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# SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY ELZAKI TRANSFORM

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***ABSTRACT***

*In this paper, we introduce a computational algorithm for solving partial differential equations such as Heat Equation, Wave Equation, Laplace Equation and Telegrapher's Equation etc. by using the modified versions of Laplace and Sumudu transforms which is called Elzaki transform. The Elzaki transform, whose fundamental properties are presented in this paper. Illustrative examples are presented to illustrate the effectiveness of its applicability.*

***Keywords:*** *Elzaki transform Partial differential equations.*

***2010 Mathematics Subject Classification:*** *35A22, 35C05, 44A15.*

## 1. INTRODUCTION

The term "differential equations" was was proposed in 1676 by G. Leibniz. The first studies of these equations were carried out in the late 17th century in the context of certain problems in mechanics and geometry. Ordinary and partial differential equations have important applications and are a powerful tool in the study of many problems in the natural sciences and in technology; they are extensively employed in mechanics, astronomy, physics, and in many problems of chemistry and biology. The reason for this is the fact that objective laws governing certain phenomena (processes) can be written as ordinary and partial differential equations, so that the equations themselves are a quantitative expression of these laws. In physics and engineering, there are many partial differential equations such as Heat Equation, Wave Equation, Laplace Equation and Telegrapher's Equation etc. These equations are quite useful and applicable in engineering, physics and science. In the last few decades, researchers have paid their attentions to find the solution of ordinary, partial, linear, nonlinear, homogeneous and non-homogeneous differential equations by using various integral transform, see [1-10]. One of such transforms know as Elzaki transform, introduced by Tarig M. Elzaki in 2011, is also very useful for solving ordinary and partial differential equations in the time domain, see [11-14]. In this paper, we apply Elzaki transform in solving various useful partial differential equations such as Heat Equation, Wave Equation, Laplace Equation and Telegrapher's Equation.

A new transform called the Elzaki transform defined for function of exponential order, we consider functions in the set A defined by

$$A=\left\{f(t);\ \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, if\ t \in (-1)^j \times [0,\infty)\right\}. \tag{1}$$

For a given function in the set A, the constant M must be finite number, $k_1, k_2$ may be finite or infinite. The Elzaki transform denoted by the operator $E(.)$ defined by the integral equations

$$E[f(t)] = T(v) = v\int_0^\infty f(t)e^{-\frac{t}{v}}dt,\ t \geq 0, k_1 \leq v \leq k_2, 0 \leq t < \infty\ . \tag{2}$$

For more results, theorems, various existing conditions concern to Elzaki transform, see [11-14]. Even after that, we mention here Elzaki transform of some functions, which are required in this paper.

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ELzaki Transform of some Functions

| $f(t)$ | $E[f(t)] = T(v)$ |
|---|---|
| $1$ | $v^2$ |
| $t$ | $v^3$ |
| $t^n$ | $n!\ v^{n+2}$ |
| $\frac{t^{a-1}}{\Gamma(a)}$ , $a > 0$ | $v^{a+1}$ |
| $e^{at}$ | $\frac{v^2}{1-av}$ |
| $te^{at}$ | $\frac{v^3}{(1-av)^2}$ |
| $\frac{t^{n-1}e^{at}}{(n-1)!}$ , $n = 1,2,\ldots$ | $\frac{v^{n+1}}{(1-av)^n}$ |
| $sin\ at$ | $\frac{av^3}{1+a^2v^2}$ |
| $cos\ at$ | $\frac{v^2}{1+a^2v^2}$ |
| $sinh\ at$ | $\frac{av^3}{1-a^2v^2}$ |
| $cosh\ at$ | $\frac{av^2}{1-a^2v^2}$ |
| $e^{at} sin\ bt$ | $\frac{bv^3}{(1-av)^2+b^2v^2}$ |
| $e^{at} cos\ bt$ | $\frac{(1-av)v^2}{(1-av)^2+b^2v^2}$ |
| $t\ sin\ at$ | $\frac{2av^4}{1+a^2v^2}$ |
| $J_0(at)$ | $\frac{v^2}{\sqrt{1+a^2v^2}}$ |
| $H(t-a)$ | $v^2e^{-\frac{a}{v}}$ |
| $\delta(t-a)$ | $ve^{-\frac{a}{v}}$ |

