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THE ⊛-COMPOSITION OF INTUITIONISTIC FUZZY IMPLICATIONS: AN ALGEBRAIC STUDY OF POWERS AND FAMILIES

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ABSTRACT

Viewing a binary operation ⊛ between pair of fuzzy implications (FIs), the pair of nth powers of self FIs and the pair of a FI and its n th power proposed by Vemuri and Jayaram [15, 16, 17] is extended to the binary operation ⊛ between pair of intuitionistic fuzzy implications (IFIs), the pair of n th powers of self IFIs and the pair of a IFI and its n th power. Finally, the basic properties like neutrality, ordering, exchange principles, the powers w.r.t. ⊛ and their convergence, and the closures of some families of IFIs w.r.to the operation ⊛ are studied.

Keywords: *Intuitionistic fuzzy logic connectives, Intuitionistic fuzzy implications, Algebras, Semi-group, Monoid, Lattice.*

AMS Classification: *03E72.*

1. INTRODUCTION

IFIs are the generalizations of the FIs to the multi-valued setting. They play an important role in decision theory, control theory, approximate reasoning, expert systems, etc. The different ways of obtaining FIs and the better understanding of the behavior of different models for FIs, and their algebraic structures are studied in the book [3]. Recently, some authors studied FIs from different perspectives [3, 16]. Yager [19] studied some new classes of implication operators and their role in approximate reasoning. Naturally, investigating the common properties of some important FIs used in fuzzy logic is meaningful. The implicative represen- tation of the rule base is less frequent in applications, but more challenging. In this case, each single rule is understood as an individual condition/restriction imposed on a modeled input-output relation. Different rules correspond to the different restrictions and are required to hold simultaneously. The view of the stored knowledge is logically driven and the related fuzzy relation can be seen as a kind of theory in the logical sense. The issue of coherence (logical consistency) of the rule base is critical in this case [15]. Deschrijver and Kerre [6] established the relationships between fuzzy sets, L-fuzzy sets [10], interval-valued fuzzy sets [7], intuitionistic fuzzy sets (IFSs), and interval-valued IFSs. The major difference among the three classes lies in the choice of FI operators. In general, fuzzy systems choose t-norm as implication operators, such as the min or product operator, while Boolean fuzzy systems utilize genuine multi-valued implications that mainly contain R-implication, S-implication and QL-implication [1]. Deschrijver and Kerre [8] introduced some aggregation operators on the lattice L and defined the special classes of binary aggregation operators based on t-norms on the unit interval. Liu and Wang [11] gave corrections of some limitations of the article [8]. Deschrijver *et al.* [5] introduced the concepts of IF t-norm and t-conorm and investigated under which conditions a similar representation theorem can be obtained. Cornelis *et al.* [4] constructed a representation theorem for Lukasiewicz implicators on the lattice, which serves as the underlying algebraic structure for both IFSs and interval-valued fuzzy sets. Shi *et al.* [12] investigated constructive methods for IFI operators. Atanassov [2] introduced five new IF operations on IFSs containing multiplication and studied their proper- ties. In [9], the Atanassov's operator was considered together with representable Atanassov's Intuitionistic De Morgan triples in order to generate new De Morgan triples, providing an extension of the notions of De Morgan triples and automorphisms on unitary interval for IFSs. Many approximate reasoning and interesting results in IF environment are reported [18]. In this work, our aim is two-fold. On the one hand, we are interested to develop a binary operation ⊛ between two IFIs and studied their properties and algebraic structures. On the other hand, we are interested to the same binary operation ⊛ among n-th power self IFIs and studied the same properties as in two different IFIs.

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The rest of the paper is organized as follows. Section 2 is devoted to introduce the basic definitions and useful theorems relevant to the proposed work. In Section 3, the monoid of IFIs are discussed. The basic properties of IFIs w.r.to ⊛-composition are introduced in Section 4 and 5. In Section 6, 7, 8 and 9, self composition w.r.to ⊛-$I_I^{[n]}$, convergence of $I_I^{[n]}$, closure of $I_I^{[n]}$, w.r.to the basic properties and closure of $I_I^{[n]}$ w.r.t. functional equations are discussed. Finally, we conclude this paper in Section 10. In Table 1, the functions sg and $\overline{sg}$ are defined by,

$$\mathrm{sg(a)} = \begin{cases} 1, & a > 0, \\ 0, & a \le 0, \end{cases} \quad and \quad \overline{sg}(\mathrm{a}) = \begin{cases} 0, & a > 0, \\ 1, & a \le 0. \end{cases}$$

Table-1: List of some IFIs.

