

**A COMPARATIVE STUDY  
OF DISTANCE MEASURES OF INTUITIONISTIC FUZZY MULTI SETS**

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**ABSTRACT**

*The Hausdroff, Normalized Geometric and Normalized Hamming distance measures of Intuitionistic Fuzzy Multi sets (IFMS) are compared in this work. Also this paper constitutes of identifying the best Distance measures of IFMSs. The comparison analysis is done by applying them in pattern recognition problems.*

**Key Words:** *Intuitionistic fuzzy set, Intuitionistic Fuzzy Multi sets, Hausdroff distance, Geometric distance, Normalized Hamming distance.*

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**1. INTRODUCTION.**

In 1965, Lofti A. Zadeh [11] introduced the concept of Fuzzy sets (*FS*), was the generalisation of Crisp sets. There the fuzzy set allows the object to partially belong to a set with a membership degree ( $\mu$ ) between 0 and 1. Later, The generalization of Fuzzy sets, introduced by Krassimir T. Atanassov [1, 2] was the Intuitionistic Fuzzy sets (*IFS*), which represent the uncertainty with respect to both membership ( $\mu \in [0,1]$ ) and non membership ( $\vartheta \in [0,1]$ ) such that  $\mu + \vartheta \leq 1$ . The number  $\pi = 1 - \mu - \vartheta$  is called the hesitation degree or intuitionistic index. As they can present the degrees of membership and non membership, the *IFSs* are widely applied in the area of logic programming, decision making, pattern recognition and medical diagnosis. Also *IFSs* defined on the same universe are compared using the Distance Measures. (Szmidt and Kacprzyk [7][8]and [9]).

R. R. Yager [10] introduced the Fuzzy Multi Sets (*FMSs*), as Multi sets [3] allow the repeated occurrences of any element. In the *FMSs*, the occurrences are more than one with the possibility of the same or the different membership functions. Later T.K Shinoj and Sunil Jacob John [6] generalised the new concept of Intuitionistic Fuzzy Multi Sets (*IFMSs*) from the Fuzzy Multi Sets (*FMSs*) in 2012 consisting of the uncertainties membership, non membership and hesitation functions.

As the Numerical results [4] and [5] show that the *IFMSs* distances measures are well suited one to real time application; in this paper, we extend to identify the best distance measure of *IFMSs*. Hence, The Hausdroff, Normalized Geometric and Normalized Hamming distance measures of *IFMSs* are applied to examine the capabilities to cope in pattern recognition problems.

The organization of this paper is: In section 2, the Fuzzy Multi sets, Intuitionistic Fuzzy Multi sets, Distance measure of *IFMSs* are explained. The Analysis of the distance measures of the *IFMSs* are proposed in Section 3. The section 4, the optimal distance measure of *IFMSs* is determined. The comparison analysis of the *IFMSs* Distance Measures is made in Section 5. And the Section 6 states the Conclusion.

**II. PRELIMINARIES**

Some basic concepts and definitions are given here,

**Definition 2.1 [1] [2]:**

An *Intuitionistic fuzzy set (IFS)*,  $A$  in  $X$  is given by  $A = \{(x, \mu_A(x), \vartheta_A(x)) / x \in X\}$  (2.1)  
where  $\mu_A: X \rightarrow [0,1]$  and  $\vartheta_A: X \rightarrow [0,1]$  with the condition  $0 \leq \mu_A(x) + \vartheta_A(x) \leq 1, \forall x \in X$

Here  $\mu_A(x)$  and  $\vartheta_A(x) \in [0, 1]$  denote the membership and the non membership functions of the fuzzy set A;

For each Intuitionistic fuzzy set in X,  $\pi_A(x) = 1 - \mu_A(x) - [1 - \mu_A(x)] = 0$  for all  $x \in X$  that is  $\pi_A(x) = 1 - \mu_A(x) - \vartheta_A(x)$  is the hesitancy degree of  $x \in X$  in A. Always  $0 \leq \pi_A(x) \leq 1, \forall x \in X$ .

