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GENERAL VARIATIONAL-LIKE INEQUALITIES

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ABSTRACT

In this paper, we consider a new concept of relaxed η-α-pseudomonotonicity with respect to g. By using KKM technique, we proved some existence results for general variational like inequalities in reflexive Banach spaces.

Keywords and Phrases: *Variational-like inequality, relaxed η-α-pseudomonotone mapping, KKM mapping.*

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1. INTRODUCTION

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. This theory has been generalized and extended in many directions using novel and innovative techniques. In recent years, considerable interest has been shown in developing various classes of variational inequalities both for its own sake and for its applications. Several numerical methods have been developed for solving variational inequalities and related optimization problems. See [1, 3, 6, 7, 8, 9] and references therein.

Inspired and motivated by [4] and references therein, in this paper, we consider general variational-like inequalities in the reflexive Banach spaces. By using KKM technique, some existence results are established for general variational-like inequalities involving relaxed η-α-pseudomonotone mappings.

2. PRELIMINARIES

Let E be a real reflexive Banach space with dual space E^* and $\langle .,. \rangle$ denote the pairing between E and E^*. Let K be a nonempty subset of E, and 2^E denote the family of all the nonempty subset of E.

Definition: 2.1 Let f, g: K → K, η : K × K → E be the mappings. A mapping T: K→ 2^{E^*} is said to be relaxed η-α-pseudomonotone with respect to g if there exists a function α : E → R with $\alpha(tz) = t^p \alpha(z)$ for all $t > 0$ and $z \in E$ such that for any $x \in K$, $u \in T(x)$

$\langle f(u), \eta(g(x), y)\rangle \geq 0$ implies that there exists $y \in K$, $u \in T(y)$ such that

$\langle f(u), \eta(g(x), y)\rangle \geq \alpha(g(x) - y)$ where $p > 1$ is a constant.

If f, g ≡ I, T is single-valued mapping, then our Definition 2.1 coincides with Definition 2.1 of [4].

The following definitions are important for the proof of main results of this paper.

Definition: 2.2 Let g: K → K and η : K × K → E be two mappings. A map T: K→ 2^{E^*} is said to be η- hemicontinuous with respect to g, if for any fixed x, y ∈ K, the mapping P : [0, 1] → (−∞,∞) defined by

P (t) = $\langle T(g(x) + t(y - g(x))), \eta(y, g(x))\rangle$ is continuous at 0^+.

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Definition: 2.3[2] A mapping F: K → 2^E is said to be a KKM mapping, if for any $\{x_1, x_2 ...., x_n\} \subset K,\ co\{x_1, x_2 ...., x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$ where $co\{x_1, x_2 ...., x_n\}$ denotes the convex hull of $x_1, x_2 ...., x_n$.

Lemma: 2.1[2] Let K be a nonempty subset of a Hausdorff topological vector space E and let F : K → 2^E be a KKM mapping. If F(x) is closed in E for every x ∈ K and compact for some x_0 ∈ K, then $\bigcap_{x\in k} F(x) \neq \phi$.

3. EXISTENCE RESULTS

In this section, we consider the following general variational-like inequality problem:

Let f, g : K → K, η : K ×K → E and T : K →2^{E*} be the mapping. Find x ∈ K, u ∈ T (x) such that

$$\langle f(u), \eta(y, g(x))\rangle \geq 0,\ for\ all\ \ y \in K \tag{3.1}$$

Where K is a nonempty closed convex subset of a real reflexive Banach space E.

Theorem: 3.1 Let K be a nonempty closed convex subset of a real reflexive Banach space E and f, g: K → K be the mapping. Let T: K →2^{E*} be a η-hemicontinuous with respect to g, relaxed η-α-pseudomonotone with respect to g. Assume that

(i) η (g(x), x) = 0 for all x ∈ K;
(ii) for any fixed y ∈ K and u ∈ T(x), the mapping x →$\langle f(u), \eta(g(x), y)\rangle$ is convex.'

Then the set of elements (x, u) is a solution of (3.1) if and only if

$$\langle f(u), \eta(y, g(x))\rangle \geq \alpha(y - g(x)),\ for\ all\ \ y \in K \tag{3.2}$$

Proof: Suppose that (x, u) is a solution of (3.1). Since T is relaxed η-α-pseudomonotone with respect to g, we have

$$\langle f(u), \eta(y, g(x))\rangle \geq \alpha(y - g(x)),\ for\ all\ \ y \in K$$

Hence (x, u) is a solution of (3.2).

Conversely, suppose that (x, u) is a solution of (3.2) and y ∈ K be any point.