Name	Formula
Zadeh 1	$(I_I)_{ZD1}(x,y) = (\max(x_2, \min(x_1,y_1)), \min(x_1,y_2))$
Zadeh 2	$(I_I)_{ZD2}(x,y) = (\max(x_2, \min(x_1,y_1)), \min(x_1, \max(x_2,y_2)))$
Gaines-Rescher	$(I_I)_{GR}(x,y) = (1 - \mathrm{sg}(x_1 - y_1), y_2 sg(x_1 - y_1))$
Gödel	$(I_I)_{GD}(x,y) = (1 - (x_1 - y_1)\mathrm{sg}(x_1 - y_1), y_2 sg(x_1 - y_1))$
Fodor's 1	$(I_I)_{FD1}(x,y) = (\overline{sg}(x_1 - y_1) + \mathrm{sg}(x_1 - y_1)\max(x_2,y_1), sg(x_1 - y_1)\min(x_1,y_2))$
Kleene-Dienes	$(I_I)_{KD}(x,y) = (\max(x_2,y_1), \min(x_1,y_2))$
Lukasiewicz	$(I_I)_{LK}(x,y) = (\min(1, x_2 + y_1), \max(0, x_1 + y_2 - 1))$
Reichenbach	$(I_I)_{RB}(x,y) = (x_2 + x_1 y_1, x_1 y_2)$
Klir and Yuan 1	$(I_I)_{KY1}(x,y) = (x_2 + x_1 y_1, x_1 x_2 + x_1^2 y_2)$
Atanassov 1	$(I_I)_{A1}(x,y) = (1 - (1 - y_1)\mathrm{sg}(x_1 - y_1), y_2 sg(x_1 - y_1) sg(y_2 - x_2))$
Atanassov 2	$(I_I)_{A2}(x,y) = (\max(x_2,y_1), 1 - \max(x_2,y_1))$
Atanassov and Kolev	$(I_I)_{AK}(x,y) = (x_2 + y_1 - x_2 y_1, x_1 y_2)$

2. PRELIMINARIES

In this section, some basic definitions, arithmetic operations and notion of fuzzy and IF inequalities are presented, which are taken from the articles

Definition 1: Let X be a universe of discourse. Then an IFS $\tilde{A}^I$ in X is defined by the set

$$\tilde{A}^I = \{< x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) >: x \in X\},$$

$\mu_{\tilde{A}^I}, \nu_{\tilde{A}^I}: X \to [0,1]$ are functions such that $0 \le \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \le 1, \forall\, x \in X$. The value $\mu_{\tilde{A}^I}(x)$ represents the degree of membership and $\nu_{\tilde{A}^I}(x)$ represents the degree of non-membership of x ∈ X being in $\tilde{A}^I$. The degree of hesitation for the element x ∈ X being in$\tilde{A}^I$ is denoted by $\pi_{\tilde{A}^I}(x)$ and is defined by

$$\pi_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x) \in [0,1], \ \forall\, x \in X.$$

Deschrijver and Kerre [6] have shown that IFSs can also be seen as L-fuzzy sets in the sense of Goguen [10]. Consider the set $\mathcal{L}$ and operation $\le_{\mathcal{L}}$ defined by

$$\mathcal{L} = \{(x_1,x_2): (x_1,x_2) \in [0,1]^2 \,\&\, x_1 + x_2 \le 1\}$$
$$(x_1,x_2) \le_{\mathcal{L}} (y_1,y_2) \Leftrightarrow x_1 \le y_1 \,\&\, x_2 \ge y_2, \forall\, (x_1,x_2),(y_1,y_2) \in \mathcal{L}.$$

Then, $(\mathcal{L}, \le_{\mathcal{L}})$ is a complete lattice [8]. For each nonempty $\mathcal{A} \subseteq_{\mathcal{L}} \mathcal{L}$, we have

$sup\mathcal{A} = (\sup\{x_1: x_1 \in [0,1] \,\&\, (\exists\, x_2 \in [0,1-x_1])((x_1,x_2) \in \mathcal{A})\}, \inf\{x_2: x_2 \in [0,1] \,\&\, (\exists\, x_1 \in [0,1-x_2])((x_1,x_2) \in \mathcal{A})\}$

