International Journal of Mathematical Archive-9(1), 2018, 90-94
Available online through www.ijma.info ISSN 2229 – 5046

AN IMPROVEMENT ON AN APPROXIMATE FUNCTIONAL EQUATION FOR $e^{i\theta}\mathfrak{H}(\frac{1}{2}+it)$

K. PUNNIAKRISHNAN

Head of the Department of Mathematics,
Bharathidasan College of Arts and Science, Erode-638316, India.

(Received On: 12-12-17; Revised & Accepted On: 30-12-17)

ABSTRACT

In this paper the approximate functional equation for $e^{i\theta}\mathfrak{H}(\frac{1}{2}+it)$ due to E.C. Titchmarash has been analysed. A minor simplification of the above equation has been obtained. New forms of the above equation in a similar way are derived.

Keywords: *Riemann zeta function, Functional equation, Approximate functional equation.*

AMS Code: *30D05, 97I80.*

INTRODUCTION

One of the important theories in the study of complex analysis is the theory of Riemann zeta function. In 1921 and subsequently, Hardy and title wood [5,6,7] developed the approximate functional equation for the Riemann zeta function. They regarded the functional equation as a "Compromise" between the series expansion $\mathfrak{H}(s) = \sum_{n=1}^{\infty} n^{-s}$ and the functional equation $\mathfrak{H}(s) = \chi(s)\mathfrak{H}(1-s)$ [1, 2, 3]. This paper contains, the approximate functional equation as given by E.C.Titchmarsh in his book, "The theory of the Riemann zeta-function" published in 1951 [10, 11, 12].

This paper is useful to understand and further simplify a theory of E.C.Titchmarsh on the mean square of $\left|\mathfrak{H}(\frac{1}{2}+it)\right|$.

DEFINITION OF $\mathfrak{H}(s)$: The Riemann zeta function $\mathfrak{H}(s)$ has its origin in the identity expressed by the two formulae

$$\mathfrak{H}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where n runs through all intervals and

$$\mathfrak{H}(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

where p runs through all primes and s is a complex variable $s = \sigma + it$

Theorem: Let f(x) be a real function with continuous derivatives upto the third order. Let $f^{'}(x)$ be steadily decreasing in $a \le x \le b$ and $f^{'}(b) = \alpha, f^{'}(a) = \beta$. Let x_V be defined by $f^{'}(x_V) = V < \alpha < V \le \beta >$. Let $2\pi\lambda_2 \le f^{''}(x) < A\lambda_2$, $\left|f^{'''}(x)\right| < A\lambda_3$. Then

$$\sum_{a<n\le b} e^{2\pi i f(n)} = e^{-\frac{\pi i}{4}} \sum_{\alpha<V\le\beta} \frac{e^{2\pi i < f(x_V) - V x_V >}}{|f^{''}(x_V)|^{\frac{1}{2}}} + O\left(\lambda_2^{-\frac{1}{2}}\right) + O < \log[2 + (b-a)\lambda_2] > + O < [(b-a)\lambda_2^{1/5}\lambda_3^{1/5}] > \quad (1)$$

The general form of the approximate functional equation

$$\mathfrak{H}(s) = \sum_{n\le x} \frac{1}{n^s} + \chi(s) \sum_{h\le y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O < |t|^{\frac{1}{2}-\sigma} y^{\sigma-1} > \quad for\ 0 < \sigma < 1 \quad (2)$$

Where $\mathfrak{H}(s) = \chi(s)\mathfrak{H}$ (1-S), the functional equation $\chi(S) = \frac{2^{s-1}\pi^s \sec\left(\frac{s\pi}{2}\right)}{\Gamma(s)}$ [4,13,14]

Corresponding Author: K. Punniakrishnan
Head of the Department of Mathematics,
Bharathidasan College of Arts and Science, Erode-638316, India.

