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FRACTIONAL INTEGRALS INVOLVING ℵ-FUNCTION AND THE GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

The object of the present paper is to study a number of new and useful fractional integrals involving a product of three general class of polynomials and the Aleph(ℵ)- function with $x^k (x^h + d^h)^{-w}$ as general arguments in view of both the operators introduced by Saigo in 1978.

The fractional integrals established here are quite general in character and on specializing the parameters of Aleph(ℵ)- function and the arbitrary coefficients occurring in three general class of polynomials, a large number of fractional integrals involving various classes of orthogonal polynomials, generalized hypergeometric polynomials and elementary functions (or product of several such functions) can be obtained from them. Thus, our results provide interesting unifications and extensions of a number of known and new results. Some special cases have also been recorded.

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Key words: *Aleph function, Fractional Calculus, Mellin-Barnes type integrals, General class of Polynomials, H-function, I-function.*

1. INTRODUCTION AND PRELIMINARIES

The Aleph-function is a new generalization of the familiar H-function [11] and the I-function [15].

The Aleph-function is defined by means of a Mellin-Barnes type integral in the following manner [13, 15]:

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z \left| \begin{matrix} (a_j,A_j)_{1,n},\dots,[\tau_j(a_j,A_j)]_{n+1,p_i} \\ (b_j,B_j)_{1,m},\dots,[\tau_j(b_j,B_j)]_{m+1,q_i} \end{matrix} \right. \right] := \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)\, z^{-s}\, ds, \tag{1}$$

where $z \neq 0$, $i = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s).\prod_{j=1}^{n}\Gamma(1-a_j - A_j s)}{\sum_{i=1}^{r}\tau_i \prod_{j=m+1}^{q_i}\Gamma(1-b_{ji} - B_{ji}s).\prod_{j=n+1}^{p_i}\Gamma(a_{ji} + A_{ji}s)}. \tag{2}$$

The integration path $L = L_{i\gamma\infty}$, $(\gamma \in \Re)$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles of $\Gamma(1-a_j - A_j s)$, $j = (\overline{1,n})$ (the symbol $(\overline{1,n})$ is used for 1, 2,…,n) do not coincide with the poles of $\Gamma(b_j + B_j s)$, $j = (\overline{1,m})$. The parameters p_i, q_i are non-negative integers satisfying the condition $0 \le n \le p_i$, $1 \le m \le q_i$, $\tau_i > 0$ for $i = \overline{1,r}$. The parameters $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$. The empty product in (2) is interpreted as unity. The existence conditions for the defining integral (1.1) are given below:

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$$\varphi_l > 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_l, \; l = \overline{1,r}; \tag{3}$$

$$\varphi_l \geq 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_l \text{ and } \Re\{\zeta_l\} + 1 < 0, \tag{4}$$

Where $$\varphi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l \left(\sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl} \right) \tag{5}$$

$$\zeta_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl} \right) + \frac{1}{2}(p_l - q_l), \quad l = \overline{1,r}. \tag{6}$$

Remark 1: For $\tau_i = 1, \; i = \overline{1,r}$ in (1), we get the I-function due to V.P. Saxena [18], defined in the following manner:

$$I^{m,n}_{p_i,q_i;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}\left[z \left| \begin{matrix} (a_j,A_j)_{1,n},\ldots,(a_j,A_j)_{n+1,p_i} \\ (b_j,B_j)_{1,m},\ldots,(b_j,B_j)_{m+1,q_i} \end{matrix} \right. \right] := \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,1;r}(s) z^{-s}\, ds, \tag{7}$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;r}(s)$ is given in (2). The existence conditions for the integral in (7) are the same as given in (3) - (6) with $\tau_i = 1, \; i = \overline{1,r}$.

Remark 2: If we further set r = 1, then (7) reduces to the familiar H- function [3,10,12]:

$$H^{m,n}_{p,q}[z] = \aleph^{m,n}_{p_i,q_i,1;1}[z] = \aleph^{m,n}_{p_i,q_i,1;1}\left[z \left| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix} \right. \right] := \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,1;1}(s)\, z^{-s}\, ds, \tag{8}$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;1}(s)$ can be obtained from (2).

Remark 3: Fractional integration of Aleph function is discussed by Saxena and Pogány [16]. A detailed account of ℵ-function is given in the papers by Saxena *et.al* [9, 16, 17].

