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HOMOTOPY PERTURBATION METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, Homotopy perturbation method is applied to solve some partial differential equations. Examples of PDEs are presented and give some useful help to solution of PDEs.

Keywords: *Homotopy perturbation method (HPM); Mathematical modeling; Partial differential equation.*

1. INTRODUCTION

In recent years, many mathematicians have developed the new techniques to find exact and approximate solutions for linear and nonlinear partial differential equations which describe in different fields. Some important methods have been appeared like homotopy perturbation method which is analytic technique for solving linear and nonlinear problems. The first mathematician proposed this was Ji-Huan in 1999[1]. This method gives analytical exact and approximate solutions of nonlinear partial differential equations easily without transforming the equation or linearizing the problem with very good results. Some mathematician's author has used the homotopy perturbation method for solving partial differential problem [2-10]. In this work, we represent the solution for partial differential equations by homotopy Perturbation method in linear, nonlinear and some type.

2. ANALYSIS OF HE'S HOMOTOPY PERTURBATION METHOD

To explain this method, we consider the following differential equation

$$D_o(u) - f(r) = 0, \qquad r \in \Omega \tag{1}$$

with the boundary conditions

$$B_o\left(u,\ \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{2}$$

where D_o is a general differential operator, B_o is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. In general, the operator D_o can be divided into a linear part L and a non-linear part N. Eqn. (1) can therefore be written as

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

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By the homotopy technique, we construct a homotopy $v(r,p):\Omega\times[0,1]\rightarrow\Re$ that satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[D_o(v)-f(r)]=0. \tag{4}$$

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0. \tag{5}$$

where $p\in[0,1]$ is an embedding parameter, and u_0 is an initial approximation of Eqn. (1) that satisfies the boundary conditions. From Eqns. (4) and (5), we have

$$H(v,0)=L(v)-L(u_0)=0 \tag{6}$$

$$H(v,1)=D_o(v)-f(r)=0 \tag{7}$$

When p=0, Eqn. (4) and Eqn. (5) become linear equations. When p =1, they become non-linear equations. The process of changing p from zero to unity is that of $L(v)-L(u_0)=0$ to $D_o(v)-f(r)=0$. We first use the embedding parameter p as a "small parameter" and assume that the solutions of Eqns. (4) and (5) can be written as a power series in p:

$$v=v_0+pv_1+p^2v_2+\ldots \tag{8}$$

Setting $p=1$ results in the approximate solution of Eqn. (1):

$$u=\lim_{p\rightarrow 1}v=v_0+v_1+v_2+\ldots \tag{9}$$

This series is convergent for most cases.

However, the convergence rate depends on nonlinear operator.

3. EXAMPLES

Example 1: Consider the following linear PDEs

$$\frac{\partial u}{\partial t}=u_{xx}+\cos x \tag{10}$$

with initial condition

$$u(0,x)=2(\sin x+\cos x) \tag{11}$$

and a given solution

$$u(t,x)=2\sin x\,e^{-t}+\cos x\,(e^{-t}+1).$$

To solve Eqn.(10) by Homotopy perturbation method, we will have

$$\frac{\partial u}{\partial t}-\frac{\partial u_0}{\partial t}=p\left[\frac{\partial^2 u}{\partial x^2}+\cos x\right] \tag{12}$$

Suppose that the solution of Eqn.(10) is the form

$$u=u_0+pu_1+p^2u_2+\ldots \tag{13}$$

Substituting Eqn. (13) into Eqn. (12) and equating the coefficients of like power p, we will have the set of differential equations:

$$\begin{aligned}
p^0 &: \frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}=0\\
p^1 &: \frac{\partial u_1}{\partial t}=\frac{\partial^2 u_0}{\partial x^2}+\cos x-\frac{\partial u_0}{\partial t}\\
p^2 &: \frac{\partial u_2}{\partial t}=\frac{\partial^2 u_1}{\partial x^2}\\
p^3 &: \frac{\partial u_3}{\partial t}=\frac{\partial^2 u_2}{\partial x^2}\\
p^4 &: \frac{\partial u_4}{\partial t}=\frac{\partial^2 u_3}{\partial x^2}
\end{aligned} \tag{14}$$

and so on. Solve the system of Eqns.(14) to get the solutions

$$u_1 = (2\sin x + \cos x)\,t$$
$$u_2 = -(2\sin x + \cos x)\frac{t^2}{2!} \tag{15}$$
$$u_3 = (2\sin x + \cos x)\frac{t^3}{3!}$$
$$u_4 = -(2\sin x + \cos x)\frac{t^4}{4!}$$

Solution of Eqn. (10) will be derived by adding these terms, so

$$u(x,t) = u_0 + u_1 + u_2 + \ldots\ldots\ldots\ldots\ldots\ldots$$
$$= \left(2\sin x - 2\sin x\frac{t}{1!} + 2\sin x\frac{t^2}{2!} - \ldots\ldots\ldots\right) + \left(2\cos x - \cos x\frac{t}{1!} + \cos x\frac{t^2}{2!} - \ldots\ldots\ldots\right)$$
$$= 2\sin x\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \ldots\ldots\ldots\right) + \cos x + \cos x\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \ldots\ldots\ldots\right)$$
$$u(t,x) = 2\sin x\, e^{-t} + \cos x\,(e^{-t} + 1).$$

Example 2: Consider the following nonlinear PDEs

$$u_t = \frac{u}{2}u_x - 2x \tag{16}$$

with initial condition

$$u(0,x) = 5 \tag{17}$$

and a given solution

$$u(t,x) = 5\sec h\, t - 2x\tanh t\,.$$

To solve Eqn.(16) by Homotopy perturbation method, we will have

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p\left[\frac{u}{2}\frac{\partial u}{\partial x} - 2x\right] \tag{18}$$