$inf\mathcal{A} = (\inf\{x_1: x_1 \in [0,1] \,\&\, (\exists\, x_2 \in [0,1-x_1])((x_1,x_2) \in \mathcal{A})\}, \sup\{x_2: x_2 \in [0,1] \,\&\, (\exists\, x_1 \in [0,1-x_2])((x_1,x_2) \in \mathcal{A})\}$

Definition 2 ([5], Definition 2.2): A function $I_I: \mathcal{L}^2 \to \mathcal{L}$ *is* called an IFI if for x, x', x'', y, y', y'' in $\mathcal{L}$, it satisfies the following conditions:

$$if\ x' \le_{\mathcal{L}} x'', then\ I_I(x',y) \ge_{\mathcal{L}} I_I(x'',y), i.e., I_I(.,y)\ is\ decreasing \quad (11)$$
$$if\ y' \le_{\mathcal{L}} y'', then\ I_I(x,y') \le_{\mathcal{L}} I_I(x,y''), i.e., I_I(x,.)\ is\ increasing \quad (12)$$
$$I_I(0_{\mathcal{L}},0_{\mathcal{L}}) = 1_{\mathcal{L}}, I_I(1_{\mathcal{L}},1_{\mathcal{L}}) = 1_{\mathcal{L}}, I_I(1_{\mathcal{L}},0_{\mathcal{L}}) = 0_{\mathcal{L}} \quad (13)$$

We also define the following set for further usage: $\mathcal{D} = \{(x_1,x_2): (x_1,x_2) \in \mathcal{L} \,\&\, x_1 + x_2 = 1\}$, and the first and second projection mapping pr_1 and pr_2 on $\mathcal{L}$, defined as $pr_1(x_1,x_2) = x_1$ and $pr_2(x_1,x_2) = x_2$, for all $(x_1,x_2) \in \mathcal{L}$.

Definition 3: An IFI $I_I: \mathcal{L}^2 \to \mathcal{L}$ is said to satisfy

(i). the left neutrality property (NP), if $I_I(1_{\mathcal{L}}, y) = y,\ y \in \mathcal{L}$;

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(ii). the ordering property (OP), if $x \leq_{\mathcal{L}} y \iff I_I(x, y) = 1_{\mathcal{L}}, \quad x,\ y \in \mathcal{L}$;
(iii). the identity principle (IP), if $I_I(x, x) = 1_{\mathcal{L}}, \quad x \in \mathcal{L}$;
(iv). the exchange principle (EP), if $I_I\big(x, I_I(y, z)\big) = I_I\big(y, I_I(x, z)\big),\ x, y, z \in \mathcal{L}$.

Definition 4 ([5], Definition 3.1): A function $\mathcal{N}: \mathcal{L} \to \mathcal{L}$ is called an IF negation if

$$\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}, \mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}, \mathcal{N} \text{ is decreasing.} \tag{2.1}$$

Definition 5: A function $\mathcal{T}: \mathcal{L}^2 \to \mathcal{L}$ is called a triangular norm (shortly t-norm) if it satisfies the commutative, associative, increasing in both components and $\mathcal{T}(x, 1_{\mathcal{L}}) = x, \forall\, x \in \mathcal{L}$.

A function $\mathcal{S}: \mathcal{L}^2 \to \mathcal{L}$ is called a triangular conorm (shortly t-conorm) if it satisfies the commutative, associative, increasing in both components and $\mathcal{S}(x, 0_{\mathcal{L}}) = x, \forall\, x \in \mathcal{L}$.

Definition 6 ([4], Definition 5): (t-representability) A t-norm $\mathcal{T}$ on $\mathcal{L}$ (respectively t-conorm $\mathcal{S}$) is called t-representable if there exists a t-norm T and a t-conorm S on [0, 1] (respectively a t-conorm S' and a t-norm T' on [0, 1]) such that, for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$,

$$\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2)),\ \mathcal{S}(x,y) = (S'(x_1,y_1), T'(x_2,y_2))$$

T and S (respectively S' and T') are called the representants of $\mathcal{T}$ (respectively $\mathcal{S}$).

Theorem 1 ([4]: Theorem 2)**.** *Given a t-norm T and t-conorm S on [0, 1] satisfying $T(a, b) \leq 1 - S(1 - a, 1 - b)$ for all $a, b \in [0,1]$, the mappings $\mathcal{T}$ and $\mathcal{S}$ defined by*

$$\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2)),\ \mathcal{S}(x,y) = (S'(x_1,y_1), T'(x_2,y_2))$$

for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$ are a t-norm and a t-conorm on $\mathcal{L}$, respectively.