The series representation of Aleph(ℵ)- function is given by

$$\aleph^{m',n'}_{p_i',q_i',c_i';r'}[u] = \aleph^{m',n'}_{p_i',q_i',c_i';r'}\left[u \left| \begin{matrix} (a_j',A_j')_{1,n'}, \left[c_i'(a_{ji}',A_{ji}')_{n'+1,p_i';r'}\right] \\ (b_j',B_j')_{1,m'}, \left[c_i'(b_{ji}',B_{ji}')_{m'+1,q_i';r'}\right] \end{matrix} \right. \right]$$
$$= \sum_{h=1}^{m'} \sum_{k'=0}^{\infty} \frac{(-1)^{k'} \phi'(\eta_{h,k'})\, u^{-\eta_{h,k'}}}{B_h'\, k'!}, \tag{9}$$

where $$\phi'(\eta_{h,k'}) = \frac{\prod_{j=1}^{m'} \Gamma(b_j' + B_j'\eta_{h,k'}) \prod_{j=1}^{n'} \Gamma(1 - a_j' - A_j'\eta_{h,k'})}{\sum_{i=1}^{r'} c_i' \left\{ \prod_{j=m'+1}^{q_i'} \Gamma(1 - b_{ji}' - B_{ji}'\eta_{h,k'}) \prod_{j=1}^{n'} \Gamma(a_{ji}' + A_{ji}'\eta_{h,k'}) \right\}}, \tag{10}$$

and $\eta_{h,k'} = \frac{b_h' + k'}{B_h'}$, $p_i' < q_i'$, $|u| < 1$.

FRACTIONAL INTEGRALS

Let α, β and η are complex numbers and let $y \in R_+ (0, \infty)$. Following [5, 14, 15] the fractional integral (Re (α) > 0) and derivative (Re (α) < 0) of the first kind of a function f(y) on R_+ are defined respectively in the following forms

$$I^{\alpha,\beta,\eta}_{0,y} f = \frac{y^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{y}\right) f(t)\,dt; \; \mathrm{Re}(\alpha) > 0 \tag{11}$$

$$= \frac{d^n}{dy^n} I^{\alpha+n,\beta-n,\eta-n}_{0,y} f, \; 0 < \mathrm{Re}(\alpha) + n \leq 1 \; (n = 1, 2, \ldots) \tag{12}$$

where ${}_2F_1$ (a, b; c; .) is Gauss's hypergeometric function.

The fractional integral (Re (α) > 0) and derivative (Re (α) < 0) of the second kind are given by

$$J_{y,\infty}^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_y^\infty (t-y)^{\alpha-1}\ t^{-\alpha-\beta}\ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right) f(t)\ dt,\ \text{Re}(\alpha) > 0 \quad (13)$$

$$= (-1)^n \frac{d^n}{dy^n} I_{y,\infty}^{\alpha+n,\beta-n,\eta} f,\ \ 0 < \text{Re}(\alpha)+n \le 1\ (n = 1,2,...) \quad (14)$$

A GENERAL CLASS OF POLYNOMIALS

Srivastava [19, p., Eq. (1), 17] introduced the general class of polynomials

$$S_{N_1}^{M_1}[x] = \sum_{K_1=0}^{[N_1/M_1]} \frac{(-N_1)_{M_1K_1}}{K_1!} A_{N_1,K_1} x^{K_1}, \qquad N_1 = 0,1,2,\ldots \ldots \quad (15)$$

where M_1 is an arbitrary positive integer and the coefficients $A_{N_1,K_1} (N_1, K_1 \ge 0)$ are arbitrary constants, real or complex.

