt\ \theta'^{\,2}\right) \quad (C3)$$

$$(1-p)(\phi''+Le\ \phi')=h\,p\left(\phi''+Le\ \phi'-Le\phi' exp(-\eta)+(Nt/Nb)\theta''\right) \quad (C4)$$

The approximate solution of the equations (C1) and (C2) are as follows:

$$\theta=\theta_0+\theta_1 p+\theta_2 p^2+.......... \quad (C5)$$

$$\phi=\phi_0+\phi_1 p+\phi_2 p^2+........... \quad (C6)$$

where *p* is the embedding parameters and $p=[0,1]$. Substituting (C5) and (C6) in (C3) and (C4) and equating the like coefficients of *p* on both sides we get,

$$p^0:\theta_0''+\Pr\theta_0'=0 \quad (C7)$$

$$p^1:\theta_1''+\Pr\ \theta_1'-\ \theta_0''-\Pr\theta_0'=h\left(\theta_0''+\Pr\ \theta_0'-\Pr\ \exp(-\eta)\theta_0'+\Pr Nb\ \theta_0'\ \phi_0'+\Pr\ Nt\ \theta_0'^{2}\right) \quad (C8)$$

$$p^2:\theta_2''+P_r\ \theta_2'-\ \theta_1''-P_r\ \theta_1'=h\begin{pmatrix}\theta_1''+\Pr\theta_1'-\Pr\ \exp(-\eta)\theta_1'+\Pr\ Nb(\theta_0'\ \phi_1'+\theta_1'\ \phi_0')\\+2\Pr\ Nt\theta_0'\theta_1'\end{pmatrix} \quad (C9)$$

$$\mathrm{p}^0:\phi_0''+\mathrm{Le}\ \phi_0'=0 \quad (C10)$$

$$p^1:\phi_1''+Le\ \phi_1'-\phi_0''-Le\phi_0'=h(\phi_0''+Le\phi_0'-Le\phi_0' exp(-\eta)+(Nt/Nb)\ \theta_0'') \quad (C11)$$

$$p^2:\phi_2''+Le\ \phi_2'-\phi_1''-Le\phi_1'=h(\phi_1''+Le\phi_1'-Le\phi_1' exp(-\eta)+(Nt/Nb)\ \theta_1'') \quad (C12)$$

With the boundary conditions,

$$\theta_0(0)=1,\quad \theta_0(\infty)=0 \quad (C13)$$

$$\theta_1(0)=0,\quad \theta_1(\infty)=0 \quad (C14)$$

$$\theta_2(0)=0,\quad \theta_2(\infty)=0 \quad (C15)$$

$$\phi_0(0)=1,\quad \phi_0(\infty)=0 \quad (C16)$$

$$\phi_1(0)=0,\quad \phi_1(\infty)=0 \quad (C17)$$

$$\phi_2(0)=0,\quad \phi_2(\infty)=0 \quad (C18)$$

Solving the eqn (C7) and using boundary conditions (C13) we get,

$$\theta_0(\eta)=\exp(-\Pr\ \eta) \quad (C19)$$

Solving the eqn (C10) and using boundary conditions (C16) we get,

$$\varphi_0(\eta)=\exp(-Le\ \eta) \quad (C20)$$

Substituting for $\theta_0(\eta)$ and $\phi_0(\eta)$ in (C8) and using boundary conditions (C14) we get,

$$\theta_1(\eta)=-h\left(\frac{\Pr^2}{1+\Pr}+\frac{\Pr^2 Nb}{Le+\Pr}+\frac{\Pr Nt}{2}\right)\exp(-\Pr\eta)+h\left(\frac{\Pr^2}{1+\Pr}\exp(-(1+\Pr)\eta)+\frac{\Pr^2 Nb}{Le+\Pr}\exp(-(\Pr+Le)\eta)+\frac{\Pr Nt}{2}\exp(-2\Pr\ \eta)\right) \quad (C21)$$

Substituting for $\theta_0(\eta)$ and $\phi_0(\eta)$ in (C11) and using boundary conditions (C17) we get,

$$\phi_1(\eta) = -h\left(\frac{Le^2}{Le+1} + \frac{Pr\,Nt}{Nb(Pr-Le)}\right)exp(-Le\,\eta) + h\left(\frac{Le^2}{Le+1}exp(-(Le+1)\eta) + \frac{Pr\,Nt}{Nb(Pr-Le)}exp(-Pr\,\eta)\right) \tag{C22}$$

