

**GEOMETRICAL INTERPRETATION
 OF AN ANALYTIC HP-TRANSFORMATIONS IN ALMOST KAEHLERIAN SPACES**

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(Received On: 12-11-17; Revised & Accepted On: 17-12-17)

ABSTRACT

In this paper, we have defined and studied on geometrical interpretation of an analytic holomorphically projective transformations in almost Kaehlerian spaces and several theorems have been obtained.

Key Words: *Kaehlerian space, H- Projective, Recurrent, Symmetric, Transformation.*

2010 MSC: *32C15, 46A13, 46M40, 53B35, 53C55.*

1. INTRODUCTION

An almost Kaehlerian space is first of all an almost complex space, that is, a $2n$ -dimensional space with an almost complex structure F_i^h :

$$F_j^i F_i^h = -\delta_j^h, \tag{1.1}$$

And always admits a positive definite Riemannian metric tensor g_{ji} satisfying:

$$F_j^a F_i^b g_{ab} = g_{ji}, \tag{1.2}$$

From which

$$F_{ji} = -F_{ij}, \tag{1.3}$$

Where

$$F_{ji} \stackrel{\text{def}}{=} F_j^a g_{ai} \tag{1.4}$$

And finally has the property that the differential form $F_{ji} d\xi^j \wedge d\xi^i$ is closed, that is,

$$F_{jih} \stackrel{\text{def}}{=} \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0$$

And finally has the property that the skew-symmetric F_{ih} is a Killing tensor

$$\nabla_j F_{ih} + \nabla_i F_{hj} = 0 \tag{1.5}$$

From which

$$\nabla_j F_i^j + \nabla_i F_j^h = 0 \tag{1.6}$$

And

$$F_i = -\nabla_j F_i^j = 0 \tag{1.7}$$

Here ∇ denotes the operation of covariant differentiation with respect to the Riemannian connection $\{j^h_i\}$.

The Nijenhuis tensor N_{ji}^h is written in the form:

$$N_{ji}^h = -4 (\nabla_j F_i^t) F_t^h + 2G_{ji}^t F_i^h + F_j^t G_{ti}^h - F_i^t G_{tj}^h. \tag{1.8}$$

A contravariant almost analytic vector field is defined as a vector field v^i , satisfying Tachibana (1959):

$$\mathcal{L}_v F_i^h \equiv v^j \partial_j F_i^h - F_i^j \partial_j v^h + F_j^h \partial_i v^j = 0,$$

Where \mathcal{L}_v stands for the Lie-derivative with respect to v^i .

Let R_{kji}^h be the Riemannian curvature tensor and put

$$R_{ji} = R_{rji}^r, R_{kjih} = R_{kji}^r g_{rh}, R = R_{ji} g^{ji} \text{ and } S_{ji} = F_j^r R_{ri},$$

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Then the following identities are satisfied (Yano 1957)

$$R^r_{kji} F^h_r = R^h_{kjr} F^r_i, R_{kji} F^r_h = R_{kjhr} F^r_i \tag{1.9}$$

$$R_{kjih} = R_{kjir} F^t_i F^r_h, R_{ji} = R_{tr} F^t_j F^r_i \tag{1.10}$$

$$S_{ji} + S_{ij} = 0, S_{ji} = S_{tr} F^t_j F^r_i, S_{ji} = -\frac{1}{2} F^{tr} R_{trji}. \tag{1.11}$$

The holomorphically projective curvature tensor P^h_{kji} , which will be briefly called HP-curvature tensor, is given by

$$P^h_{kji} = R^h_{kji} + \frac{1}{n+2} (R_{ki} \delta^h_j - R_{ji} \delta^h_k + S_{ki} F^h_j - S_{ji} F^h_k + 2 S_{kj} F^h_i) \tag{1.12}$$