The **complementary set**  $A^c$  of A is defined as  $A^c = \{ \langle x, \vartheta_A(x), \mu_A(x) \rangle / x \in X \}$  (2.2)

**Definition 2.2 [10]:** Let X be a nonempty set. A **Fuzzy Multi set (FMS)** A in X is characterized by the count membership function Mc such that Mc:  $X \rightarrow Q$  where Q is the set of all crisp multi sets in  $[0,1]$ . Hence, for any  $x \in X$ , Mc(x) is the crisp multi set from  $[0, 1]$ . The membership sequence is defined as

$$(\mu_A^1(x), \mu_A^2(x), \dots, \dots, \mu_A^p(x)) \text{ where } \mu_A^1(x) \geq \mu_A^2(x) \geq \dots \geq \mu_A^p(x).$$

Therefore, A FMS A is given by  $A = \{ \langle x, (\mu_A^1(x), \mu_A^2(x), \dots, \dots, \mu_A^p(x)) \rangle / x \in X \}$  (2.3)

**Definition 2.3 [6]:** Let X be a nonempty set. A **Intuitionistic Fuzzy Multi set (IFMS)** A in X is characterized by two functions namely count membership function Mc and count non membership function NMc such that Mc :  $X \rightarrow Q$  and NMc :  $X \rightarrow Q$  where Q is the set of all crisp multi sets in  $[0,1]$ . Hence, for any  $x \in X$ , Mc(x) is the crisp multi set from  $[0, 1]$  whose membership sequence is defined as

$(\mu_A^1(x), \mu_A^2(x), \dots, \dots, \mu_A^p(x))$  where  $\mu_A^1(x) \geq \mu_A^2(x) \geq \dots \geq \mu_A^p(x)$  and the corresponding non membership sequence NMc (x) is defined as  $(\vartheta_A^1(x), \vartheta_A^2(x), \dots, \dots, \vartheta_A^p(x))$  where the non membership can be either decreasing or increasing function. such that  $0 \leq \mu_A^i(x) + \vartheta_A^i(x) \leq 1, \forall x \in X$  and  $i = 1, 2, \dots, p$ .

Therefore, An **IFMS** A is given by  $A = \{ \langle x, (\mu_A^1(x), \mu_A^2(x), \dots, \dots, \mu_A^p(x)), (\vartheta_A^1(x), \vartheta_A^2(x), \dots, \dots, \vartheta_A^p(x)) \rangle / x \in X \}$  (2.4) where  $\mu_A^1(x) \geq \mu_A^2(x) \geq \dots \geq \mu_A^p(x)$

The **complementary set**  $A^c$  of A is defined as

$$A^c = \{ \langle x, (\vartheta_A^1(x), \vartheta_A^2(x), \dots, \dots, \vartheta_A^p(x)), (\mu_A^1(x), \mu_A^2(x), \dots, \dots, \mu_A^p(x)) \rangle / x \in X \}$$

where  $\vartheta_A^1(x) \geq \vartheta_A^2(x) \geq \dots \geq \vartheta_A^p(x)$  (2.5)

**Definition 2.4 [6]:** The **Cardinality** of the membership function Mc(x) and the non membership function NMc (x) is the length of an element x in an **IFMS** A denoted as  $\eta$ , defined as  $\eta = |Mc(x)| = |NMc(x)|$  If A, B, C are the **IFMS** defined on X, then their cardinality  $\eta = \text{Max} \{ \eta(A), \eta(B), \eta(C) \}$ .

**Definition 2.5 [4]: HAUSFROFF DISTANCE MEASURE**

In Hamming metrics, the Hausdroff distance is defined as

$$d_h(A, B) = \frac{1}{\eta} \sum_{j=1}^{\eta} \{ \frac{1}{n} \sum_{i=1}^n \max [ (|\mu_A^j(x_i) - \mu_B^j(x_i)|, |\vartheta_A^j(x_i) - \vartheta_B^j(x_i)|) ] \}$$
 (2.6)

and with all three degrees, it is

$$d_h(A, B) = \frac{1}{\eta} \sum_{j=1}^{\eta} \{ \frac{1}{n} \sum_{i=1}^n \max [ (|\mu_A^j(x_i) - \mu_B^j(x_i)|, |\vartheta_A^j(x_i) - \vartheta_B^j(x_i)|, |\pi_A^j(x_i) - \pi_B^j(x_i)|) ] \}$$
 (2.7)