Here $(\lambda)_n$ denotes the Pochhamer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$$

$$= \begin{cases} 1, & \text{if } n = 0 \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & \forall\ n \in \{1,2,3,\ldots\} \end{cases}$$

By suitably specializing the coefficients A_{N_1,K_1} occurring in (15), the general class polynomials $S_{N_1}^{M_1}[x]$ can be reduced to the classical orthogonal polynomials and the generalized hypergeometric polynomials.

$${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},\ \text{Re}(\gamma-\alpha-\beta) > 0,\ \text{Re}(\gamma) > 0 \quad (16)$$

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n\ n!} z^n \quad \ldots \quad (17)$$

2. THE FOLLOWING TWO LEMMAS ARE PROVED TO EMPLOY IN THE SEQUEL

Lemma 1: If $\text{Re}(a) > 0,\ k \in \Omega_{b,c}$, d is a positive number and h = 1, 2, 3, … and w, p, q are complex numbers, m_1, m_2 are arbitrary positive integers and the coefficients $A_{n_1,s_1}(n_1,s_1 \ge 0), A_{n_2,s_2}(n_2,s_2 \ge 0)$ are arbitrary constants, real or complex, then

$$I_{0,x}^{a,b,c}\left[x^k (x^h+d^h)^{-w} S_{n_1}^{m_1}[yx^{\alpha}(x^h+d^h)^{-p}] S_{n_2}^{m_2}[Dx^{\beta}(x^h+d^h)^{-q}]\right]$$

$$= \sum_{s_1=0}^{[n_1/m_1]} \sum_{s_2=0}^{[n_2/m_2]} \frac{(-n_1)_{m_1s_1}}{s_1!} A_{n_1,s_1} y^{s_1} \frac{(-n_2)_{m_2s_2}}{s_2!} A_{n_2,s_2} D^{s_2}$$

$$\cdot \frac{\Gamma(k+\alpha s_1+\beta s_2+1)\Gamma(k+\alpha s_1+\beta s_2-b+c+1)\, x^{k+\alpha s_1+\beta s_2-b}}{\Gamma(k+\alpha s_1+\beta s_2-b+1)\ \Gamma(k+\alpha s_1+\beta s_2+a+c+1)\, d^{hw+hps_1+hqs_2}}$$

$$\cdot {}_{2h+1}F_2\left[\begin{matrix} w+ps_1+qs_2, \frac{k+\alpha s_1+\beta s_2+1}{h},..., \frac{k+\alpha s_1+\beta s_2+h}{h}, \frac{k+\alpha s_1+\beta s_2-b+c+1}{h},..., \\ \frac{k+\alpha s_1+\beta s_2-b+1}{1},..., \frac{k+\alpha s_1+\beta s_2-b+h}{h}, \frac{a+k+\alpha s_1+\beta s_2+c+1}{h},..., \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{k+\alpha s_1+\beta s_2-b+c+h}{h}; \\ \frac{a+k+\alpha s_1+\beta s_2+c+h}{h}; \end{matrix} -\frac{x^h}{d^h}\right] \tag{18}$$

Proof: To establish Lemma 1, we first operate the fractional integral operator by (11) for

$$f(t) = t^k (t^h + d^h)^{-w} S_{n_1}^{m_1}[y\, t^{\alpha}(t^h + d^h)^{-p}] S_{n_2}^{m_2}[D t^{\beta}(t^h + d^h)^{-q}] \tag{19}$$

and express Gauss function by (17) and $S_{n_1}^{m_1}, S_{n_2}^{m_2}$ by (15) with $(t^h + d^h)^{-w}$ in series form by the formula

$$(t^h + d^h)^{-w} = d^{-hw} \sum_{\sigma=0}^{\infty} \frac{(w)_\sigma}{\sigma!}\left(-\frac{t^h}{d^h}\right)^{\sigma} \quad \ldots \tag{20}$$

Interchanging the order of summations and integration, this is permissible due to the absolute convergence involved in the process, evaluating the integral by the following result

$$\int_0^x (x-t)^{\alpha+m-1} t^{\lambda+kn}\, dt = x^{\alpha+m+\lambda+kn} \frac{\Gamma(\alpha+m)\,\Gamma(\lambda+kn+a)}{\Gamma(\alpha+m+\lambda+kn+1)} \tag{21}$$