We can obtain the following identities

$$P^h_{(kji)} = 0, P^h_{[kji]} = 0, \tag{1.13}$$

$$P^r_{rji} = 0, \tag{1.14}$$

$$P^r_{kji} F^h_r = P^h_{kjr} F^r_i, P^h_{rji} F^r_k = P^h_{rki} F^r_j \tag{1.15}$$

From which, we have

$$P^r_{kjr} = 0, \tag{1.16}$$

$$P^t_{rji} F^r_t = 0, P^t_{kjr} F^r_t = 0. \tag{1.17}$$

A necessary and sufficient condition for $P^h_{kji} = 0$, is that the space is a space of constant holomorphically curvature (Tashiro 1957), i.e., a space whose curvature tensor R^h_{kji} takes the form

$$R^h_{kji} = -\frac{R}{n(n+2)} (g_{ki} \delta^h_j - g_{ji} \delta^h_k + F_{ki} F^h_j - F_{ji} F^h_k + 2 F_{kj} F^h_i) \tag{1.18}$$

For a vector field V^i and a tensor field α^h_i , the following identities are known (Yano 1957)

$$\mathcal{E}_v \nabla_j \alpha^h_i - \nabla_j \mathcal{E}_v \alpha^h_i = \alpha^r_i \mathcal{E}_v \{^h_r\} - \alpha^h_r \mathcal{E}_v \{^r_i\} \tag{1.19}$$

$$\nabla_k \mathcal{E}_v \{^h_i\} - \nabla_j \mathcal{E}_v \{^h_i\} = \mathcal{E}_v R^h_{kji} \tag{1.20}$$

Where \mathcal{E}_v denotes the operator of Lie-differentiation with respect to V^i .

A Killing vector or an infinitesimal isometry V^i is defined by

$$\mathcal{E}_v g_{ji} = \nabla_j V_i + \nabla_i V_j = 0.$$

Here we shall identify a contravariant vectors V^i with a covariant vector $V_i = g_{ir} V^r$. Hence we shall say V_i is a Killing vector, or that ρ^i is gradient, for example.

An infinitesimal affine transformation V^i is defined by

$$\mathcal{E}_v \{^h_i\} = \nabla_j \nabla_i V^h + R^h_{rji} V^r = 0.$$

We shall say a vector field V^i an infinitesimal holomorphically projective transformation or, for simplicity, an HP-transformation, if it satisfies

$$\mathcal{E}_v \{^h_i\} = \rho_j \delta^h_i + \rho_i \delta^h_j - \bar{\rho}_j F^h_i - \bar{\rho}_i F^h_j,$$

Where ρ_i is a certain vector and $\bar{\rho}_i = F^r_i \rho_r$. In this case, we shall called ρ_i the associated vector of the transformation, If ρ_i vanishes, then the HP-transformation reduces to an affine one.

Contracting the last equation with respect to h and i, we get

$$\nabla_j \nabla_r V^r = (n+2) \rho_j,$$

Which shows that the associated vector is gradient.

A vector field V^i is called Contravariant analytic or, for simplicity, analytic, if it satisfies

$$\mathcal{E}_v F^h_i \equiv -F^r_i \nabla_r V^h + F^h_r \nabla_i V^r = 0.$$

2. GEOMETRICAL INTERPRETATION OF AN ANALYTIC HP-TRANSFORMATION

In a differentiable space M, we consider a tensor valued function V depending not only on a point P of M but also on k vectors u_1, u_2, \dots, u_k at the point and denote it by $V(P, u_1, u_2, \dots, u_k)$. We assume that the value of this function V lies in the tensor space associated to the tangent space of M at P and that it depends differentially on its arguments.

Assuming the space M to be affinely connected, we take an arbitrary curve C: $x^i = x^i(t)$ and denote its successive derivatives by

$$\frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \frac{d^3x^i}{dt^3} \tag{2.1}$$

Then if we substitute (2.1) into the function V instead of u_1, u_2, \dots, u_k . We have a family of tensors

$$V(C) = V \left(x^i, \frac{dx^i}{dt}, \dots, \frac{d^kx^i}{dt^k} \right)$$

along the curve C.