**Definition 2.6 [5]: GEOMETRIC DISTANCE MEASURE**

The Geometric distance of the Intuitionistic Multi Fuzzy set is defined as

$$D_g(A, B) = \frac{1}{\eta} \sum_{j=1}^{\eta} \{ \frac{1}{n} \sum_{i=1}^n \sqrt{(\mu_A^j(x_i) - \mu_B^j(x_i))^2 + (\vartheta_A^j(x_i) - \vartheta_B^j(x_i))^2} \}$$
 (2.8)

and when all degrees are taken under consideration , it

$$D_g(A, B) = \frac{1}{\eta} \sum_{j=1}^{\eta} \{ \frac{1}{n} \sum_{i=1}^n \sqrt{(\mu_A^j(x_i) - \mu_B^j(x_i))^2 + (\vartheta_A^j(x_i) - \vartheta_B^j(x_i))^2 + (\pi_A^j(x_i) - \pi_B^j(x_i))^2} \}$$
 (2.9)

Where the Normalized Geometric distance is  $D_G(A, B) = \frac{1}{\sqrt{2}} D_g(A, B)$  (2.10)

**Definition 2.7 [5]: NORMALIZED HAMMING DISTANCE MEASURE**

In the **IFMS**, the Normalized Hamming distance is

$$N_D^*(A, B) = \frac{1}{\eta} \sum_{j=1}^{\eta} \{ \frac{1}{2n} \sum_{i=1}^n (|\mu_A^j(x_i) - \mu_B^j(x_i)| + |\vartheta_A^j(x_i) - \vartheta_B^j(x_i)|) \}$$
 (2.11)

and with all three degrees taken under consideration it becomes

$$N_D^*(A, B) = \frac{1}{\eta} \sum_{j=1}^{\eta} \{ \frac{1}{2n} \sum_{i=1}^n (|\mu_A^j(x_i) - \mu_B^j(x_i)| + |\vartheta_A^j(x_i) - \vartheta_B^j(x_i)| + |\pi_A^j(x_i) - \pi_B^j(x_i)|) \}$$
 (2.12)

### III. ANALYSIS OF DISTANCE MEASURES

The Distance Measures have gained much attention for their wide applications in real world, such as pattern recognition, machine learning, decision making and market prediction. Although, all the proposed distance Measure of *IFMS* provide an effective way to deal with the real life situations, in some cases it obtains results where the accuracy is doubtful. Hence, we have analysed the proposed Distance Measures of *IFMS* to identify an efficient measure of Distance.

#### Analysis of Distance of *IFMS* of equal cardinality:

The illustrated problem illustrates that the *IFMS* of equal cardinality's Distance and Similarity Measures numerical value do not significantly deviate, and all the proposed measures are applicable for any real life situation.

Let  $X = \{A_1, A_2, A_3, A_4, A_5, \dots, A_n\}$  with  $A = \{A_1, A_2, A_3, A_4, A_5\}$  and  $B = \{A_6, A_7, A_8, A_9, A_{10}\}$  such that the *IFMSs*  $A$  and  $B$  are defined in terms of membership and non-membership functions (two parametric functions).

$$A = \{ \langle A_1: (0.6,0.4), (0.5, 0.5) \rangle, \langle A_2: (0.5,0.3), (0.4, 0.5) \rangle, \langle A_3: (0.5, 0.2), (0.4, 0.4) \rangle, \langle A_4: (0.3, 0.2), (0.3, 0.2) \rangle, \langle A_5: (0.2,0.1), (0.2, 0.2) \rangle \}$$

$$B = \{ \langle A_6: (0.8,0.1), (0.4, 0.6) \rangle, \langle A_7: (0.7, 0.3), (0.4, 0.2) \rangle, \langle A_8: (0.4, 0.5), (0.3, 0.3) \rangle, \langle A_9: (0.2,0.7), (0.1, 0.8) \rangle, \langle A_{10}: (0.2,0.6), (0, 0.6) \rangle \}$$

Here, the cardinality  $\eta = 5$  as  $|Mc(A)| = |NMc(A)| = 5$  and  $|Mc(B)| = |NMc(B)| = 5$