3. MONOID OF IFIs

In this section, we introduced the monoid structure of IFIs.

Definition 7: For any two IFIs I_I, J_I, we define $I_I \mathrm{J_I}$ as

$$I_I \mathrm{J_I}(x,y) = I_I(x, J_I(x,y)),\ x,\ y \in \mathcal{L}. \tag{3.1}$$

Example 1: Let us consider two IFI operators, viz., Gaines-Reschere: $(I_I)_{GR}(x, y) = (1 - \mathrm{sg}(x_1 - y_1), y_2 sg(x_1 - y_1))$ and Reichenbach: $(I_I)_{RB}(x, y) = (x_2 + x_1 y_1, x_1 y_2)$ (see Table 1). Then $((I_I)_{GR}(I_I)_{RB})(x, y) = \big(1 - sg(x_1(1 - y_1) - x_2),\ x_1 y_2 sg(x_1(1 - y_1) - x_2)\big)$.

Theorem 2: *The function $I_I \mathrm{J_I}$ is an IFI.*

Next aim is to investigate the algebraic structure of I_I on the operation .

Theorem 3: $(\mathcal{I_J},)$ *forms a monoid whose identity element is given by*

$$I_I(x, y) = \begin{cases} 1_{\mathcal{L}}, & x = 0_{\mathcal{L}}, \\ y, & otherwise. \end{cases}$$

Remark 1: Let us take Reichenbach IFI $(I_I)_{RB}(x, y) = (x_2 + x_1 y_1, x_1 y_2)$.
Then $((I_I)_{RB}(I_I)_{RB})(x, y) = (x_2 + x_1(x_2 + x_1 y_1),\ x_1^2 y_2)$, which is not same as $(I_I)_{RB}(x, y)$. Thus is not idempotent in $\mathcal{I_J}$ and consequently, $(\mathcal{I_J},)$ is a non-idempotent monoid.

Theorem 4: *An $I_I \in \mathcal{I_J}$ is invertible w.r.t. if and only if*

$$I_I(x, y) = \begin{cases} 1_{\mathcal{L}}, & x = 0_{\mathcal{L}}, \\ \phi(y), & x >_{\mathcal{L}} 0_{\mathcal{L}}. \end{cases} \tag{3.2}$$

where the function $\phi: \mathcal{L} \to \mathcal{L}$ is an increasing bijection.

Definition 8: The pair of IFIs (I_I, J_I) are said to be mutually exchangeable (ME) if

$$I_I(x, J_I(y,z)) = J_I(y, I_I(x,z)),\ x, y, z \in \mathcal{L}.$$

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Example 2: Let us suppose for fixed $\rho = (\rho_1, \rho_2)$, $\sigma = (\sigma_1, \sigma_2) \in \mathcal{L}$. Let us consider two IFI operators

$$I_I(x,y) = \begin{cases} 1_{\mathcal{L}}, & x \le_{\mathcal{L}} \rho, \\ y, & x >_{\mathcal{L}} \rho, \end{cases} \quad and \quad J_I(x,y) = \begin{cases} 1_{\mathcal{L}}, & x \le_{\mathcal{L}} \sigma, \\ y^2, & x >_{\mathcal{L}} \sigma, \end{cases}$$

where $x, y \in \mathcal{L}$. It easy to check that (I_I, J_I) satisfies ME.

Remark 2: If I_I, J_I are ME, then I_I, J_I are commutative w.r.t. . To see this, let us suppose x = y in (ME), which then becomes $I_I(x, J_I(y,z)) = J_I(y, I_I(x,z))$, i.e., $(I_I J_I)(x,z) = (J_I I_I)(x,z)$, $y, z \in \mathcal{L}$.

Theorem 5. *Let $I_I, J_I \in \mathfrak{I}_{\mathfrak{J}}$ satisfy EP and ME. Then $I_I J_I$ satisfies EP.*

Definition 9: An IFI I_I is said to satisfy the law of importation (LI) w.r.t. a t-norm $\mathcal{T}$ if

$$I_I(x, I_I(y,z)) = I_I(\mathcal{T}(x,y), z), \quad x, y, z \in \mathcal{L}.$$

Remark 3: Note that even if $I_I, J_I \in \mathfrak{I}_{\mathfrak{J}}$ satisfy LI w.r.to the same t-norm $\mathcal{T}$, $I_I J_I$ may not satisfy LI w.r.t. any t-norm or may satisfy LI w.r.t. the same t-norm or even a different t-norm $\mathcal{T}'$.