Letting

x_t = ty + (1 − t)g(x), t ∈ (0, 1] such that u_t ∈ T (x_t) then x_t ∈ K. It follows from (3.2) that

$$\begin{aligned}\langle f(u_t), \eta(x_t, g(x))\rangle &\geq \alpha(x_t - g(x))\\ &= \alpha(t(y - g(x)))\\ &= t^p \alpha(y - g(x))\end{aligned} \tag{3.3}$$

By condition (i) and (ii), we have

$$\begin{aligned}\langle f(u_t), \eta(x_t, g(x))\rangle &\leq t\langle f(u_t), \eta(y, g(x))\rangle + (1-t)\langle f(u_t), \eta(x, g(x))\rangle\\ &= t\langle f(u_t), \eta(y, g(x))\rangle\end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\langle f(u_t), \eta(y, g(x))\rangle \geq t^{p-1}\alpha(y - g(x) \quad (3.5)$$

for all y ∈ K. Since T is η-hemicontinuous with respect to g and p > 1, letting t → 0^+in (3.5), we get

$$\langle f(u), \eta(y, g(x))\rangle \geq 0, \quad \textit{for all } y \in K.$$

Therefore (x, u) is a solution of (3.1).

Theorem: 3.2 Let K be a bounded closed convex subset of the reflexive Banach space E. Let f, g: K → K, η: K × K → E be the mapping and T: K →2^{E^*} be a η-hemicontinuous with respect to g and relaxed η-α- pseudomonotone with respect to g. Assume that

(i) η (g(x), y) + η(y, g(x)) = 0, for all x, y ∈ K;
(ii) for any fixed y ∈ K and u ∈ T(x), the mapping $x \mapsto \langle f(u), \eta(g(x), y)\rangle$ is convex and lower semicontinuous;
(iii) α: E → R is weakly lower semicontinuous, i.e. for any net $\{x_\beta\} \subset E$, x_β converges weakly to x_0 implies that $\alpha(x_0) \leq \liminf_\beta \alpha(x_\beta)$

Then there exists at least one solution for (3.1).

Proof: For any y ∈ K, define two set-valued mappings P,Q : K → 2^K as follows:
P(y) = {x ∈ K: ∃ u ∈ T(x) such that $\langle f(u), \eta(y, g(x))\rangle \geq 0\}$
Q(y) = {x ∈ K: ∃ u ∈ T(y) such that $\langle f(u), \eta(y, g(x))\rangle \geq \alpha(y - g(x))\}$

We claim that P is a KKM mapping. If P is not a KKM mapping, then there exists $\{y_1, y_2, ..., y_n\} \subset K$ such that co $\{y_1, y_2, ..., y_n\} \not\subset \bigcup_{i=1}^{n} P(y_i)$

i.e., there exists a $y_0 \in \{y_1, y_2, ..., y_n\}$, $y_0 = \sum_{i=1}^{n} t_i y_i$, where $t_i \geq 0, i = 1, ..., n$, $\sum_{i=1}^{n} t_i = 1$, but $y_0 \notin \bigcup_{i=1}^{n} P(y_i)$.
By the definition of P, we have

$$\langle f(u_0), \eta(y_i, g(y_0))\rangle < 0 \quad \textit{for } i = 1, ..., n.$$

It follows from conditions (i) and (ii) that

$$\begin{aligned} 0 = \langle f(u_0), \eta(y_0, g(y_0))\rangle &= \left\langle f(u_0), \eta(\sum_{i=1}^{n} t_i y_i, g(y_0))\right\rangle \\ &\leq \sum_{i=1}^{n} t_i \langle f(u_0), \eta(y_i, g(y_0))\rangle \\ &< 0 \end{aligned}$$

which is a contradiction. This implies that P is a KKM mapping. Now we prove that

P(y) ⊂ Q(y), for all y ∈ K.

For any given y ∈ K, letting x ∈ P(y), then
$\langle f(u), \eta(y, g(x))\rangle \geq 0.$

Since T is relaxed η-α-pseudomonotone with respect to g, we have there exists u ∈ T (y) such that

$$\langle f(u), \eta(y, g(x))\rangle \geq \alpha(y - g(x))$$

It follows that $x \in Q(y)$ and so

$P(y) \subset Q(y)$, for all $y \in K$.

This implies that Q is also a KKM mapping.

Since the mapping $x \mapsto \langle f(u), \eta(g(x), y)\rangle$ is convex, lower semicontinuous, it follows that it is weakly δ lower semicontinuous. From the definition of Q and the weakly lower semicontinuity of α, it is easy to see that P(y) is weakly closed for all $y \in K$. Since K is bounded closed and convex, we know that K is weakly compact, and so Q(y) is weakly compact in K for each $y \in K$. Therefore, the conditions of Lemma 2.1 are satisfied in the weak topology. It follows from Lemma 2.1 and Theorem 3.1 that

$$\bigcap_{y\in k} P(y) = \bigcap_{y\in k} Q(y) \neq \phi.$$

Hence, there exists $x \in K$ and $u \in T(x)$ such that

$$\langle f(u), \eta(y, g(x))\rangle \geq 0, \text{ for all } y \in K.$$

This completes the proof.

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