Let V^i be an infinitesimal transformation, i.e., a vector field, and $\dot{x}^i = x^i + \varepsilon v_i$ be the infinitesimal point transformation determined by v^i , ε being an infinitesimal constant. Given a curve $C: x^i = x^i(t)$, the image \dot{C} of C is expressed by $\dot{x}^i = x^i(t) + \varepsilon V^i(x(t))$.

We shall call the limiting value

$$\mathcal{E}_v V(C) \equiv \lim_{\varepsilon \rightarrow 0} \frac{V(\dot{C}) - V(C)}{\varepsilon}$$

The Lie-derivative of $V(C)$ with respect to V^i , where we have denoted by $\dot{V}(C)$ the family of tensors induced from $V(C)$ by the transformation

$$\dot{x}^k = x^k + \varepsilon V^i$$

In a Almost Kaehlerian space, a curve $x^i = x^i(t)$ defined by

$$\frac{d^2 x^h}{dt^2} + \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} \frac{dx^j}{dt} \frac{dx^i}{dt} = \alpha \frac{dx^h}{dt} + \beta F_j^h \frac{dx^j}{dt} \tag{2.2}$$

is, by definition, a holomorphically planar curve, or an H-plane curve, where α and β are certain functions of t .

Let V^i be an infinitesimal transformation and assume that any ε the infinitesimal point transformation $\dot{x}^i = x^i + \varepsilon V^i$ maps any H-plane curves.

Now we ask for the condition that V^i preserve that H-plane curves. For such a vector V^i taking account of (2.2), we have

$$\mathcal{E}_v \left[\frac{d^2 x^h}{dt^2} + \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} \frac{dx^j}{dt} \frac{dx^i}{dt} - \alpha \frac{dx^h}{dt} - \beta F_j^h \frac{dx^j}{dt} \right] = \gamma \frac{dx^h}{dt} + \delta F_j^h \frac{dx^j}{dt} \tag{2.3}$$

along any H-plane curve, where γ and δ are certain functions of t .

Denoting the Lie-derivative of the Christoffel's symbols and the complex structure F^h_i , respectively, by

$$t^h_{ji} = \mathcal{E}_v \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\}, \quad \alpha^h_i = \mathcal{E}_v F^h_i,$$

We have from (2.3)

$$t^h_{ji} \dot{x}^j \dot{x}^i + \alpha \dot{x}^h + b F^h_j \dot{x}^j - \beta \alpha^h_j \dot{x}^j = 0 \tag{2.4}$$

Where we have put

$$a = -(\gamma + \mathcal{E}_v \alpha), \quad b = -(\delta + \mathcal{E}_v \beta), \quad \dot{x}^i = \frac{dx^i}{dt}$$

Since the relation (2.4) holds for any H-plane curve C , it must hold identically for any values of x^i and \dot{x}^i .

By means of the definition of the H-plane curve, we see further that the identity (2.4) holds for any value of the coefficient β .

Taking account of these arguments, we can easily see that relation

$$a^h_j \dot{x}^j = f \dot{x}^h + g F^h_j \dot{x}^j, \tag{2.5}$$

$$t^h_{ji} \dot{x}^j \dot{x}^i = p \dot{x}^h + q F^h_j \dot{x}^j, \tag{2.6}$$

hold for any values x^i and \dot{x}^i , where f, g, p and q are certain functions of x^i and \dot{x}^i .

