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ORTHOGONALITY OF GENERALIZED DERIVATIONS ON SEMIPRIME NONACCESSIBLE RINGS

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ABSTRACT

This paper gives the properties of orthogonal generalized derivations of semiprime nonaccessible rings. We prove that if (D, d) and (G, g) are generalized derivations of R, then the following conditions are equivalent:

- *(i) (D, d) and (G, g) are orthogonal.*
- *(ii) For all x, y∈R, the following relations hold: (a) D(x)G(y) + G(x)D(y) = 0 (b) d(x)G(y) + g(x)D(y) = 0*
- *(iii) D(x)G(y) = d(x)G(y) = 0 for all x, y∈R (iv) D(x)G(y) = 0 for all x, y∈R and dG = dg = 0 (v) (DG,dg) is a generalized derivation and D(x)G(y) = 0 for all x, y∈R*
- *(iv) There exist ideals U and V of R such that (a) U⋂V=0 and U⊕V is an essential ideal of R. (b) D(R),d(R)⊂U and G(R), g(R)⊆V (c) D(V) = d(V) = 0 and G(U) = g(U) = 0.*

Also we prove that the following conditions are equivalent:

- *(i) (DG, dg) is a generalized derivation.*
- *(ii) (GD, gd) is a generalized derivation.*
- *(iii) D and g are orthogonal, and G and d are orthogonal.*

Mathematics Subject Classification*: 16N60, 16W25.*

Keywords*: Semiprime ring, Derivation, Orthogonality, Nonaccessible ring, Generalized Derivation.*

INTRODUCTION:

Argac, Nakajima and Albas [2] proved some results concerning orthogonal generalized derivations of semiprime associative rings. Albas [1] generalized some results of [2] for a nonzero ideal of a semiprime associative ring. Vijaya lakshmi and Suvarna [4] obtained some results for Orthogonality of derivations and biderivations in Semiprime Accessible ring. Also these results are a generalization of results of Bresar and Vukman [3].

In this paper, we prove the properties of orthogonal generalized derivations of semiprime nonaccessible rings. If (D, d) and (G, g) are generalized derivations of R, then the following conditions are equivalent:

- (i) (D, d) and (G, g) are orthogonal.
- (ii) For all x, y∈R, the following relations hold : (a) D(x)G(y) + G(x)D(y) = 0 (b) d(x)G(y) + g(x)D(y) = 0
- (iii) D(x)G(y) = d(x)G(y) = 0 for all x, y∈R (iv) D(x)G(y) = 0 for all x,y∈R and dG = dg = 0 (v) (DG, dg) is a generalized derivation and D(x)G(y) = 0 for all x,y∈R
- (iv) There exist ideals U and V of R such that (a) U⋂V=0 and U⊕V is an essential ideal of R. (b) D(R),d(R)⊂U and G(R),g(R)⊆V (c) D(V) = d(V) = 0 and G(U) = g(U) = 0.

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Also we prove that the following conditions are equivalent:

(i) (DG, dg) is a generalized derivation.
(ii) (GD, gd) is a generalized derivation.
(iii) D and g are orthogonal, and G and d are orthogonal.

A ring R is nonaccessible if the following two identities hold:

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \text{ and}$$
$$((w, x), y, z) = 0 \text{ for all } x, y, z \text{ in } R.$$

PRELIMINARIES

Throughout this paper R represents nonaccessible ring. R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$ and R is semiprime if $xRx = 0$ implies $x = 0$. An additive mapping $d: R\to R$ is a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y\in R$. An additive mapping $D:R\to R$ is said to be a generalized derivation if there exists a derivation $d:R\to R$ such that $D(xy) = D(x)y + xd(y)$ for all $x,y\in R$. If U is an ideal of a ring R, then $Ann(U) = \{x\in R / xU = Ux = 0\}$ is called an Annihilator of U on R. Two generalized derivations (D, d) and (G, g) of R are called orthogonal if $D(x)RG(y) = 0 = G(y)RD(x)$ for all x, y in R. A ring R is defined to be accessible if it satisfies the following two identities:
$(x, y, z) + (z, x, y) – (x, z, y) = 0$ or $(x, y, z) +(y, z, x) + (z, x, y) = 0$ and $((w, x), y, z) = 0$, for all $w, x, y, z \in R$.