Let $u(x,t)$ be a function of two independent variables $x\ and\ t$, then

(i) $E[u(x,t)] = T(x,v)$

(ii) $E\left[\frac{\partial u}{\partial t}\right] = \frac{T(x,v)}{v} - v\,u(x,0)$

(iii) $E\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{T(x,v)}{v^2} - u(x,0) - v\,u_t(x,0)$

(iv) $E\left[\frac{\partial u}{\partial x}\right] = \frac{dT(x,v)}{dx}$

(v) $E\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2T(x,v)}{dx^2}$

## 2. SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

In this section we solve partial differential equations such as Heat Equation, Wave Equation, Laplace Equation and Telegrapher's Equation which are known as four fundamental equations in mathematical physics and occur in many branches of physics, in applied mathematics as well as in engineering.

**Heat Conduction Equation**

The most important physical phenomenon of heat conduction in a metallic rod of finite length can explicitly be described by using the partial differential equation.

Let $u(x,t)$ be the temperature distribution in a rod at any distance '$x'$ measured from initial point and at any time '$t'$ then the event is governed by the following PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial t^2}$$

where $k = \frac{K}{c\rho}$; $K$ = Thermal Conductivity, $c$ = Specific Heat of the metal and $\rho$ is the density(mass per unit volume). Various types of boundary conditions can be imposed to solve this equation. For illustration,

**Example 1:** Consider the heat equation in one dimension:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2};\ u(x,0) = sin2\pi x, u(0,t) = 0, u(1,t) = 0, 0 < x < 1, t > 0.$$

**Solution:** Given PDE is an example of heat conduction equation in a metallic rod of unit length. Taking Elzaki transform on both sides of given equation and making use of conditions, we get

$$\frac{T(x,t)}{v} - v\,u(x,0) = \frac{d^2T(x,v)}{dx^2}$$

$$\Rightarrow \frac{d^2T(x,v)}{dx^2} - \frac{T(x,v)}{v} = -vsin2\pi x \qquad (3)$$

this is a linear differential equation of second order, therefore solution of (3)

$$T(x,v) = c_1 e^{\frac{x}{v}} + c_2 e^{-\frac{x}{v}} + \frac{v^2}{(1+4\pi^2 v)} sin2\pi x$$

Using boundary conditions, we get $c_1 = c_2 = 0$,

$$\Rightarrow E[u(x,t)] = T(x,v) = \frac{v^2}{(1+4\pi^2 v)} sin2\pi x \qquad (4)$$

If we take the inverse Elzaki transform of (4), we obtain solution of PDE, as given in example 1