We use (1) with an extra factor g(n) in the sum and if we ignore error terms for the moment. Then

$$\sum_{a<n\le b} g(n)e^{2\pi i f(n)} \approx e^{-\frac{\pi i}{4}}\sum_{\alpha<v\le\beta}\frac{e^{2\pi i\{f(x_v)-Vx_v\}}}{[f''(x_V)]^{1/2}}\,g(x_v)$$

Taking g(u)= $u^{-\sigma}, f(u)=\frac{t\log u}{2\pi}$

$$f'(u)=\frac{t}{2\pi u},\qquad f''(u)=\frac{-t}{2\pi u^2}$$

$$x_V=\frac{t}{2\pi v},\quad f(x_V)=\frac{t\log(x_V)}{2\pi},\quad f''(x_V)=\frac{-2\pi v^2}{t}$$

We have $\alpha=f'(b)=\frac{t}{2\pi b}$, $\beta=f'(a)=\frac{t}{2\pi a}$

$$g(x_V)=(x_V)^{-\sigma}$$
$$=(\frac{t}{2\pi v})^{-\sigma}$$

and consider the functional equation $\mathfrak{H}(s)=\chi(s)\mathfrak{H}(1-s)$ *where*

$$\chi(s)=\frac{2^{s-1}\pi^s\sec(\frac{s\pi}{2})}{\Gamma(s)}\ and\ I_n\ any\ fixed\ strip\ \alpha\le\sigma\le\beta\ as\ t\to\infty$$

Replacing a, b by x, N and i by –i we obtain

$$\mathfrak{H}(s)=\sum_{n\le x}\frac{1}{n^s}+\chi(s)\sum_{n\le y}\frac{1}{n^{1-s}}+O(x^{-\sigma})+O<|t|^{\frac{1}{2}-\sigma}Y^{\sigma-1}>for\ 0<\sigma<1$$

This is known as the approximate functional equation (3)

As a special case of the approximate functional equation we have the following theorem

Theorem: We have $e^{i\theta}\,\mathfrak{H}\left(\frac{1}{2}+it\right)=2\sum_{n=1}^{m}\frac{\cos(\theta_1-t\log n)}{\sqrt{n}}$ + $0(t^{-1/4})$ (4)
where

$$\theta=\left(-\frac{1}{2}\right)a_m\chi<\frac{1}{2}+it>=(1/2)t\log<\frac{t}{2\pi}>-\frac{t}{2}-\frac{\pi}{\theta}+o\left(\frac{1}{t}\right)\theta_{1=\frac{1}{2}t\log\left(\frac{t}{2\pi}\right)-\frac{t}{2}-\pi/\theta}$$

m = $\sqrt{(\frac{t}{2\pi})}$ and $\chi(s)=\pi^{s-(\frac{1}{2})}\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})}$

Proof: Consider the approximate functional equation (3)

Taking $\sigma=\frac{1}{2}$ and X = Y = $\left(\frac{t}{2\pi}\right)^{\frac{1}{2}}$

We get

$$\mathfrak{H}(\tfrac{1}{2}+it)=\sum_{n\le x}n^{-\frac{1}{2}-it}+\ \chi<\frac{1}{2}+it>\sum_{n\le x}n^{-\frac{1}{2}+it}+O\left(x^{-\frac{1}{2}}\right)+O<|t|^{<\frac{1}{2}>--(\frac{1}{2})}y^{\frac{1}{2}-1}>$$
$$=\sum_{n\le x}n^{-\frac{1}{2}-it}+\chi\left(\frac{1}{2}+it\right)\sum_{n\le x}n^{-\frac{1}{2}+it}+O<\{\left(\frac{t}{2\pi}\right)^{\frac{1}{2}}\}^{-1/2}>+O<\{\left(\frac{t}{2\pi}\right)^{\frac{1}{2}}\}^{-1/2}>$$
$$=\sum_{n\le x}n^{-\frac{1}{2}-it}+\chi<\frac{1}{2}+it>\sum_{n\le x}n^{-\frac{1}{2}+it}+O(t^{-\frac{1}{4}})\quad[8,9]\qquad(5)$$