Lemma 1: If (D, d) and (G, g) are orthogonal generalized derivations of a semiprime nonaccessible ring, then the following relations hold.

i) $D(x)G(y) = G(x)D(y) = 0$, hence $D(x)G(y) + G(x)D(y) = 0$ for all $x, y\in R$.
ii) d and G are orthogonal, and $d(x)G(y) = G(y)d(x) = 0$ for all $x, y\in R$.
iii) g and D are orthogonal, and $g(x)D(y) = D(y)g(x) = 0$ for all $x, y\in R$.
iv) d and g are orthogonal derivations.
v) $dG = Gd = 0$ and $gD = Dg = 0$.
vi) $DG = GD = 0$.

Proof:
(i) By the hypothesis we have

$$D(x)zG(y) = 0 \text{ for all } x, y, z\in R. \quad (1)$$

By ([4] lemma 1), we have

$$D(x)G(y) = G(x)D(y) = 0, \text{ for all } x, y\in R. \quad (2)$$

Hence $D(x)G(y) + G(x)D(y) = 0$ for all $x, y\in R$.

(ii) From (1) we have $D(x)G(y) = 0 = G(y)D(x)$ for all $x, y\in R$. (3)

Now we substitute $x = rx$ in the equation (3). Then $0 = D(rx)G(y)$.

$$0 = (D(r)x)G(y) + (rd(x))G(y).$$

By ([4] lemma 2), we have $D(r)xG(y) + (r\, d(x))G(y) = 0$.

By using (1), we get $(rd(x))G(y) = 0$, for all $x, y, r\in R$. (4)

Similarly we write $x = rx$ in $G(y)D(x) = 0$.

So $G(y)D(rx) = 0$.

i.e., $G(y)(D(r)x + rd(x)) = 0$.
$G(y)(D(r)x) + G(y)(rd(x)) = 0$.

From (3), we have $G(y)(rd(x)) = 0$.

By using ([4] lemma 2), (4) and this last equation imply that

$$rd(x)G(y) = 0 \quad (5)$$
$$\text{and} \quad G(y)rd(x) = 0. \quad (6)$$

By left multiplying equation (5) with $d(x)G(y)$, we have

$$d(x)G(y)rd(x)G(y) = 0.$$

Since R is semiprime, $d(x)G(y) = 0$, for all $x, y\in R$. (7)

Now we take $x = xr$ in the equation (7). Then we have

$d(xr)G(y) = 0$, for all $x, r, y\in R$.
$(d(x)r)G(y) + (xd(r))G(y) = 0$.
$d(x)rG(y) + xd(r)G(y) = 0$, by ([4] lemma 2).

By using (7) in this last equation, we get $d(x)rG(y) = 0$ for all $x, y\in R$.

From (6) and this equation it follows that d and G are orthogonal.

Also by ([4] lemma 1), we obtain $G(y)d(x) = 0$ for all $x, y\in R$. This prove (ii). The proof of (iii) is similar.

(iv) From (2), we use the relation $D(x)G(y) = 0$.

Now we substitute $x = xz$, $y = yw$ in the last equation. Then

$D(xz)G(yw) = 0$.
$(D(x)z + xd(z))(G(y)w + yg(w)) = 0$.
$(D(x)z)(G(y)w) + (D(x)z)(yg(w)) + (xd(z))(G(y)w) + (xd(z))(yg(w)) = 0$.

By ([4] lemma 2), $D(x)zG(y)w + D(x)zyg(w) + xd(z)G(y)w + xd(z)yg(w) = 0$.

Thus we get $xd(z)yg(w) = 0$ for all $x, y, z, w\in R$ by (i),(ii) and (iii).

By left multiplying with $d(z)yg(w)$ in this last equation, we have

$d(z)yg(w)xd(z)yg(w) = 0$.

Since R is semiprime, $d(z)yg(w) = 0$ for all $y, z, w\in R$ which shows that d and g are orthogonal.