With a little simplification, we get

$$\sum_{s_1=0}^{[n_1/m_1]} \sum_{s_2=0}^{[n_2/m_2]} \frac{(-n_1)_{m_1 s_1}}{s_1!} A_{n_1,s_1} y^{s_1} \frac{(-n_2)_{m_2 s_2}}{s_2!} A_{n_2,s_2} D^{s_2}$$

$$\cdot \frac{x^{k+\alpha s_1+\beta s_2-b}}{d^{hw+hps_1+hqs_2}} \frac{\Gamma(k+\alpha s_1+\beta s_2+1)}{\Gamma(a+\alpha s_1+\beta s_2+k+1)}$$

$$\cdot \sum_{\sigma=0}^{\infty} \frac{(k+\alpha s_1+\beta s_2+1)_{h\sigma} (w+ps_1+qs_2)_\sigma}{(a+\alpha s_1+\beta s_2+k+1)_{h\sigma}\ \sigma!}$$

$${}_2F_1\left[\begin{matrix} a+b\ , -c \\ a+k+\alpha s_1+\beta s_2+h\sigma+1\,; \end{matrix} ;\, 1\right]\left(-\frac{x^h}{d^h}\right)^{\sigma} \tag{22}$$

Then we get the desired result by applying Gauss theorem (17) and multiplication formula Rainville [13, cf. Theo.18] with a little simplification.

Lemma 2: If $\mathrm{Re}(a) > 0,\ k \in \Omega_{b,c}$, d is a positive number and h = 1, 2, 3, … and w, p, q are complex numbers m_1, m_2 are arbitrary positive integers and the coefficients $A_{n_1,s_1}(n_1, s_1 \geq 0), A_{n_2,s_2}(n_2, s_2 \geq 0)$ are arbitrary constants, real or complex, then

$$J_{x,\infty}^{a,b,c}\left[x^k (x^h + d^h)^{-w} S_{n_1}^{m_1}[yx^{\alpha}(x^h + d^h)^{-p}] S_{n_2}^{m_2}[Dx^{\beta}(x^h + d^h)^{-q}]\right]$$

$$= \sum_{s_1=0}^{[n_1/m_1]} \sum_{s_2=0}^{[n_2/m_2]} \frac{(-n_1)_{m_1 s_1}}{s_1!} A_{n_1,s_1} y^{s_1} \frac{(-n_2)_{m_2 s_2}}{s_2!} A_{n_2,s_2} D^{s_2}$$

$$\cdot \frac{x^{k+\alpha s_1+\beta s_2-b}}{d^{hw+hps_1+hqs_2}} \frac{\Gamma(b-k-\alpha s_1-\beta s_2)\Gamma(c-k-\alpha s_1-\beta s_2)}{\Gamma(-k-\alpha s_1-\beta s_2)\,\Gamma(a+b+c-k-\alpha s_1-\beta s_2)}$$

$$\cdot {}_{2h+1}F_2\left[\begin{array}{l} w+ps_1+qs_2, \frac{k+\alpha s_1+\beta s_2+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+h}{h},\frac{k+\alpha s_1+\beta s_2-a-b-c+1}{h},\ldots, \\ \frac{k+\alpha s+\beta s_2-b+1}{1} \qquad ,\ldots,\frac{k+\alpha s_1+\beta s_2-b+h}{h},\frac{k+\alpha s_1+\beta s_2-c+1}{h},\ldots, \\ \qquad \frac{k+\alpha s_1+\beta s_2-a-b-c+h}{h}; \ -\frac{x^h}{d^h} \\ \qquad \frac{k+\alpha s_1+\beta s_2-c+h}{h} \qquad ; \end{array}\right] \qquad \ldots \qquad (23)$$

Proof: To prove Lemma 2, we take

$$f(t) = t^k (t^h+d^h)^{-w} S_{n_1}^{m_1}[y\, t^{\alpha}(t^h+d^h)^{-p}] S_{n_2}^{m_2}[Dt^{\beta}(t^h+d^h)^{-q}]$$

in equation (13) and write series expansions for the Gauss function by (17) and $S_{n_1}^{m_1}, S_{n_2}^{m_2}$ by (15) with $(t^h+d^h)^{-w}$, then interchanging the order of integration and summations which is permissible due to the absolute convergence involved in the process, solve the integral by the following result

$$\int_x^{\infty} (t-x)^{\alpha+m-1} t^{\gamma-\alpha-\beta-m+kn}\,dt = x^{\gamma-\beta+kn} \frac{\Gamma(\alpha+m)\,\Gamma(\beta-\gamma-kn)}{\Gamma(\alpha+\beta+m-\gamma-kn)}, \qquad (24)$$