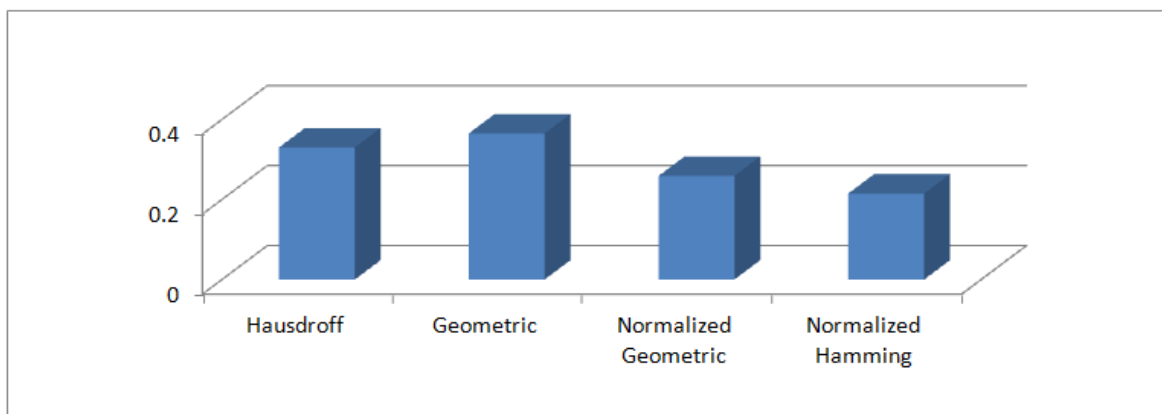
The *IFMSs*  $A$  and  $B$  defined are of five Intuitionistic fuzzy sets, each consisting of two elements. Here the *IFMSs*  $A$  and  $B$  consist of **equal cardinality** with the value of  $n = 2$  and  $\eta = 5$ .

The consolidated *IFMS* Distance Measures of equal cardinality are presented in a table and to highlight that the values are alike. Further, it is observed that the numerical value of Geometric Distance Measure is comparatively greater than that of the other measures.

<i>IFMS</i> Hausdorff Distance Measure	$d_h(A, B) = 0.33$
<i>IFMS</i> Geometric Distance Measure	$D_g(A, B) = 0.365$
<i>IFMS</i> Normalized Geometric Distance Measure	$D_G(A, B) = 0.25887$
<i>IFMS</i> Normalized Hamming Distance Measure	$N_D^*(A, B) = 0.215$

**Table-3.1:** *IFMS* Distance Measures of equal cardinality

The Figure 3.1 construes that the **Geometric Distance Measure** is comparatively greater and the next greater measure is the **Hausdorff Distance Measure**.



**Figure-3.1:** *IFMS* Distance Measures of equal cardinality

**Analysis of Distance of IFMS of unequal cardinality:**

As the subsequent problem illustrates that the IFMS of unequal cardinality’s Distance and Similarity Measures numerical value do not significantly deviate, all the proposed measures are applicable for any real life situation.

Let  $X = \{A_1, A_2, A_3, A_4, \dots, A_n\}$  with  $A = \{A_1, A_2, A_3\}$  and  $B = \{A_6\}$  such that the IFMSs  $A$  and  $B$  are  
 $A = \{ \langle A_1: (0.6, 0.2, 0.2), (0.4, 0.3, 0.3), (0.1, 0.7, 0.2), (0.5, 0.4, 0.1), (0.2, 0.6, 0.2) \rangle, \langle A_2: (0.7, 0.1, 0.2), (0.3, 0.6, 0.1), (0.2, 0.7, 0.1), (0.6, 0.3, 0.1), (0.3, 0.4, 0.3) \rangle, \langle A_3: (0.5, 0.4, 0.1), (0.4, 0.4, 0.2), (0, 0.8, 0.2), (0.7, 0.2, 0.1), (0.4, 0.4, 0.2) \rangle \}$   
 $B = \{ \langle A_6: (0.8, 0.1, 0.1), (0.2, 0.7, 0.1), (0.3, 0.5, 0.2), (0.5, 0.3, 0.2), (0.5, 0.4, 0.1) \rangle \}$