(v), (vi) Using (ii) and (iv) we have

$0 = G(d(x)zG(y))$
$0 = G(d(x))zG(y) + d(x)g(zG(y))$
$0 = G(d(x))zG(y)$.

We replace y by d(x) in the above relation. Then we have

$G(d(x))zG(d(x)) = 0$.
i.e., $Gd(x)zGd(x) = 0$.

Since R is semiprime, $Gd = 0$.

Similarly, we can show that

$d(G(x)zd(y)) = 0$, $D(g(x)zD(y)) = 0$,
$g(D(x)zg(y)) = 0$, $D(G(x)zD(y)) = 0$ and $G(D(x)zG(y)) = 0$ holds for all $x, y, z\in R$.

Thus we have $dG = Dg = DG = GD = 0$, respectively.

This completes the proof of the lemma.

By lemma 1, we have the following corollary.

Corollary 1: If (D, d) and (G, g) are orthogonal generalized derivations of semiprime nonaccessible ring R, then dg is a derivation and $(DG,dg) = (0,0)$ is a generalized derivation.

Lemma 2: Let R be a semiprime nonaccessible ring. Let U be an ideal of R and V = Ann (U). If (D, d) is a generalized derivation of R such that $D(R),d(R)\subset U$, then $D(V) = d(V) = 0$.

Proof: By hypothesis we have $V = Ann\,(U) = \{U\in R/\ xU=Ux=0\}$ for all $x\in V$ and $D(R), d(R)\subset U$. This implies $D(U),d(U) \subset U$, for all $U\in R$. Since (D, d) is a generalized derivation of R, we have
$0 = D(xy) = D(x)y + xd(y)$ for all $y\in U$.

Since $xd(y)\in V$, we have $0 = D(x)y$ for all $y\in U$, Therefore $D(x)\subset V$ for all $x\in V$.

So $D(x)\in U\cap V$.

By right multiplying with D(x) in 0 = D(x)y, we get

$$D(x)yD(x) = 0 \text{ for all } x,y\in R.$$

Since R is semiprime, D(x) = 0 for all $x\in V$.

Thus D(V) = 0. Similarly we obtain d(V) = 0.

Theorem 1: Let (D, d) and (G, g) be generalized derivations of a semiprime nonaccessible ring R. Then the following conditions are equivalent.

i) (D, d) and (G, g) are orthogonal.

ii) For all $x,y\in R$, the following relations hold :
   a) D(x)G(y) + G(x)D(y) = 0.
   b) d(x)G(y) + g(x)D(y) = 0.

iii) D(x)G(y) = d(x)G(y) = 0 for all $x,y\in R$.

iv) D(x)G(y) = 0 for all $x,y\in R$ and dG = dg = 0.

v) (DG, dg) is a generalized derivation and D(x)G(y) = 0 for all $x,y\in R$.

vi) There exist ideals U and V of R such that
   (a) $U\cap V = 0$ and $U\oplus V$ is an essential ideal of R.
   (b) $D(R),d(R) \subset U$ and $G(R),g(R)\subseteq V$.
   (c) D(V) = d(V) = 0 and G(U) = g(U) = 0.

Proof: (i) $\Rightarrow$ (ii), (iii), (iv) and (v) are proved by lemma 1 and corollary 1

Now we prove (ii) $\Rightarrow$ (i)

From (ii) (a), we have D(x)G(y) + G(x)D(y) = 0.

By replacing x by xz, we get

$$D(xz)G(y) + G(xz)D(y) = 0.$$
$$(D(x)z + xd(z))G(y) + (G(x)z + xg(z))D(y) = 0.$$
$$(D(x)z)G(y) + (xd(z))G(y) + (G(x)z)D(y) + (xg(z))D(y) = 0.$$

By lemma 2, we have D(x)zG(y) + xd(z)G(y) + G(x)zd(y) + xg(z)d(y) = 0.

Using (ii) (b) we have D(x)zG(y) + G(x)zD(y) = 0 for all $x,y,z \in R$.