Substituting for $\theta_0(\eta)$, $\phi_0(\eta)$, $\theta_1(\eta)$ and $\phi_1(\eta)$ in (C9) and using boundary conditions (C15) we get,

$$\begin{aligned}\theta_2(\eta) = {} & A\,exp(-Pr\,\eta) + \frac{B}{1+Pr}exp(-(1+Pr)\eta) + \frac{C}{Le(Pr+Le)}exp(-(Pr+Le)\eta) \\ & + \frac{D}{2Pr^2}exp(-2Pr\,\eta) + \frac{E}{(1+Le)(1+Le+Pr)}exp(-(1+Pr+Le)\eta) + \frac{h^2\,Pr^3\,Nb\,Nt}{Pr+Le} \\ & exp(-(2Pr+Le)\eta) + \frac{h^2\,Pr^3\,Nt}{(Pr+1)}exp(-(1+2Pr)\eta) + \frac{h^2\,Pr^3}{2(2+Pr)}exp(-(2+Pr)\eta) \\ & + \frac{h^2\,Pr^3\,Nb^2}{2(Pr+2Le)}exp(-(Pr+2Le)\eta) + h^2\,Pr^2\,Nt^2\,exp(-(3Pr)\eta)\end{aligned} \tag{C23}$$

By taking the constants A, B, C, D and E are defined as

$$A = -\left[\begin{aligned}&\frac{B}{1+Pr} + \frac{C}{Le(Pr+Le)} + \frac{D}{2Pr^2} + \frac{E}{(1+Le)(1+Le+Pr)} + \frac{h^2\,Pr^3\,Nb\,Nt}{Pr+Le} + + \frac{h^2\,Pr^3\,Nt}{(Pr+1)} + \frac{h^2\,Pr^3}{2(2+Pr)} \\ &+ \frac{h^2\,Pr^3\,Nb^2}{2(Pr+2Le)} + h^2\,Pr^2\,Nt^2\end{aligned}\right]$$

$$B = h\,Pr^2\left(1 + h - h\,Pr\left(\frac{Pr}{(Pr+1)} + \frac{Pr\,Nb}{Pr+Le} + \frac{Nt}{(2)}\right)\right)$$

$$C = h\,Pr^2\,Nb\,Le\left(1 + h - h\,Pr\left(\frac{Pr}{(Pr+1)} + \frac{Pr\,Nb}{Pr+Le} + \frac{Nt}{(2)}\right) - h\,Le\left(\left(\frac{Le^2}{Le+1} + \frac{Pr(Nt/Nb)}{Pr-Le}\right)\right)\right)$$

$$D = h\,Pr^3\,Nt\left(1 + h - 2h\,Pr\left(\frac{Pr}{(Pr+1)} + \frac{Pr\,Nb}{Pr+Le} + \frac{Nt}{(2)}\right) + \frac{h\,Pr}{Pr-Le}\right),$$

$$E = h\,Pr^2\,Nb\left(h\,Pr(Le+1) + hLe^2\right)$$

Substituting in (C5) and taking the limit as p tends to 1 we get

$$\theta(\eta) = \theta_0(\eta) + \theta_1(\eta) + \theta_2(\eta) \tag{C24}$$

$$\theta(\eta)=\exp(-\Pr\eta)-h\left(\frac{\Pr^2}{1+\Pr}+\frac{\Pr^2 Nb}{Le+\Pr}+\frac{\Pr Nt}{2}\right)\exp(-\Pr\eta)+h\left(\frac{\Pr^2}{1+\Pr}\exp(-(1+\Pr)\eta)+\frac{\Pr^2 Nb}{Le+\Pr}\exp(-(\Pr+Le)\eta)+\frac{\Pr Nt}{2}\exp(-2\Pr\eta)\right)+A\exp(-\Pr\eta)+\frac{B}{1+\Pr}\exp(-(1+\Pr)\eta)+\frac{C}{Le(\Pr+Le)}\exp(-(\Pr+Le)\eta)+\frac{D}{2\Pr^2}\exp(-2\Pr\eta)+\frac{E}{(1+Le)(1+Le+\Pr)}\exp(-(1+\Pr+Le)\eta)+\frac{h^2\Pr^3 NbNt}{\Pr+Le}\exp(-(2\Pr+Le)\eta)+\frac{h^2\Pr^3 Nt}{(\Pr+1)}\exp(-(1+2\Pr)\eta)+\frac{h^2\Pr^3}{2(2+\Pr)}\exp(-(2+\Pr)\eta)+\frac{h^2\Pr^3 Nb^2}{2(\Pr+2Le)}\exp(-(\Pr+2Le)\eta)+h^2\Pr^2 Nt^2\exp(-(3P)\eta) \tag{C25}$$