Here  $L(A, B) = \eta = 3$  as  $|Mc(A)| = |NMc(A)| = 3$  and  $|Mc(B)| = |NMc(B)| = 1$

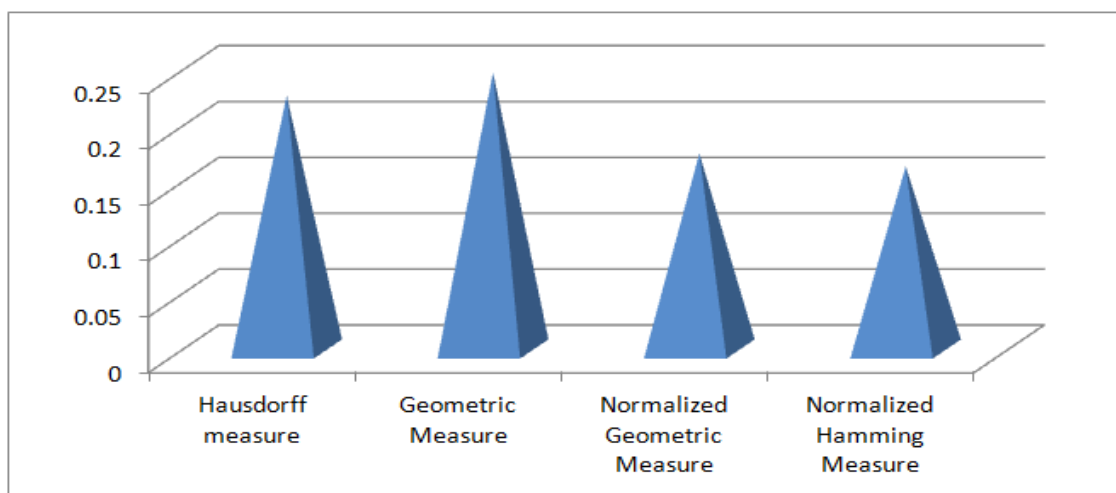
Hence, their cardinality  $\eta = \text{Max}\{\eta(A), \eta(B)\} = \text{max}\{3, 1\} = 3$ .

The above defined IFMSs  $A$  and  $B$  are in terms of membership, non-membership and hesitation functions (three parametric functions). Here, IFMS  $A$  has three Intuitionistic fuzzy sets consisting of five elements, whereas the IFMS  $B$  has one Intuitionistic fuzzy set consisting of five elements. Hence, the IFMSs  $A$  and  $B$  are of unequal cardinality (3 and 1).

IFMS Hausdorff Distance Measure	$d_h(A, B) = 0.22667$
IFMS Geometric Distance Measure	$D_g(A, B) = 0.2475$
IFMS Normalized Geometric Distance Measure	$D_G(A, B) = 0.1750$
IFMS Normalized Hamming Distance Measure	$N_D^*(A, B) = 0.1633$

**Table-3.2:** IFMS Distance Measures of unequal cardinality

The consolidated Distance Measures are presented in the form a table and a figure to make it obvious that the values are similar. Further, it is observed that the numerical value of Geometric Distance Measure is comparatively greater than that of the other measures.



**Figure-3.2:** IFMS Distance Measures of unequal cardinality

**IV. OPTIMUM DISTANCE MEASURE**

In order to study the ability of the proposed Distance Measures, a set of experiments has been conducted. For this purpose, a pattern recognition problem, according to which a test sample has to be recognised by classifying it to a specific category, is selected.

For the Comparative analysis of the proposed Distance Measures, a performance index called **Degree of Confidence** is considered. This index measures the confidence of each Distance Measures in recognizing a specific pattern that belongs to the pattern(x) in the following form

$$\text{Degree of Confidence} = \sum_{i=1, i \neq j}^n | \text{Distance}(P_j, T) - \text{Distance}(P_i, T) |$$

The greater the Degree of confidence the more accurate the Distance Measure in recognizing the patterns correctly.

In the following example, attributes correspond to the measurements that are used to describe each class, which classes are represented by specific pattern. This procedure constitutes the main operation of the maximum-similarity classifier. (i.e) the test sample is assigned to the pattern for which its similarity is higher.