and using the relation

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n}, \qquad (25)$$

We get

$$\sum_{s_1=0}^{[n_1/m_1]} \sum_{s_2=0}^{[n_2/m_2]} \frac{(-n_1)_{m_1 s_1}}{s_1!} A_{n_1,s_1} y^{s_1} \frac{(-n_2)_{m_2 s_2}}{s_2!} A_{n_2,s_2} D^{s_2}$$

$$\cdot \quad x^{k+\alpha s_1+\beta s_2-b} d^{-hw-hps_1-hqs_2} \frac{\Gamma(b-k-\alpha s_1-\beta s_2)}{\Gamma(a+b-k-\alpha s_1-\beta s_2)}$$

$$\cdot \sum_{\sigma=0}^{\infty} \frac{(w+\beta s_1+qs_2)_{\sigma}}{\sigma!} \frac{(1-a-b+k+\alpha s_1+\beta s_2)_{h\sigma}}{(1-b+k+\alpha s_1+\beta s_2)_{h\sigma}}$$

$${}_2F_1\left[\begin{array}{l} a+b, -c \qquad ; \\ a+b-k-\alpha s_1-\beta s_2-h\sigma\ ; \end{array} 1\right]\left(-\frac{x^h}{d^h}\right)^{\sigma}. \qquad (26)$$

Now using Gauss theorem (4) and multiplication formula [13] and with a little simplification, we obtain the desired result.

3. THE FRACTIONAL INTEGRAL FORMULAE

If

$$f(t) = t^k (t^h+d^h)^{-w} S_{n_1}^{m_1}[y\, t^{\alpha}(t^h+d^h)^{-p}] S_{n_2}^{m_2}[Dt^{\beta}(t^h+d^h)^{-q}]$$

$$S_{n_3}^{m_3}[Et^{\gamma}(t^h+d^h)^{-r}].\,\aleph_{p_i',q_i',c_i';r'}^{m',n'}\left[zt^{\lambda}(t^h+d^h)^{-\eta}\left|\begin{array}{l}(a_j',A_j')_{1,n'},\left[c_i'(a_{ji}',A_{ji}')_{n'+1,p_i';r'}\right] \\ (b_j',B_j')_{1,m'},\left[c_i'(b_{ji}',B_{ji}')_{m'+1,q_i';r'}\right]\end{array}\right.\right]$$

then, we get

$$I_{0,x}^{a,b,c}[f(x)] = \sum_{s_1=0}^{[n_1/m_1]} \sum_{s_2=0}^{[n_2/m_2]} \sum_{s_3=0}^{[n_3/m_3]} \sum_{h=1}^{m'} \sum_{k'=0}^{\infty} \varphi(s_1,s_2,s_3,h,k')$$

$$\cdot \frac{x^{R-b}\,\Gamma(R+1)\,\Gamma(R+c-b+1)}{d^T\,\Gamma(R-b+1)\,\Gamma(R+c+a+1)}$$

$$\cdot {}_{2h+1}F_{2h}\left[\begin{matrix} w+ps_1+qs_2+rs_3-n\eta_{h,k'}, \frac{R+1}{h},\ldots,\frac{R+h}{h}, \frac{R+c-b+1}{h},\ldots,\frac{R+c-b+h}{h}\,; \\ \frac{R-b+1}{h},\ldots,\frac{R-b+h}{h}, \quad \frac{R+c+a+1}{h},\ldots,\frac{R+c+a+h}{h}\,; \end{matrix} -\frac{x^h}{d^h}\right] \quad (27)$$

where

$$\varphi(s_1,s_2,s_3,h,k') = \frac{(-n_1)_{m_1s_1} A_{n_1,s_1} (-n_2)_{m_2s_2} B_{n_2,s_2} (-n_3)_{m_3s_3} C_{n_3,s_3}}{s_1!\ \ s_2!\ \ s_3!}$$

$$\times \frac{(-1)^{k'}\, \varphi'(\eta_{h,k'})\, y^{s_1} D^{s_2} E^{s_3} z^{-\eta_{h,k'}}}{k'!\ B_{h'}},$$