This can also be put into another form which is sometimes useful, we have

$$\chi\left(\frac{1}{2}+it\right)\chi\left(\frac{1}{2}-it\right)=1$$

Then $|\chi(\frac{1}{2}+it)|=1$

Let $\theta=\theta(t)=-(\frac{1}{2})$ arg $\chi(\frac{1}{2}+it)$ then

$$-2\theta=\arg\chi<\frac{1}{2}+it>$$
$$\chi\left(\frac{1}{2}+it\right)=|\chi(\frac{1}{2}+it|e^{-i2\theta}=e^{-i2\theta}\qquad(6)$$

We have $e^{i\theta}\mathfrak{H}<\frac{1}{2}+it>=[\chi(1/2+it)]^{-1/2}\,\mathfrak{H}\left(\frac{1}{2}+it\right)$ (7)

Now $\chi(s)=\frac{\pi^{s-1/2}\Gamma<\frac{1}{2}-\frac{s}{2}>}{\Gamma(\frac{s}{2})}$

Then $\chi\left(\frac{1}{2}+it\right) = \pi^{\frac{1}{2}+it-1/2}\frac{\Gamma<\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}+it\right)>}{\Gamma<\frac{1}{2}\left(\frac{1}{2}+it\right)>}$

$$= \frac{\pi^{it}\Gamma<\frac{1}{2}-\frac{1}{4}-\frac{it}{2}>}{\Gamma<\frac{1}{4}+\frac{it}{2}>}$$

$$= \frac{\pi^{it}\Gamma<\frac{1}{4}-\frac{it}{2}>}{\Gamma(\frac{1}{4}+\frac{it}{2}>} \quad (8)$$

$$[\chi(1/2+it)]^{-1/2} = \pi^{-it/2}\left[\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}\right]^{-\frac{1}{2}}$$

$$= \pi^{-it/2}\left[\frac{[\Gamma(\frac{1}{4}+\frac{it}{2})]^{1/2}[\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)]^{1/2}}{[\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)]^{1/2}[\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)]^{1/2}}\right] = \pi^{-it/2}\frac{\Gamma<\frac{1}{4}+\frac{it}{2}>}{[\Gamma(\frac{1}{4}-\frac{it}{2})\Gamma(\frac{1}{4}+\frac{it}{2}]^{1/2}}$$

$$= \pi^{-it/2}\frac{\Gamma(\frac{1}{4}+\frac{it}{2})}{[|\Gamma(\frac{1}{4}+\frac{it}{2})|^2]^{1/2}} \quad (9)$$

We have $\xi(s) = (1/2\_s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\mathfrak{H}(s)$

Then $\xi<\frac{1}{2}+it>=\left(\frac{1}{2}\right)<\frac{1}{2}+it><\frac{1}{2}+it-1>\pi^{(-\frac{1}{2})(\frac{1}{2}+it)}\Gamma\left\{\left(\frac{1}{2}\right)\left(\frac{1}{2}+it\right)\right\}\mathfrak{H}\left(\frac{1}{2}+it\right)$

$$= <1/2><1/2+it><it-1/2>\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma<\frac{1}{4}+\frac{it}{2}>\mathfrak{H}<\frac{1}{2}+it>$$

$$= <1/2><1/2+it><it-1/2>\pi^{<-\frac{1}{4}-\frac{it}{2}>}\Gamma<\frac{1}{4}+\frac{it}{2}>\mathfrak{H}\left(\frac{1}{2}+it\right) \quad (10)$$

$$\mathfrak{H}\left(\frac{1}{2}+it\right) = \frac{\xi<\frac{1}{2}+it>}{<\frac{1}{2}>\left\{-\left(t^2+\frac{1}{4}\right)\right\}\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma(\frac{1}{4}+\frac{it}{2})}$$

$$= \frac{-2\pi^{\frac{1}{4}}\pi^{\frac{it}{2}}\xi(\frac{1}{2}+it)}{\left(t^2+\frac{1}{4}\right)\Gamma(\frac{1}{4}+\frac{it}{2})} \quad (11)$$