Let  $X = \{A_1, A_2, A_3, A_4, \dots, A_n\}$  with  $A = \{A_1, A_2\}$ ;  $B = \{A_4, A_6\}$ ;  $C = \{A_1, A_{10}\}$ ;  $D = \{A_4, A_6\}$ ;  $E = \{A_4, A_6\}$ .

A, B, C, D and E are the IFMSs defined as  
 $A = \{\langle A_1 : (0.1, 0.2) \rangle, \langle A_2 : (0.3, 0.3) \rangle\}$ ;  
 $B = \{\langle A_4 : (0.2, 0.2) \rangle, \langle A_6 : (0.3, 0.2) \rangle\}$ ;  
 $C = \{\langle A_1 : (0.1, 0.2) \rangle, \langle A_{10} : (0.2, 0.3) \rangle\}$ ;  
 $D = \{\langle A_3 : (0.2, 0.1) \rangle, \langle A_4 : (0.3, 0.2) \rangle\}$ ;  
 $E = \{\langle A_1 : (0.5, 0.4) \rangle, \langle A_4 : (0.8, 0.1) \rangle\}$

The IFMS Pattern  $Y = \{\langle A_1 : (0.1, 0.2) \rangle, \langle A_{10} : (0.2, 0.3) \rangle\}$

Here, the cardinality  $\eta = 2$  as  $|Mc(A)| = |NMc(A)| = 2$  and  $|Mc(B)| = |NMc(B)| = 2$ ,

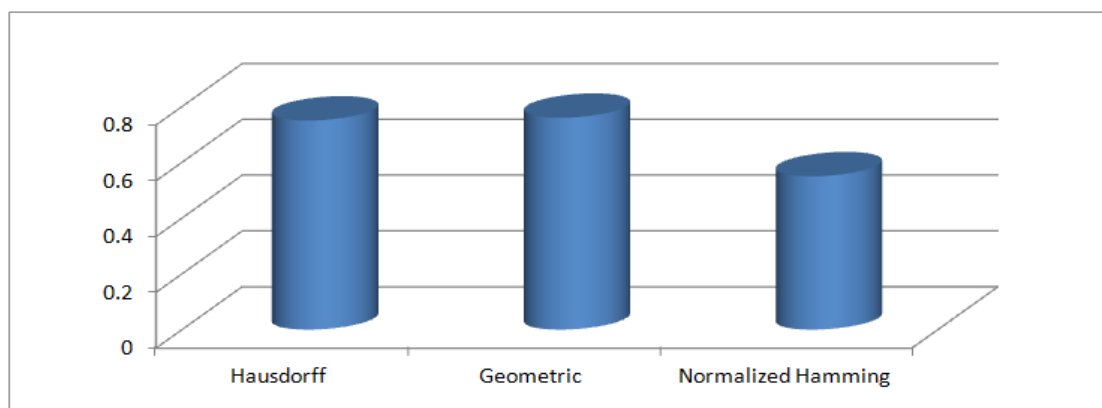
In this problem, the IFMSs defined have IFs of two parametric functions—membership and non membership functions. The IFMS Patterns defined are of two Intuitionistic fuzzy sets, each consisting of one element. Hence, the IFMS Patterns are of equal cardinality with the value of  $n = 1$  and  $\eta = 2$ .

The consolidated proposed IFMS Distance Measures with the Degree of Confidence are tabulated to construe that **Geometric Distance Measure is the best** as it is comparatively greater than that of other measures. The next best measure is the **Hausdorff Distance Measure**.

Distance Measures	(A,Y)	(B,Y)	(C,Y)	(D,Y)	(E,Y)	Degree of Confidence
Hausdorff 's Measure	0.05	0.1	0	0.1	0.5	0.75
Geometric Measure	0.035	0.085	0	0.1	0.5398	<b>0.7598</b>
Normalized Hamming Measure	0.025	0.075	0	0.1	0.35	0.55

**Table-4.1:** Degree of Confidence for the IFMS Distance Measures

Higher Degree of Confidence gives a more accurate measurement of the distance's behaviour in pattern recognition rate and from table and figure it is obvious that the **Geometric distance is highly confident**.