Theorem 6: *Let $I_I, J_I \in \mathfrak{I}_{\mathfrak{J}}$* satisfy LI w.r.to the same t-norm $\mathcal{T}$. If I_I, J_I satisfy ME, then $I_I J_I$ *satisfies LI w.r.t. the same t-norm $\mathcal{T}$.*

4. THE ⊛-COMPOSITION w.r.TO THE BASIC PROPERTIES

In this section, as mentioned before, we do the following: Given $I_I, J_I \in \mathfrak{I}_{\mathfrak{J}}$ satisfying a certain properties **P**, we now investigate whether $I_I J_I$ also satisfies the same property or not. If not, then we attempt to characterize I_I, J_I such that $I_I J_I$ also satisfies the same property. Towards this end we have the following result.

Theorem 7: *Let $I_I, J_I \in \mathfrak{I}_{\mathfrak{J}}$ and $\phi_I \in \Phi_I$. Then $(I_I J_I)_{\phi_I} = (I_I)_{\phi_I} (J_I)_{\phi_I}$.*

Theorem 8: *If $I_I, J_I \in \mathfrak{I}_{\mathfrak{J}}$ satisfy (NP) ((IP), self conjugacy, continuity), then $I_I J_I$ satisfies the same property.*

5. THE - COMPOSITION w.r.t. PROPERTY OP

While the composition preserves the properties NP, IP, self-congugacy and continuity, this is not true with either the OP or EP.

Theorem 9: Let $I_I, J_I \in \mathfrak{I}_{\mathfrak{J}}$ satisfy OP. Then the following statements are equivalent:

(i) *$I_I \circledast J_I$ satisfies OP.*

(ii) J_I *satisfies the following for all $x, y \in \mathcal{L}$:*

$x >_{\mathcal{L}} J_I(x,y)$, whenever $x >_{\mathcal{L}} y$.

(iii) $J_I(x,y) \le_{\mathcal{L}} y$, whenever $x >_{\mathcal{L}} y$.

Proof: Let $I_I, J_I \in \mathfrak{I}_{\mathfrak{J}}$ satisfy OP.

***(i)*⇒ (*ii*):** $I_I J_I$ satisfies OP. Then $(I_I J_I)(x,y) = 1_{\mathcal{L}} \Leftrightarrow x \le_{\mathcal{L}} y$, i.e., $I_I(x, J_I(x,y)) = 1_{\mathcal{L}} \Leftrightarrow x \le_{\mathcal{L}} y$, i.e., $x \le_{\mathcal{L}} J_I(x,y)$ $\Leftrightarrow x \le_{\mathcal{L}} y$, which implies that $x >_{\mathcal{L}} J_I(x,y)$, for all $x >_{\mathcal{L}} y$

***(ii)*⇒ (*iii*):** Let J_I satisfy (5.1). If $x >_{\mathcal{L}} y$, then there exists $\varepsilon = (\varepsilon_1, \varepsilon_2) >_{\mathcal{L}} 0_{\mathcal{L}}$, arbitrarily small, such that

$$x >_{\mathcal{L}} y + \varepsilon >_{\mathcal{L}} y.$$

Now, from the antitonicity of J_I in the first variable and (5.1), we have $J_I(x,y) \le_{\mathcal{L}} J_I(y+\varepsilon, y) <_{\mathcal{L}} y+\varepsilon$. Since $\varepsilon = (\varepsilon_1, \varepsilon_2) >_{\mathcal{L}} 0_{\mathcal{L}}$ is arbitrary, we see that $x >_{\mathcal{L}} J_I(x,y) \le_{\mathcal{L}} y$ for all $x >_{\mathcal{L}} y$.

***(iii)*⇒ (*i*):** Let J_I satisfy $x >_{\mathcal{L}} J_I(x,y) \le_{\mathcal{L}} y$ for all $x >_{\mathcal{L}} y$.

- Let $x \le_{\mathcal{L}} y$. Then, since J_I satisfies OP, $J_I(x,y) = 1_{\mathcal{L}}$ and consequently, $(I_I J_I)(x,y) = I_I(x, J_I(x,y)) = 1_{\mathcal{L}}$.
- Let $x >_{\mathcal{L}} y$. Then we have $J_I(x,y) \le_{\mathcal{L}} y <_{\mathcal{L}} x$. From OP of I_I, it follows that $I_I(x, J_I(x,y)) <_{\mathcal{L}} 1_{\mathcal{L}}$. In other words, we have $x >_{\mathcal{L}} y \Leftrightarrow (I_I J_I)(x,y) <_{\mathcal{L}} 1_{\mathcal{L}}$ and hence $I_I J_I$ satisfies OP.