where

$$A_1=1-h\left(\frac{\Pr^2}{1+\Pr}+\frac{\Pr^2 Nb}{Le+\Pr}+\frac{\Pr Nt}{2}\right)+\mathrm{A},\quad \mathrm{A}_2=D+\frac{h(\Pr Nt)}{2},\quad \mathrm{A}_3=h^2\Pr^2 Nt^2,$$
$$A_4=\left(\frac{B}{1+\Pr}\right)+h\left(\frac{\Pr^2}{1+\Pr}\right),\qquad A_5=\left(\frac{\mathrm{C}}{\mathrm{Le}(\mathrm{Le}+\Pr)}\right)+h\left(\frac{\Pr^2 Nb}{Le+\Pr}\right),$$
$$\mathrm{A}_6=\frac{E}{(1+Le)(1+Le+\Pr)},\qquad \mathrm{A}_7=\frac{h^2\Pr^3 NbNt}{\Pr+Le},\ \mathrm{A}_8=\frac{h^2\Pr^3 Nb^2}{2(\Pr+2Le)},$$
$$\mathrm{A}_9=\frac{h^2\Pr^3 Nt}{(\Pr+1)},\qquad \mathrm{A}_{10}=\frac{h^2\Pr^3}{2(2+\Pr)}, \tag{C26}$$

the temperature distribution is given in (9).

Substituting for $\theta_0(\eta)$, $\phi_0(\eta)$, $\theta_1(\eta)$ and $\phi_1(\eta)$ in (C12) and using boundary conditions (C18) we get,

$$\varphi_2(\eta)=F\exp(-Le\eta)+\frac{G}{1+Le}\exp(-(1+Le)\eta)+\frac{H}{\Pr(\Pr-Le)}\exp(-\Pr\eta)+\frac{I}{2(2+Le)}\exp(-(2+Le)\eta)+\frac{J}{(1+\Pr)^2-Le(1+\Pr)}\exp(-(1+\Pr)\eta)+\frac{K}{2\Pr(2\Pr-Le)}\exp(-2\Pr\eta)+\frac{L}{\Pr(\Pr-Le)}\exp(-(Le+\Pr)\eta) \tag{C27}$$

where the constants F, G, H, I, J, K and L are defined as

$$\mathrm{F}=-\left[\frac{G}{1+Le}+\frac{H}{\Pr(\Pr-Le)}+\frac{I}{2(2+Le)}+\frac{J}{(1+\Pr)^2-Le(1+\Pr)}+\frac{K}{2\Pr(2\Pr-Le)}+\frac{L}{\Pr(\Pr-Le)}\right],$$
$$G=hLe^2\left(1+h-\frac{hLe^2}{(Le+1)}-\frac{h\Pr(Nt/Nb)}{\Pr-Le}\right),\quad H=h\Pr^2(Nt/Nb)\left(1+h-\frac{h\Pr^2}{(\Pr+1)}-\frac{h\Pr^2 Nb}{\Pr+Le}-\frac{h\Pr Nt}{(2)}\right),$$
$$I=h^2Le^3,\ J=h^2\Pr^2(Nt/Nb)\left((\Pr+1)+\frac{Le}{\Pr-Le}\right),\ K=\left(2h^2\Pr^3(Nt/Nb)Nt\right),$$
$$L=\left(h^2\Pr^2(Nt/Nb)Nb(\Pr+Le)\right)$$