$$\Rightarrow \boldsymbol{u(x,t) = e^{-4\pi^2 t} sin2\pi x.}$$

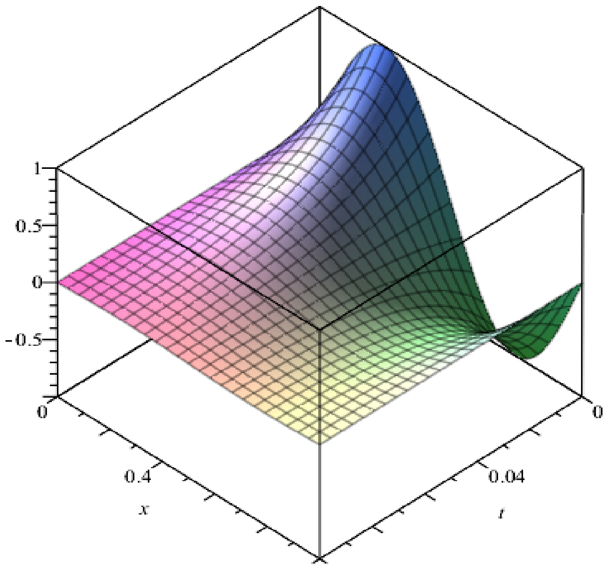


**Figure-1:**. Graph of $u(x,t) = e^{-4\pi^2 t} sin2\pi x,\ t > 0$ and $0 \le x \le 1$.

**Wave Equation**

Yet another important physical phenomenon of vibrations in tightly stretched string refers to wave motion. The small transverse vibrations of a flexible string are governed by one dimensional wave equation expressed as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ denotes the displacement of any point of the string at any time $t$. The constant $c^2 = \frac{T}{\rho}$ is described as the ratio of Tension $T$ in the string to the mass per unit length of the string which is supposed to be uniform. Various types of boundary conditions can be imposed to solve this equation. For illustration,

**Example 2:** Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = 20sin2\pi x - 10sin5\pi x,\ u_t(x,0) = 0, u(0,t) = 0,$$
$$u(2,t) = 0, 0 < x < 2, t > 0\ .$$

**Solution:** Given PDE is an example of wave equation. Taking Elzaki transform on both sides of given equation and making use of conditions, we get

$$\frac{T(x,v)}{v^2} - u(x,0) - v\,u_t(x,0) = 9\frac{d^2T(x,v)}{dx^2}$$

$$\Rightarrow 9\,\frac{d^2T(x,v)}{dx^2} - \frac{T(x,v)}{v^2} = -20sin2\pi x + 10sin5\pi x \quad (5)$$

this is a linear differential equation of second order, therefore solution of (5)

$$T(x,v) = c_1 e^{\frac{x}{3v}} + c_2 e^{-\frac{x}{3v}} + \frac{20v^2}{(1+36\pi^2v^2)} sin2\pi x - \frac{10v^2}{(1+225\pi^2v^2)} sin5\pi x$$

Using boundary conditions, we get $c_1 = c_2 = 0$,

$$\Rightarrow E[u(x,t)] = T(x,v) = \frac{20v^2}{(1+36\pi^2v^2)} sin2\pi x - \frac{10v^2}{(1+225\pi^2v^2)} sin5\pi x \quad (6)$$

If we take the inverse Elzaki transform of (6), we obtain solution of PDE, as given in example 2

$$\Rightarrow u(x,t) = 20sin2\pi xcos\,6\pi t - 10sin5\pi xcos\,15\pi t.$$

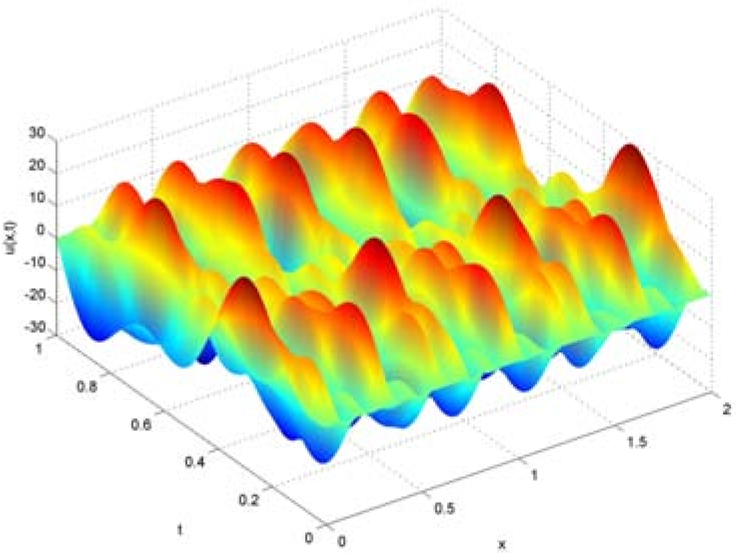


**Figure-2:** Graph of $u(x,t) = 20sin2\pi xcos\,6\pi t - 10sin5\pi xcos\,15\pi t,\ t > 0$ and $0 \leq x \leq 2$.

**Laplace Equation**

We want to study the steady-state temperature distribution in a thin, flat, rectangular plate. Without any loss of generality let the boundaries of the plate be $x = 0, x = a, y = 0$ and $y = b$.