Using (9) & (11) in (7). We get

$$e^{i\theta}\mathfrak{H}<\frac{1}{2}+it>=\pi^{-it/2}\frac{\Gamma(\frac{1}{4}+\frac{it}{2})}{|\Gamma<\frac{1}{4}+\frac{it}{2}>|}\frac{-2\pi^{1/4}\pi^{it/2}\xi<\frac{1}{2}+it>}{<t^2+\frac{1}{4}>\Gamma<\frac{1}{4}+\frac{it}{2}>}$$

$$= \frac{-2\pi^{1/4}}{t^2+1/4}\frac{\xi(\frac{1}{2}+it)}{|\Gamma<\frac{1}{4}+\frac{it}{2}>|}$$

$$= \frac{-2\pi^{1/4}}{t^2+1/4}\frac{\Sigma(t)}{|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)|} \quad (12)$$

Where $\quad \Sigma(t) = \xi\left(\frac{1}{2}+it\right)$

Thus the function $e^{i\theta}\mathfrak{H}\left(\frac{1}{2}+It\right)$ is real for real t and from (7), we have

$$|e^{i\theta}\mathfrak{H}\left(\frac{1}{2}+it\right)| = |\mathfrak{H}\left(\frac{1}{2}+it\right)|$$

Multiplying (5) by $e^{i\theta}$, we get

$$e^{i\theta}\,\mathfrak{H}\left(\frac{1}{2}+it\right) = e^{i\theta}\sum_{n\le x}n^{-\frac{1}{2}-it} + e^{i\theta}.e^{-2i\theta}\sum_{n\le x}n^{-\frac{1}{2}+it} + O(t^{-\frac{1}{4}})$$

$$= e^{i\theta}\sum_{n\le x}n^{-\frac{1}{2}-it} + e^{-i\theta}\sum_{n\le x}n^{-\frac{1}{2}+it} + O(t^{-\frac{1}{4}})$$

$$= \sum_{n\le x}e^{i\theta}n^{-1/2}n^{-it} + \sum_{n\le x}e^{-i\theta}n^{-1/2}n^{it} + O(t^{-\frac{1}{4}})$$

$$= \sum_{n\le x}[e^{i\theta}n^{-\frac{1}{2}}n^{-it} + e^{-i\theta}n^{-\frac{1}{2}}n^{it}] + O(t^{-\frac{1}{4}})$$

$$= \sum_{n\le x}[e^{i\theta}n^{-1/2}e^{logn^{-it}} + e^{-i\theta}n^{-1/2}e^{logn^{it}}] + O(t^{-\frac{1}{4}})$$

$$= \sum_{n\le x}n^{-1/2}\left[e^{i(\theta-tlog\,n)} + e^{-i(\theta-tlog\,n)}\right] + O(t^{-\frac{1}{4}})$$

$$= \sum_{n\le x}n^{-\frac{1}{2}}[2\cos(\theta-tlogn)] + O\left(t^{-\frac{1}{4}}\right)$$

$$= 2\sum_{n\le x}n^{-\frac{1}{2}}[\cos(\theta-tlogn)] + O\left(t^{-\frac{1}{4}}\right)$$

$$= 2\sum_{n=1}^{m}n^{-\frac{1}{2}}[\cos(\theta-tlogn] + O(t^{-\frac{1}{4}})$$

$$= 2\sum_{n=1}^{m}[\frac{\cos(\theta-tlogn)}{n^{\frac{1}{2}}}] + O\left(t^{-\frac{1}{4}}\right) \quad (13)$$

Where m=[x] and $\theta = -1/2\ \arg\,\chi(\frac{1}{2}+it)$

Taking logarithm on both sides of (8) and applying

$$\log\Gamma(\sigma+it)=\left(\sigma+it-\frac{1}{2}\right)\log it-it+\frac{1}{2}(log2\pi)+O\left(\frac{1}{t}\right) \tag{14}$$