Substituting in (C6) and taking the limit as p tends to 1 we get

$$\phi(\eta)=\phi_0(\eta)+\phi_1(\eta)+\phi_2(\eta) \tag{C28}$$

$$\begin{aligned}\phi(\eta)=\, & exp(-Le\ \eta)-h\left(\frac{Le^2}{Le+1}+\frac{Pr\,(Nt/Nb)}{Pr-Le}\right)exp(-Le\,\eta)+ \\ & h\left(\frac{Le^2}{Le+1}exp(-(Le+1)\eta)+\frac{Pr\,(Nt/Nb)}{Pr-Le}exp(-Pr\,\eta)\right)+F\ exp(-Le\,\eta)+\frac{G}{1+Le} \\ & exp(-(1+Le)\eta)+\frac{H}{Pr(Pr-Le)}exp(-Pr\,\eta)+\frac{I}{2(2+Le)}exp(-(2+Le)\eta) \\ & +\frac{J}{(1+Pr)^2-Le(1+Pr)}exp(-(1+Pr)\eta)+\frac{K}{2\,Pr(2\,Pr-Le)}exp(-2\,Pr\,\eta) \\ & +\frac{L}{Pr(Pr-Le)}exp(-(Le+Pr)\eta)\end{aligned} \tag{C29}$$

By taking

$$B_1=\left(1-h\left(\frac{Le^2}{Le+1}+\frac{\mathrm{Pr}(Nt/Nb)}{\mathrm{Pr}-Le}\right)+F\right),\qquad B_5=h\left(\frac{\mathrm{Pr}(Nt/Nb)}{\mathrm{Pr}-Le}\right)+\frac{H}{\mathrm{Pr}(\mathrm{Pr}-Le)}$$

$$B_2=h\left(\frac{Le^2}{Le+1}\right)+\left(\frac{G}{1+Le}\right),\qquad B_6=\left(\frac{K}{2\,\mathrm{Pr}(2\,\mathrm{Pr}-Le)}\right),$$

$$B_3=\left(\frac{I}{2(2+Le)}\right),\ B_4=\left(\frac{L}{\mathrm{Pr}(\mathrm{Pr}-Le)}\right),\ B_7=\left(\frac{J}{(1+\mathrm{Pr})^2-Le(1+\mathrm{Pr})}\right) \tag{C30}$$

the concentration profile is given in (10)

APPENDIX D: Determining the validity region of h

The analytical solution represented by (13), (14), (15) and (16) contains the auxiliary parameter h, which gives the convergence region and rate of approximation for homotopy analysis method. The analytical solution should converge. It should be noted that the auxiliary parameter h controls the convergence and accuracy of the solution series. In order to define region such that the solution series is independent of h, a multiple of h curves are plotted. The region where the distribution of θ and ϕ versus h is a horizontal line is known as the convergence region for the corresponding function. The common region among concentrations is known as the overall convergence region. To study the influence of h on the convergence of solution, the h curves of θ (Pr =1, Le = 5, Nt=Nb=0.1) and ϕ (Pr =1, Le = 5, Nt=Nb=0.1) are plotted in Figure 9. This figure clearly indicates that the valid region of h is about (−0.77). Similarly we can find the value of the convergence-control parameter h for different values of constant parameters.

APPENDIX E: MATLAB program to find the numerical solution of eqns(1)-(4).

```
function sol = ex7
ex7 init=bvp init(linspace(0,10,11),[0 1 0 1 -1 1 -2.5]);
sol=bvp4c(@ex7ode,@ex7bc,ex7init)
end
function dydx=ex7 ode(x,y)
Pr=1;
Le=5;
Nt=0.1;
Nb=0.1;
dydx=[
y(2)
y(3)
    -y(1)*y(3)+y(2)*y(2)
y(5)
```

```
    (-y(1)*y(5)-Nb*y(7)*y(5)-Nt*y(5)*y(5))*Pr
y(7)
    -Le*y(1)*y(7)-Nt/Nb*(-y(1)*y(5)-Nb*y(7)*y(5)
    -Nt*y(5)*y(5))*Pr
    ];
end
function res=ex7bc(ya,yb)
res=[ya(1)
ya(2)-1
yb(2)
ya(4)-1
yb(4)
ya(6)-1
yb(6)];
end
```

To be typed in the command window

```
solution=ex7
   x=solution.x;
y=solution.y;
y1=solution.y(1,:);
y4=solution.y(4,:);
y6=solution.y(6,:);

Plot(x,y1,'r',x,y4,'g',x,y6,’b’);
```

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