The differential equation modeling the steady-state temperature distribution is given by the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \qquad 0 < y < b.$$

Various types of boundary conditions can be imposed to solve this equation. For illustration,

**Example 3:** Consider the Laplace's equation:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$, which satisfies the conditions

$u(0,t) = 0 = u(l,t), u(x,0) = 0, u(x,a) = \frac{n\pi}{l} sin\frac{n\pi x}{l},\ u_t(x,0) = \frac{n\pi}{l} cosech\frac{n\pi a}{l} sin\frac{n\pi x}{l}$

$0 < x < l, 0 < t < a.$

**Solution:** Given PDE is an example of Laplace's equation. Taking Elzaki transform on both sides of given equation and making use of conditions, we get

$$\frac{d^2T(x,v)}{dx^2} + \frac{T(x,v)}{v^2} - u(x,0) - v\,u_t(x,0) = 0$$

$$\Rightarrow \frac{d^2T(x,v)}{dx^2} + \frac{T(x,v)}{v^2} = v\frac{n\pi}{l} cosech\frac{n\pi a}{l} sin\frac{n\pi x}{l} \quad (7)$$

this is a linear differential equation of second order, therefore solution of (7)

$$T(x,v) = c_1 cos\frac{x}{v} + c_2 sin\frac{x}{v} + \frac{v^3}{1-\frac{n^2\pi^2}{l^2}v^2}\frac{n\pi}{l} cosech\frac{n\pi a}{l} sin\frac{n\pi x}{l}$$

Using boundary conditions, we get $c_1 = c_2 = 0$,

$$\Rightarrow E[u(x,t)] = T(x,v) = \frac{n\pi}{l} cosech\frac{n\pi a}{l} sin\frac{n\pi x}{l}\frac{v^3}{1-\frac{n^2\pi^2}{l^2}v^2} \quad (8)$$

if we take the inverse Elzaki transform of (8), we obtain solution of PDE, as given in example 3

$$\Rightarrow \boldsymbol{u(x,t) = \frac{n\pi}{l} cosech\frac{n\pi a}{l} sin\frac{n\pi x}{l} sinh\frac{n\pi t}{l}}.$$

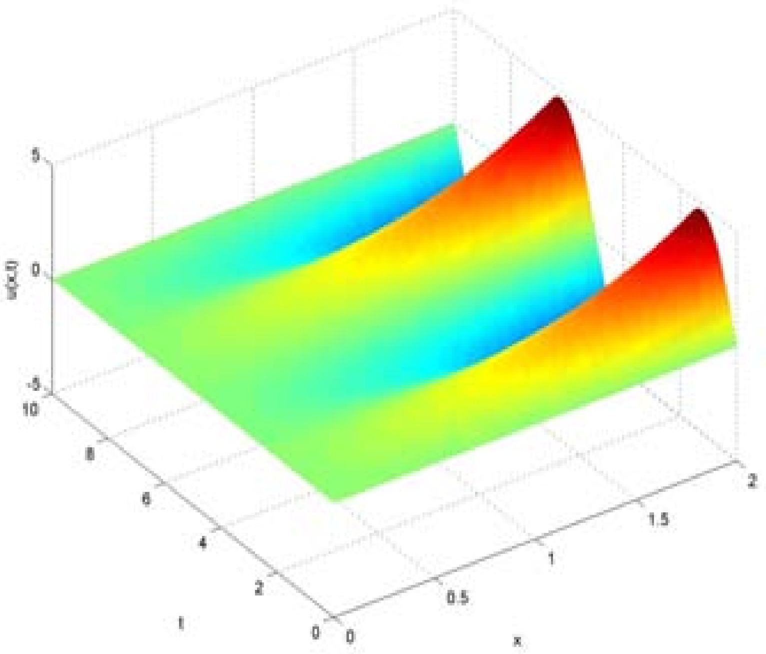


**Figure-3:** Graph of $u(x,t) = \frac{n\pi}{l} cosech \frac{n\pi a}{l} sin \frac{n\pi x}{l} sinh \frac{n\pi t}{l},\ t > 0$ and $0 \leq x \leq 1, l = 5, n = 2, a = 1$.