Let α^i_j be a tensor on V such that $F^r_j \alpha^i_r + \alpha^r_j F^i_r = 0$, we obtain by means of (2.5)

$$\alpha^h_i \equiv \mathcal{E}_v F^h_i = 0. \tag{2.7}$$

On the other hand, If we substitute (2.7) and $\nabla_j F^i_j = 0$ into the identity

$$\nabla_j \mathcal{E}_v F^h_i - \mathcal{E}_v \nabla_j F^h_i = F^h_r \mathcal{E}_v \{ F^r_i \} - F^r_i \mathcal{E}_v \{ F^h_r \},$$

Then we get

$$t^r_{ji} F^h_r = t^h_{jr} F^r_i. \tag{2.8}$$

From (2.6) and (2.8), taking account of the fact that

$$t^h_{ji} = \alpha_j \delta^h_i + \alpha_i \delta^h_j - \bar{\alpha}_j F^h_i - \bar{\alpha}_i F^h_j,$$

Where α_i is certain vector and $\bar{\alpha}_i = F^r_i \alpha_r$, we get

$$t^h_{ji} = \mathcal{E}_v \{ F^h_i \} = \rho_j \delta^h_i + \rho_i \delta^h_j - \bar{\rho}_j F^h_i - \bar{\rho}_i F^h_j, \tag{2.9}$$

Where ρ_i is a certain vector field. Therefore, the infinitesimal transformation V^i is an analytic HP-transformation.

Conversely, it is obvious that an analytic HP-transformation preserves the H-plane curves.

Thus we have the following:

Theorem 2.1: In an almost Kaehlerian space, an infinitesimal transformation preserves the H-plane curves, if and only if it is an analytic HP-transformation.

3. SOME PROPERTIES OF HP-TRANSFORMATIONS.

Let V^i be an HP-transformation, then it holds

$$\mathcal{E}_V \{ j^h_i \} \equiv \nabla_j \nabla_i V^h + R_{rji}^h V^r = \rho_j \delta_i^h - \rho_i \delta_j^h - \bar{\rho}_j F_i^h - \bar{\rho}_i F_j^h. \tag{3.1}$$

Transvecting (3.1) with g^{ji} , we have

$$\nabla^r \nabla_r V^i + R_r^h V^r = 0. \tag{3.2}$$

Hence, by virtue of the well known theorem on an analytic vectors, Yano (1957), Lichnerowicz (1957), we have the following:

Theorem 3.1: In a compact almost Kaehlerian space an HP-transformation is analytic.

In a compact almost Kaehlerian space, M, it holds that

$$\int_M (R_{ji} V^j V^i) d\sigma \cong 0$$

For an analytic vector V^i , where $d\sigma$ denote the volume element of M and the equality holds when and when only V^i is parallel. Therefore, if the Ricci's from $R_{ji} \xi^j \xi^i$ is negative definite, then there exists no non-trivial HP-transformation provided that the space is compact.

Taking account of the identity (1.19), we have for a vector field V^i

$$\mathcal{E}_V \nabla_j F_i^h - \nabla_j \mathcal{E}_V F_i^h = F_i^r \mathcal{E}_V \{ j^h_r \} - F_r^h \mathcal{E}_V \{ j^r_i \},$$

Which implies

$$\nabla_j \mathcal{E}_V F_i^h = F_r^h \mathcal{E}_V \{ j^r_i \} - F_i^r \mathcal{E}_V \{ j^h_r \},$$

Because of $\nabla_j F_i^h = 0$. If the vector field V^i is an HP-transformation, it is easily verified that the right hand-side of the last equation vanishes. Thus we have the following theorems by the virtue of Obata's theorem, gives Obata (1956)

Theorem 3.2: In an irreducible almost Kaehlerian space admitting no quaternion structure, any HP-transformation is analytic.

Theorem 3.3: In an irreducible almost Kaehlerian space having non-vanishing Ricci tensor any HP-transformation is analytic.

Theorem 3.4: In an irreducible almost Kaehlerian Einstein space if its scalar curvature is non-vanishing, any HP-transformation is analytic.

Now, we shall find some formulae on analytic HP-transformation which will be useful in the further study.