We get

$$\log\chi\left(\frac{1}{2}+it\right)=it\log\pi+log\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)-log\Gamma(\frac{1}{4}+\frac{it}{2})$$

$$=\text{itlog}\pi+\left\{<\left(\frac{1}{4}-\frac{it}{2}\right)-\frac{1}{2}><\log\left(-\frac{it}{2}\right)>+\left(\frac{it}{2}\right)+\frac{1}{2}\,log2\pi+O\left(\frac{1}{t}\right)\right\}-\{<\frac{1}{4}+\frac{it}{2}-\frac{1}{2}>\log\left(\frac{it}{2}\right)$$
$$-\left(\frac{it}{2}\right)+\left(\frac{1}{2}\right)\log 2\pi+O(\frac{1}{t})\}$$

$$=\text{itlog}\pi+\{<\frac{-1}{4}-\frac{it}{2}>\log\left(-\frac{it}{2}\right)+\left(\frac{it}{2}\right)+\left(\frac{1}{2}\right)\log\ 2\pi+O(\frac{1}{t})\}-\{\left(-\frac{1}{4}+\frac{it}{2}\right)\log\left(\frac{it}{2}\right)-\left(\frac{it}{2}\right)+\frac{1}{2}\log 2\pi+O(\frac{1}{t})\}$$

$$=\text{itlog}\pi-\left(\frac{1}{4}\right)\log\left(-\frac{it}{2}\right)-\left(\frac{it}{2}\right)\log\left(-\frac{it}{2}\right)+\left(\frac{it}{2}\right)+\frac{1}{2}\log 2\pi+O\left(\frac{1}{t}\right)+\left(\frac{1}{4}\right)\log\left(\frac{it}{2}\right)-\left(\frac{it}{2}\right)\log\left(\frac{it}{2}\right)+$$
$it2-12\log 2\pi+O(1t)$

$$=\text{itlog}\pi-\frac{1}{4}\log<(\frac{t}{2})e^{-i\pi/2}>-(it/2)\log<(t/2)e^{-i\pi/2}>+<\frac{it}{2}>+\left(\frac{1}{4}\right)log<\left(\frac{t}{2}\right)e^{\frac{i\pi}{2}}>-\left(\frac{it}{2}\right)log<\left(\frac{t}{2}\right)e^{\frac{i\pi}{2}}>+\left(\frac{it}{2}\right)+O(\frac{1}{t})$$

$$=\text{itlog}\pi-\left(\frac{1}{4}\right)\{\log\left(\frac{t}{2}\right)+\log e^{-i\pi/2}\}-\left(\frac{it}{2}\right)\{\log\left(\frac{t}{2}\right)+\log e^{-i\pi/2}\}+\left(\frac{it}{2}\right)+\left(\frac{1}{4}\right)[\log\left(\frac{t}{2}\right)+\log e^{i\pi/2}\}$$
$$-(it/2)\{\log(t/2)+\log e^{i\pi/2}\}+\left(\frac{it}{2}\right)+O(\frac{1}{t})$$

$$=\text{it}\log\pi-\left(\frac{1}{4}\right)\log\left(\frac{t}{2}\right)-\left(\frac{1}{4}\right)\left(-\frac{i\pi}{2}\right)-\left(\frac{it}{2}\right)\log\left(\frac{t}{2}\right)-\left(\frac{it}{2}\right)\left(-\frac{i\pi}{2}\right)+it+\left(\frac{1}{4}\right)\log\left(\frac{t}{2}\right)+\left(\frac{1}{4}\right)\left(\frac{i\pi}{2}\right)$$
$$-\left(\frac{it}{2}\right)\log\left(\frac{t}{2}\right)-\left(\frac{it}{2}\right)\left(\frac{i\pi}{2}\right)+O(\frac{1}{t})$$

$$=\text{itlog}\pi-\left(\frac{1}{4}\right)\log\left(\frac{t}{2}\right)+i\pi/8)-(it/2)\log(t/2)-t\pi/4+it+(1/4)\log(t/2)+i\pi/8-(it/2)\log(t/2)$$
$$+(t\pi/4)+O(1/t)$$

$$=\text{it}\log\pi+\frac{i\pi}{4}-it\log\left(\frac{t}{2}\right)+it+O(\frac{1}{t})$$

$$\arg\chi<\frac{1}{2}+it>=t\,log\pi+\frac{\pi}{4}-tlog\left(\frac{t}{2}\right)+t+O(\frac{1}{t})=\text{t}+\frac{\pi}{4}-t\left\{\log\left(\frac{t}{2}\right)-log\pi\right\}+O(\frac{1}{t})$$
$$=\text{t}+\frac{\pi}{4}-t\log\left(\frac{t}{2\pi}\right)+O(\frac{1}{t})$$