**Figure -4.1:** Degree of Confidence for the IFMS Distance Measures

## V. COMPARISON OF DISTANCE MEASURES OF IFMS

To identify the best measure from all the defined frameworks which have the capabilities to cope with the uncertainty in pattern recognition problem is given below.

The IFMS patterns A, B and C defined have equal Intuitionistic fuzzy sets, each consisting of two parametric functions – membership and non membership functions. The IFMS Patterns are of two Intuitionistic fuzzy sets, each consisting of one element. Hence, the IFMS Patterns are of equal cardinality with the value of  $n = 1$  and  $\eta = 2$ .

Let  $X = \{A_1, A_2, A_3, A_4, \dots, A_n\}$  with  $A = \{A_1, A_2\}$ ;  $B = \{A_1, A_6\}$ ;  $C = \{A_1, A_4\}$ .

A, B and C are the IFMSs defined as

$$A = \{\langle A_1: (0.1, 0.2) \rangle, \langle A_2: (0.3, 0.3) \rangle\};$$

$$B = \{\langle A_1: (0.1, 0.2) \rangle, \langle A_6: (0.2, 0.3) \rangle\};$$

$$C = \{\langle A_1: (0.1, 0.2) \rangle, \langle A_4: (0.2, 0.2) \rangle\}$$

Here, the cardinality  $\eta = 2$  as  $|Mc(A)| = |NMc(A)| = 2$  and  $|Mc(B)| = |NMc(B)| = 2$

The consolidated Pattern recognition of the IFMS Distance Measures is tabulated to construe that Geometric Distance Measure is the best as it is specify the significant difference between the patterns. The next best measure is the Normalized Hamming distance.

IFMS Distance Measure of equal cardinality	(A, B) Distance Between A and B	(A, C) Distance Between A and C
Hausdorff Distance Measure	0.05	0.05
Geometric Distance Measure	<b>0.05</b>	<b>0.0707</b>
Normalized Geometric Distance Measure	0.035	0.05
Normalized Hamming Distance Measure	<b>0.025</b>	<b>0.05</b>

Table-5.1: Pattern Identification using the Distance Measures of IFMS

With the help of counter-intuitive example (IFMSs values are closer to each other. It is clear that the Geometric distance bears a greater difference among the values, and this is followed by the Normalized Hamming distance.

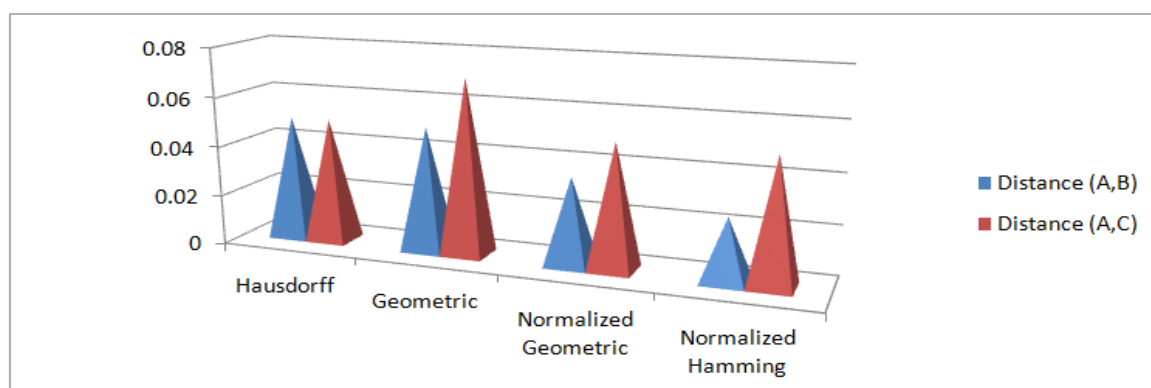


Figure-5.1: Pattern Identification using the Distance Measures of IFMS

## VI. CONCLUSION

Firstly, the analysis of IFMS Distance Measures of equal and unequal cardinalities show that they are simple, effective and can help the decision makers efficiently. Secondly, after making a comparative study of the IFMS Distance Measures, through the Optimum Distance Measure, it is established that the Geometric distance is better. Finally, a pattern recognition problem has been presented to illustrate the effectiveness of the IFMS Distance Measures.

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