$$R = k + \alpha s_1 + \beta s_2 + \gamma s_3 - \lambda\eta_{h,k'}$$

and $$T = hw + hps_1 + hqs_2 + hrs_3 - hn\eta_{h,k'} .$$

The result in (27) is valid for Re (a) > 0, $(k + \alpha s_1 + \beta s_2 + \gamma s_3 + \lambda f_i/F_i) \in \Omega_{b,c}$, i = 1, …, M. $|\arg z| < \frac{T'\pi}{2}, T' > 0$, d is a positive number and w, α, p, β, q, γ, r, λ, n are complex numbers, h = 1, 2, …, m_1, m_2, m_3 are arbitrary positive integers and the coefficients $A_{n_1,s_1}(n_1,s_1 \geq 0)$, $B_{n_2,s_2}(n_2,s_2(n_2,s_2 \geq 0)$, $C_{n_3,s_3}(n_3,s_3 \geq 0)$ are arbitrary constants, real or complex.

$$J_{x,\infty}^{a,b,c}[f(x)] = \sum_{s_1=0}^{[n_1/m_1]} \sum_{s_2=0}^{[n_2/m_2]} \sum_{s_3=0}^{[n_3/m_3]} \sum_{G=0}^{\infty} \sum_{g=1}^{M} \varphi(s_1,s_2,s_3,h,k')$$

$$\cdot \frac{x^{R-b}\,\Gamma(c-R)\,\Gamma(b-R)}{d^T\,\Gamma(a+b+c-R)\,\Gamma(-R)}$$

$$\cdot {}_{2h+1}F_{2h}\left[\begin{matrix} w+ps_1+qs_2+rs_3\text{-}n\eta_{h,k'}, \frac{R+1}{h},\ldots,\frac{R+h}{h}, \frac{R-a-b-c+1}{h},\ldots,\frac{R-a-b-c+h}{h}\,; \\ \frac{R-c+1}{h},\ldots,\frac{R-c+h}{h}, \quad \frac{R-b+1}{h},\ldots,\frac{R-b+h}{h}\,; \end{matrix} -\frac{x^h}{d^h}\right] \quad (28)$$

where d is a positive number and w, α, p, β, q, γ, r, λ, n are complex numbers, h = 1, 2, …, Re (a) > 0, $(k + \alpha s_1 + \beta s_2 + \gamma s_3 + \lambda f_i/F_i) \in \Omega_{b,\,c}$, i = 1, …, M, $|\arg z| < \frac{1}{2}T'\pi,\ T' > 0, m_1,\ m_2, m_3$ are arbitrary positive integers and the coefficients $A_{n_1,s_1}(n_1,s_1 \geq 0)$, $B_{n_2,s_2}(n_2,s_2 \geq 0)$, $C_{n_3,s_3}(n_3,s_3 \geq 0)$ are arbitrary constants, real or complex.

Proof: The proofs of the results (27) and (28) can be developed by proceeding on the lines similar to the proof of (18) and (23) respectively.

SPECIAL CASES

(I) If we take λ = 0 = n, the results in (27) and (28) reduces to the following formula

$$I_{0,x}^{a,b,c}\Big[x^k (x^h+d^h)^{-w} S_{n_1}^{m_1}[yx^{\alpha}(x^h+d^h)^{-p}]$$

$$.S_{n_2}^{m_2}[Dx^{\beta}(x^h+d^h)^{-q}]\, S_{n_3}^{m_3}[e\, x^{\gamma}(x^h+d^h)^{-r}]\Big]$$

$$=\frac{x^{k-b}}{d^{hw}}\sum_{s_1=0}^{[n_1/m_1]}\sum_{s_2=0}^{[n_2/m_2]}\sum_{b_3=0}^{[n_3/m_3]}\frac{(-n_1)_{m_1 s_1}(-n_2)_{m_2 s_2}(-n_3)_{m_3 s_3}}{s_1!\;\; s_2!\;\; s_3!}$$

$$.A_{n_1,s_1} B_{n_2,s_2} C_{n_3,s_3}\, y^{s_1} D^{s_2} E^{s_3}\left(\frac{x^{\alpha}}{d^{hp}}\right)^{s_1}\left(\frac{x^{\beta}}{d^{hq}}\right)^{s_2}\left(\frac{x^{\gamma}}{d^{hr}}\right)^{s_3}$$