Thus by ([4] equation (1)), we have

$$D(x)\, RG(y) = G(y)\, RD(x) = 0 \text{ for all } x, y\in R.$$

Hence D and G are orthogonal, which proves (i)

Next we prove (iii) $\Rightarrow$ (i)

By (iii), we have D(x)G(y) = 0 = d(x)G(y).

Now we replace x and xz in D(x)G(y) = 0. Then we get D(xz)G(y) = 0.

$$(D(x)z)G(y) + (xd(z))G(y) = 0.$$
$$D(x)zG(y) + xd(z)G(y) =0, \text{ by([4] lemma 2)}.$$
$$D(x)zG(y) = 0 \text{ for all } x, y, z\in R.$$

So D and G are orthogonal which proves (i).

We prove (iv) $\Rightarrow$ (i)

By (iv), we have D(x)G(y) = 0.

Since dG=dg = 0, we have dG(x) = 0.

Now by taking x=xy in the above equation, we get

$$dG(xy) = 0.$$
$$d(G(x)y + xg(y)) = 0.$$
$$d(G(x)y) + d(xg(y)) = 0.$$
$$d(G(x))y + G(x)d(y) + d(x)g(y) + xd(g(y)) = 0.$$

By lemma 1, $dG(x)y + G(x)d(y) + d(x)g(y) + xdg(y) = 0$.

By (iv) and lemma 1 (iv), we get $G(x)d(y) = 0$ for all $x,y\in R$.

Now we substitute x=z in the above equation. Then

$$G(xz)d(y) = 0$$
$$(G(x)z)d(y) + (xg(z))\, d(y) = 0$$
$$G(x)zd(y) + xg(z)d(y) = 0 \text{ for all } x, y, z\in R, \text{ by ([4] lemma 2).}$$

Then by lemma 1, we have d and g are orthogonal.

Therefore $G(x)zd(y) = 0$.

Hence we get $d(y)G(x) = 0$ for all $x,y,z\in R$, by ([4] lemma 1).

Then by lemma 1 (iii), we have $g(x)D(y) = 0$.

From this last equation and $d(y)G(x) = 0$, we get (D, d) and (G, g) are orthogonal generalized derivations.

Next (v) $\Rightarrow$ (i)

Since (DG, dg) is a generalized derivation, dg is a derivation. Then we obtain

$$DG(xy) = DG(x)y + xdg(y) \text{ for all } x, y\in R. \tag{8}$$

Also we have

$$DG(xy) = D(G(x)y + xg(y)) = D(G(x)y) + D(xg(y)).$$
$$DG(xy) = D(G(x))y + G(x)d(y) + D(x)g(y) + xd(g(y)).$$
$$DG(xy) = DG(x)y + G(x)d(y) + D(x)g(y) + xdg(y). \tag{9}$$

Comparing the (8) and (9), we get

$$G(x)d(y) + D(x)g(y) = 0 \text{ for all } x, y\in R.$$

We have $D(x)G(y) = 0$ for all $x,y \in R$.

Then by substituting y = yz in the last equation, we get

$$D(x)G(yz) = 0.$$
$$D(x)(G(y)z) + D(x)(yg(z)) = 0.$$
$$D(x)G(y)z + D(x)yg(z) = 0.$$
$$D(x)yg(z) = 0 \text{ for } x,y,z\in R.$$

Hence by ([4] lemma 1), we obtain $g(z)D(x) = 0$ for all $x, z\in R$.

Replacing z by yz in the last relation,we get

$$g(yz)D(x) = 0.$$
$$(g(y)z)D(x) + (yg(z))D(x) = 0.$$
$$g(y)zD(x) + yg(z)D(x) = 0.$$
$$g(y)zD(x) = 0 \text{ for all } x, y, z\in R.$$

By ([4] lemma 1), we have $D(x)g(y) = 0$ for all $x, y\in R$.

Similarly, this last equation implies $G(x)d(y) = 0$ for all $x, y\in R$.

This shows that $d(y)G(x) = 0$ for all $x, y\in R$.

Therefore by (iii) it follows that D and G are orthogonal.

Now we prove (i) $\Rightarrow$ (vi)

Let U_o be the ideal of R generated by $d(R) \cup D(R)$.