But -2$\theta=\arg\chi(\frac{1}{2}+it)$

$$\theta=\left(-\frac{1}{2}\right)\left\{t+\frac{\pi}{4}-tlog\left(\frac{t}{2\pi}\right)\right\}+O(\frac{1}{t})$$
$$=\frac{-t}{2}-\frac{\pi}{8}+\left(\frac{t}{2}\right)\log\left(\frac{t}{2\pi}\right)+O(\frac{1}{t})$$
$$=(t/2)\log(t/2\pi)-\frac{t}{2}-\frac{\pi}{8}+O(\frac{1}{t})$$

We can replace θ by $\theta_1=\left(\frac{t}{2}\right)\log\left(\frac{t}{2\pi}\right)-\frac{t}{2}-\pi/8$

With an error

$$=O[\sum_{n=1}^{m}\frac{\{cos\ (\theta-t\ log\ n)-(cos\theta_1-tlog\ n)\}}{\sqrt{n}}]$$
$$=O[\sum_{n=1}^{m}\frac{\sin\left(\frac{\theta}{t}\right)\sin\{\frac{\theta+\theta_1}{2}-t\log n\}}{\sqrt{n}}]$$
$$=O[\sum_{n=1}^{m}\frac{1}{\sqrt{n}t}]$$
$$=O[\frac{m^{1/2}}{t}]$$
$$=O[\frac{t^{1/4}}{t}]$$
$$=O(t^{-3/4}) \tag{15}$$

The theorem now follows using (15) in (14).

Remark: We can replace θ by $\theta_1$this in the other special form of the approximate functional equation in a similar way.

REFERENCES

1. Ahlfors.L.V "Complex Analysis" Mc Graw Hill, New York, 1979.
2. Aswin.A "A functional equation from the theory of the Riemann $\mathfrak{H}$(s) - function" Annals of Math. (2), 2d, 359-366, 1923.
3. Churchill.V.Ruel Brown.W.James and Roges.F.Verhery "Complex variables and Applications" Third Edition, Mc Graw-Hill Kogakusha Ltd,1934.
4. Creightou Buck.R "Advanced Calculus" International Student Edition, Mc Graw-Hill, Kogakusha Ltd, 1965.
5. Hardy.G.H "On the average order of the arithmetical functions P(u) and Δ(u)"P.L.M.S,(2),15,192-213, 1915.
6. Mordell.L.J "Some applications of fourier series in the analytic theory of numbers", P.C.P.S, 34,585-596, 1928.
7. Nevanlinna Rolf Paatero.V "Introduction to complex analysis" Addison-wesley publishing company, London,1969.
8. Paley.R.E.A.C and Wienner.N "Note on the theory and application of fourier transforms" V.Trans Ameer, Math.Soc.35, 768-781,1933.
9. Phillips,Eric "A note on the zeros of $\mathfrak{H}$(s)", Q.J.O.6,139-145, 1935.
10. Ramanujan.S, "New expression for Riemann's functions E(s) and I(t)" Quart. J of Math. 46, 253-261, 1951.
11. Thiruvenkatachaya.V "On some properties of the zeta-function", Journal Indian Math. Soc.19, 92-106, 1931.
12. Titchmarasu.E.C "The theory of the Riemann Zeta-function" Oxford, At the clarendon press, 1951.
13. Walter Rudin "Principles of mathematical analysis", Third Edition, Mc Graw-Hill, International book company, 1976.
14. Wilton.J.R "Note on the zeros of Riemann's $\mathfrak{H}$ -function" Messenger of Math, 45 c, 180-193, 1915.

Source of support: Nil, Conflict of interest: None Declared.