$$.\frac{\Gamma(k+\alpha s_1+\beta s_2+\gamma s_3+1)\Gamma(k+\alpha s_1+\beta s_2+\gamma s_3+c-b+1)}{\Gamma(k+\alpha s_1+\beta s_2+\gamma s_3-b+1)\,\Gamma(k+\alpha s_1+\beta s_2+\gamma s_3+c+a+1)}$$

$$.{}_{2h+1}F_{2h}\left[\begin{matrix} w+ps_1+qs_2+rs_3,\ \frac{k+\alpha s_1+\beta s_2+\gamma s_3+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+\gamma s_3+h}{h}, \\ \frac{k+\alpha s_1+\beta s_2+\gamma s_3-b+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+\gamma s_3-b+h}{h}, \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{k+\alpha s_1+\beta s_2+\gamma s_3-b+c+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+\gamma s_3-b+c+h}{h}; \\ \frac{k+\alpha s_1+\beta s_2+\gamma s_3+c+a+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+\gamma s_3+c+a+h}{h}; \end{matrix}\ -\frac{x^h}{d^h}\right], \tag{29}$$

and

$$J_{x,\infty}^{a,b,c}\Big[x^k (x^h+d^h)^{-w} S_{n_1}^{m_1}[yx^{\alpha}(x^h+d^h)^{-p}]$$

$$.S_{n_2}^{m_2}[Dx^{\beta}(x^h+d^h)^{-q}]\, S_{n_3}^{m_3}[E\, x^{\gamma}(x^h+d^h)^{-r}]\Big]$$

$$=\frac{x^{k-b}}{d^{hw}}\sum_{s_1=0}^{[n_1/m_1]}\sum_{s_2=0}^{[n_2/m_2]}\sum_{b_3=0}^{[n_3/m_3]}\frac{(-n_1)_{m_1 s_1}(-n_2)_{m_2 s_2}(-n_3)_{m_3 s_3}}{s_1!\;\; s_2!\;\; s_3!}$$

$$.\quad A_{n_1,s_1} B_{n_2,s_2} C_{n_3,s_3}\, y^{s_1} D^{s_2} E^{s_3}\left(\frac{x^{\alpha}}{d^{hp}}\right)^{s_1}\left(\frac{x^{\beta}}{d^{hq}}\right)^{s_2}\left(\frac{x^{\gamma}}{d^{hr}}\right)^{s_3}$$

$$.\frac{\Gamma(c-k-\alpha s_1-\beta s_2-\gamma s_3)\Gamma(b-k-\alpha s_1-\beta s_2-\gamma s_3)}{\Gamma(a+b+c-k-\alpha s_1-\beta s_2-\gamma s_3)\,\Gamma(-k-\alpha s_1-\beta s_2-\gamma s_3)}$$

$${}_{2h+1}F_{2h}\left[\begin{matrix} w+ps_1+qs_2+rs_3,\ \frac{k+\alpha s_1+\beta s_2+\gamma s_3+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+\gamma s_3+h}{h}, \\ \frac{k+\alpha s_1+\beta s_2+\gamma s_3-c+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+\gamma s_3-c+h}{h}, \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{k+\alpha s_1+\beta s_2+\gamma s_3-b-a-c+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+\gamma s_3-a-b\ c+h}{h}; \\ \frac{k+\alpha s_1+\beta s_2+\gamma s_3-b+1}{h},\ldots,\frac{k+\alpha s_1+\beta s_2+\gamma s_3-b+h}{h}; \end{matrix}\ -\frac{x^h}{d^h}\right]. \tag{30}$$

(II) Letting $w \to 0$, $n_1 \to 0$ and $n_2 \to 0$, Lemma 1 and Lemma 2 are reduced to the result of Saigo and Raina [11] and for h = y = D = E = 1, Lemma 1 and 2 with results (29) and (30) are reduced to the results of Banerji and Choudhary [1].

(III) For $n_1 \to 0$, $n_2 \to 0$, Lemma 1 and Lemma 2 along with the results (27) and (28) and suitably specializing the arbitrary coefficients of general class of polynomials are reduced to the known results obtained by Chaurasia and Gupta [6].

(IV) For $\tau_i = 1,\ i = \overline{1, r}$ and further set r = 1, then the results (27) and (28) reduced to the results obtained by Chaurasia and Jain [2].

It may be of interest to remark that the formulae derived in this chapter may be useful in obtaining formulae for various classical orthogonal polynomials.

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