Suppose that the solution of Eqn.(16) is the form

$$u = u_0 + p\,u_1 + p^2\,u_2 + \ldots \tag{19}$$

Substituting Eqn. (19) into Eqn. (16) and equating the coefficients of like power p, we will have the set of differential equations:

$$p^0 : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0$$
$$p^1 : \frac{\partial u_1}{\partial t} = \frac{u_0}{2}\frac{\partial u_0}{\partial x} - 2x$$
$$p^2 : \frac{\partial u_2}{\partial t} = \frac{u_0}{2}\frac{\partial u_1}{\partial x} + \frac{u_1}{2}\frac{\partial u_0}{\partial x} \tag{20}$$
$$p^3 : \frac{\partial u_3}{\partial t} = \frac{u_0}{2}\frac{\partial u_2}{\partial x} + \frac{u_1}{2}\frac{\partial u_1}{\partial x} + \frac{u_2}{2}\frac{\partial u_0}{\partial x}$$
$$p^4 : \frac{\partial u_4}{\partial t} = \frac{u_0}{2}\frac{\partial u_3}{\partial x} + \frac{u_1}{2}\frac{\partial u_2}{\partial x} + \frac{u_2}{2}\frac{\partial u_1}{\partial x} + \frac{u_3}{2}\frac{\partial u_0}{\partial x}$$

and so on. Solve the system of Eqns. (20) to get the solutions

$$u_1 = -2x$$
$$u_2 = -5\frac{t^2}{2!} \tag{21}$$
$$u_3 = -2x\frac{t^3}{3!}$$
$$u_4 = -\frac{5}{12}t^4$$

and so on. Solution of Eqn. (16) will be derived by adding these terms, so

$$u(x,t) = u_0 + u_1 + u_2 + \dots\dots\dots\dots\dots$$
$$= 5 - 2xt - 5\frac{t^2}{2!} + 2x\frac{t^3}{3!} + \frac{5}{12}t^4 + \dots\dots\dots\dots\dots.$$
$$= 5\left(1 - \frac{t^2}{2!} + \frac{t^4}{12} - \dots\dots\dots\right) - 2x\left(\frac{t}{1!} - \frac{t^3}{3!} + \dots\dots\dots\right)$$
$$u(t,x) = 5\,\mathrm{sec}\,h\,t - 2x\tanh t\,.$$

Example 3: Consider the following PDEs in second order

$$u_t = -\left(u_{xx} + \frac{2}{x}u_x + 2e^{-t}\sin x\right) \tag{22}$$

subject to the initial conditions

$$u(0,x) = \frac{\cos x}{x} + \sin x \tag{23}$$

and a given solution $u(t,x) = e^{-t}\left(\frac{\cos x}{x} + \sin x\right)$.

To solve Eqn.(22) by Homotopy perturbation method, we will have

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = -p\left[\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} + 2e^{-t}\sin x\right] \tag{24}$$

Suppose that the solution of Eqn.(22) is the form

$$u = u_0 + p\,u_1 + p^2\,u_2 + \dots \tag{25}$$

Substituting Eqn. (25) into Eqn. (22) and equating the coefficients of like power *p*, we will have the set of differential equations:

$$p^0 : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0$$
$$p^1 : \frac{\partial u_1}{\partial t} = -\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{2}{x}\frac{\partial u_0}{\partial x} + 2e^{-t}\sin x + \frac{\partial u_0}{\partial t}\right)$$
$$p^2 : \frac{\partial u_2}{\partial t} = -\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{2}{x}\frac{\partial u_1}{\partial x}\right) \tag{26}$$
$$p^3 : \frac{\partial u_3}{\partial t} = -\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{2}{x}\frac{\partial u_2}{\partial x}\right)$$

and so on. Solve the system of Eqns. (26) to get the solutions

$$u_1 = -\frac{\cos x}{x}t + t\sin x + 2e^{-t}\sin x$$

$$u_2 = -3\frac{\cos x}{x}\frac{t^2}{2!} + \frac{t^2}{2!}\sin x - 2e^{-t}\sin x + \frac{4\cos x}{x}e^{-t} \tag{27}$$

$$u_3 = -5\frac{\cos x}{x}\frac{t^3}{3!} + \frac{t^3}{3!}\sin x + 2e^{-t}\sin x - \frac{8\cos x}{x}e^{-t}$$

$$u_4 = -7\frac{\cos x}{x}\frac{t^4}{4!} + \frac{t^4}{4!}\sin x - 2e^{-t}\sin x + \frac{12\cos x}{x}e^{-t}$$

and so on. Solution of Eqn. (22) will be derived by adding these terms, so

$$u(x,t) = u_0 + u_1 + u_2 + \ldots\ldots\ldots\ldots\ldots$$

$$= \frac{\cos x}{x}\left[\left(1 - \frac{t}{1!} - 3\frac{t^2}{2!} - 5\frac{t^3}{3!} - 7\frac{t^4}{4!} - \ldots\ldots\right) + \left(4 - 8 + 12 - 16 + \ldots\ldots\right)e^{-t}\right]$$

$$+ \sin x\left[\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots\ldots\right) + 2\left(1 - 1 + 1 - 1 - 1 + \ldots\ldots\ldots\right)e^{-t}\right].$$

$$= \frac{\cos x}{x}e^{-t} + \sin x\, e^{-t}$$

$$u(t,x) = e^{-t}\left(\frac{\cos x}{x} + \sin x\right).$$

4. CONCLUSION

In this paper, we used the homotopy perturbation method for solving some partial differential equations. We get the result is very effective and have an exact to find the solutions for the PDEs. Furthermore, HPM was successful implemented in approximating the solutions of nonlinear systems of PDEs.

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