Let $Ann(U_o) = V$ and $Ann(V) = U$.

By lemma 1, we have $D(x)G(y) = G(x)D(y) = 0$,

$$d(x)G(y) = g(x)D(y) = 0 \text{ and } d(x)g(y) = g(y)d(x) = 0 \text{ for all } x, y \in R.$$

Since $D(R), d(R) \subset U_o$, we obtain $G(R), g(R) \subset V$.

It is seen that by lemma 2 and $U_o \subset V$, we have

$$D(V) = d(V) = 0 \text{ and } G(U) = g(U) = 0.$$

Since R is semiprime, $U \oplus V$ is an essential ideal of R, which shows (vi).

Theorem 2: Let (D, d) and (G, g) be generalized derivations of a semiprime nonaccessible ring R. Then the following conditions are equivalent.

1) (DG, dg) is a generalized derivation.
2) (GD, gd) is a generalized derivation.
3) D and g are orthogonal and G and d are orthogonal.

Proof: First we prove (i) $\Leftrightarrow$ (iii)

Now we prove (i) $\Rightarrow$ (ii).

First we assume that (DG, dg) is a generalized derivation.

Thus as in the proof of the theorem 1 (v) $\Rightarrow$ (i), we obtain

$$G(x)d(y) + D(x)g(y) = 0.$$

By substituting y = yz in the above relation, where $z \in R$, we get

$$G(x)d(yz) + D(x)g(yz) = 0 \text{ for all } x,y,z \in R.$$

$$G(x)d(y)z + G(x)yd(z) + D(x)g(y)z + D(x)yg(z) = 0.$$

$$G(x)yd(z) + D(x)yg(z) + (G(x)d(y) + D(x)g(y))z = 0.$$

$$\text{i.e., } G(x)yd(z) + D(x)yg(z) = 0. \tag{10}$$

Since (DG, dg) is a generalized derivation, dg is a derivation.

Therefore d and g are orthogonal by theorem 1.

Now we substitute $y = g(z)y$ in (10).Then we have

$$0 = G(x)g(z)yd(z) + D(x)g(z)yg(z).$$

$$0 = D(x)g(z)yg(z) \text{ for all } x, y, z \in R.$$

Hence we get $D(x)g(z)\, RD(x)g(z) = 0$ for all $x, z \in R$.

Since R is semiprime, $D(x)g(z) = 0$ for all $x, z \in R$.

Thus $D(x)yg(z) = 0$ for all $x,y,z \in R$.

From (10) we have $G(x)yd(z) = 0$ for all $x, y, z \in R$.

By ([4] lemma 1), we have $G(x)d(z) = 0$.

Hence $D(x)g(z) = G(x)d(z) = 0$ for all $x, z \in R$.

So D and g are orthogonal and G and d are orthogonal.

Next we prove (iii) $\Rightarrow$ (i)

Since D and g are orthogonal, we get $D(x)yg(z) = 0$ for all $x, y, z \in R$. (11)

We substitute rx for x in the last relation. Then we have

$$0 = D(rx)yg(y).$$
$$0 = (D(r)x + rd(x))yg(z).$$
$$0 = D(r)xyg(z) + rd(x)yg(z).$$
$$0 = rd(x)yg(z), \text{ using (11).}$$

By left multiplying with d(x)yg(z) in this last equation, we get

$$d(x)yg(z)rd(x)yg(z) = 0.$$

Since R is a semiprime, we have d(x)yg(z) = 0.

From ([4] lemma 1), we get d(x)g(z) = 0 for all $x, z \in R$.

Thus by theorem 1, we get dg is a derivation.

Moreover since D(x)yg(z) = 0 for all $x, y, z \in R$.

We also get D(x)(g(z)RD(x))g(z) = 0 and since R is semiprime, D(x)g(z) = 0 for all $x, z \in R$.

Similarly, since G and d are orthogonal, we have G(x)d(y) = 0 for all $x, y \in R$.

Thus we obtain DG(xy) = DG(x)y + xdg(y) for all $x, y \in R$, which shows that (DG,dg) is a generalized derivation.

Similarly (ii) $\Leftrightarrow$ (iii) is proved.

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