Let V^i be an HP-transformation. Substituting (3.1) into the identity

$$\mathcal{E}_V g_{ji} - \mathcal{E}_V \nabla_k g_{ji} = g_{ri} \mathcal{E}_V \{ k^r_j \} + g_{jr} \mathcal{E}_V \{ k^r_i \},$$

We find

$$\nabla_k \mathcal{E}_V g_{ji} = \rho_j g_{ki} + \rho_i g_{kj} - \bar{\rho}_j F_{ki} - \bar{\rho}_i F_{kj} + 2\rho_k g_{ji}. \tag{3.3}$$

If we substitute (3.1) into (1.20), then we have

$$\mathcal{E}_V R_{kji}^h = \delta_j^h \nabla_k \rho_i - \delta_k^h \nabla_j \rho_i - F_j^h \nabla_k \bar{\rho}_i + F_k^h \nabla_j \bar{\rho}_i - (\nabla_k \bar{\rho}_j - \nabla_j \bar{\rho}_k) F_i^h, \tag{3.4}$$

Contracting the last equation with respect to h and k we find

$$\mathcal{E}_V R_{ji} = -n \nabla_j \rho_i - 2F_j^r F_i^t \nabla_r \rho_t \tag{3.5}$$

Now we shall assume that V^i is an analytic HP-transformation. Then we have $\mathcal{E}_V R_{ji} = \mathcal{E}_V (R_{rt} F_j^r F_i^t)$

By virtue of (2.1). Hence from (3.5) it follows

$$\nabla_j \rho_i = F_j^r F_i^t \nabla_r \rho_t. \tag{3.6}$$

Since $n > 2$. The last equation also is written in the form:

$$\mathcal{E}_V F_i^h \equiv -F_i^r \nabla_r \rho^h + F_r^h \nabla_i \rho^r = 0,$$

Which shows that ρ^i is analytic. Moreover, according to (3.6) we have

$$\nabla_j \bar{\rho}_i + \nabla_i \bar{\rho}_j = F_i^r (\nabla_j \rho_r - F_j^s F_r^t \nabla_t \rho_s) = 0, \tag{3.7}$$

Which means that $\bar{\rho}^i$ is a Killing vector. Thus we get the following:

Theorem 3.5: If a vector ρ_i is the associated vector of an analytic HP-transformation, then ρ^i is analytic and $\bar{\rho}^i$ is a Killing vector.

Now, from (3.5) and (3.6) it follows

$$E_V R_{ji} = -(n+2) \nabla_j \rho_i, \tag{3.8}$$

From which we have

$$E_V S_{ji} = (n+2) \nabla_j \bar{\rho}_i. \tag{3.9}$$

On the other hand, from (3.4) and (3.7) we get

$$E_V R_{kji}^h = \delta_j^h \nabla_k \rho_i - \delta_k^h \nabla_j \rho_i - F_j^h \nabla_k \bar{\rho}_i + F_k^h \nabla_j \bar{\rho}_i - 2F_i^h \nabla_k \bar{\rho}_j. \tag{3.10}$$

If we substitute (3.8) and (3.9) into (3.10). Then we can verify Ishihara (1957)

$$E_V R_{kji}^h = 0. \tag{3.11}$$

In the next place, substitute (3.1) and (3.8) into the identify

$$E_V \nabla_k R_{ji} - \nabla_k E_V R_{ji} = -R_{rt} E_V \{ k^r j \} - R_{jr} E_V \{ k^r i \},$$

We have

$$\nabla_k R_{ji} = -(n+2) \nabla_k \nabla_j \rho_i - R_{ki} \rho_j - R_{kj} \rho_i + S_{ki} \bar{\rho}_j + S_{kj} \bar{\rho}_i - 2R_{ji} \rho_k. \tag{3.12}$$

Hence we put

$$P_{kji} = \frac{1}{n+2} (\nabla_k R_{ji} - \nabla_j R_{ki}). \tag{3.13}$$

It holds

$$E_V P_{kji} = P_{kji}^r \rho_r. \tag{3.14}$$

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Source of support: Nil, Conflict of interest: None Declared.

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