**Telegrapher's Equation**

Telegrapher equation is generally used to describe electrical phenomenon in a more practical approach by using short segments of an electrical system. For analysis of distributed electrical parameters of any electrical system, Telegrapher equation is being used because analysis of electrical parameters in lumped form is very difficult. Various types of boundary conditions can be imposed to solve this equation. For illustration,

**Example 4:** Consider the Telegrapher's Equation:

$\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}\ ,\ 0 < x < 1, t > 0$ With the initial conditions:

$u(x,0) = \cos x,\ \ u_t(x,0) = 0.$

**Solution:** Given PDE is an example of Telegrapher's Equation. Taking Elzaki transform on both sides of given equation and making use of conditions, we get

$$\frac{T(x,v)}{v^2} - u(x,0) - v\, u_t(x,0) + 2\alpha \frac{T(x,v)}{v} - 2\alpha\, v\, u(x,0) = \alpha^2 \frac{d^2 T(x,v)}{dx^2}$$

$$\Rightarrow \alpha^2 v^2 \frac{d^2 T(x,v)}{dx^2} - (1 + 2\alpha v)T(x,v) = -(2\alpha v^3 + v^2)\cos x \quad (9)$$

this is a linear differential equation of second order, therefore solution of (9)

$$\Rightarrow\ E[u(x,t)] = T(x,v) = \frac{-(2\alpha v^3 + v^2)\cos x}{\alpha^2 v^2 D^2 - (1 + 2\alpha v)}, \qquad \text{where } D^2 \equiv \frac{d^2}{dx^2}$$

$$\Rightarrow\ E[u(x,t)] = T(x,v) = \frac{\cos x (2\alpha v^3 + v^2)}{(1 + \alpha v)^2} \quad (10)$$

(10) is the particular solution of (9).

If we take the inverse Elzaki transform of (10), we obtain solution of PDE, as given in example 4

$$u(x,t) = \cos x\, E^{-1}\left[\frac{2\alpha v^3}{(1 + \alpha v)^2} + \frac{v^2}{(1 + 2\alpha v)}\right]$$

$$\Rightarrow \boldsymbol{u(x,t) = [2\alpha t e^{-\alpha t} + e^{-\alpha t}]\cos x}\ .$$

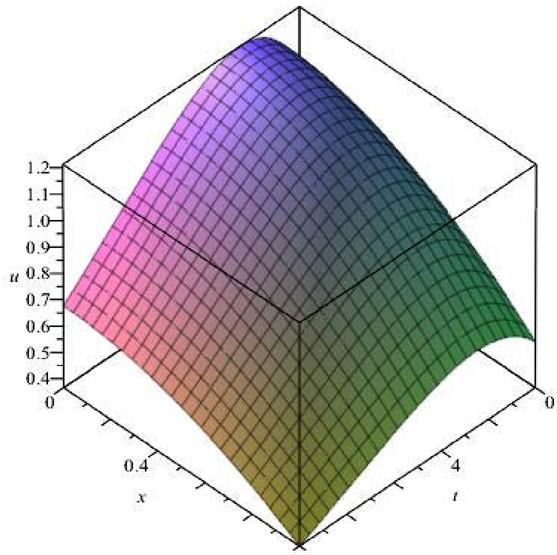


**Figure-4:** Graph of $u(x,t) = [2\alpha t e^{-\alpha t} + e^{-\alpha t}]\cos x$ , $t > 0$ and $0 \leq x \leq 1, \alpha = 0.2$.

## 3. CONCLUSION

In this paper, we apply interesting new transform “Elzaki transform” in solving various useful partial differential equations such as Heat Equation, Wave Equation, Laplace Equation and Telegrapher’s Equation. Our purpose here is to show the applicability of this interesting new transform.

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