6. SELF COMPOSITION w.r.t. $-\ \mathrm{I}_{\mathrm{I}}^{[\mathrm{n}]}$

Since we have analyzed the associativity of binary operation on IFI I_{I}, one can define the self composition of IFIs w.r.t. the same binary operation .

Definition 10: Let $\mathrm{I}_{\mathrm{I}} \in \mathcal{I}_{\mathcal{J}}$. For any $\mathrm{n} \in \mathbb{N}$, the n-th power of I_{I}w.r.to is denoted by $\mathrm{I}_{\mathrm{I}}^{[\mathrm{n}]}$ and is defined as follows:

$$\mathrm{I}_{\mathrm{I}}^{[\mathrm{n}]}(x,y) = \begin{cases} I_I(x,y), & n=1, \\ I_I\left(x,\ \mathrm{I}_{\mathrm{I}}^{[\mathrm{n-1}]}(x,y)\right) = \ \mathrm{I}_{\mathrm{I}}^{[\mathrm{n-1}]}\left(\mathrm{x}, I_I(x,y)\right), & n \geq 2. \end{cases} \tag{6.1}$$

where $x, y \in \mathcal{L}$. Noting that if $\mathrm{I}_{\mathrm{I}} \in \mathcal{I}_{\mathcal{J}}$, then $\mathrm{I}_{\mathrm{I}}^{[\mathrm{n}]} \in \mathcal{I}_{\mathcal{J}}$ for every $\mathrm{n} \in \mathbb{N}$.

Example 3: Let us consider Reichenbach IFI operator $(I_I)_{RB}(x,y) = (\mathrm{x}_2 + \mathrm{x}_1\mathrm{y}_1, \mathrm{x}_1\mathrm{y}_2)$. (see Table 1). Then the $(\mathrm{I}_{\mathrm{I}}^{[\mathrm{n}]})_{RB}(x,y)$ is given by

$$(\mathrm{I}_{\mathrm{I}}^{[\mathrm{n}]})_{RB}(x,y) = \begin{cases} (\mathrm{x}_2 + \mathrm{x}_1\mathrm{y}_1, \mathrm{x}_1\mathrm{y}_2), & n=1, \\ (\mathrm{x}_2 + \mathrm{x}_1\mathrm{x}_2 + \mathrm{x}_1^2\mathrm{x}_2 + \cdots + \mathrm{x}_1^{\mathrm{n}}\mathrm{y}_1, \mathrm{x}_1^{\mathrm{n}}\mathrm{y}_2), & n \geq 2, \end{cases}$$

where $x, y \in \mathcal{L}$.

7. CONVERGENCE OF $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}$

In this section, we investigate the convergence of $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}$ w.r.t. operation **.**

Theorem 10: Let $\mathbf{I}_{\mathbf{I}} \in \mathcal{I}_{\mathcal{J}}$ satisfy w.r.t. a t-norm $\boldsymbol{\mathcal{T}}$. Then

(i) $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}(\mathbf{x},\mathbf{y}) = \boldsymbol{I_I}\left(\boldsymbol{x}_{\mathcal{T}}^{[\boldsymbol{n}]}, \boldsymbol{y}\right)$, where $\boldsymbol{x}_{\mathcal{T}}^{[\boldsymbol{n}]} = \boldsymbol{\mathcal{T}}(\boldsymbol{x}, \boldsymbol{x}_{\mathcal{T}}^{[\boldsymbol{n-1}]})$ and $\boldsymbol{x}_{\mathcal{T}}^{[\mathbf{1}]} = \boldsymbol{x}$ for any $\boldsymbol{x} \in \boldsymbol{\mathcal{L}}$ and $\mathbf{n} \in \mathbb{N}$.

(ii) Further, let $\boldsymbol{\mathcal{T}}$ be Archimedean, i.e., for any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{L}} - \{\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}\}$ there exists an $\mathbf{n} \in \mathbb{N}$ such that $\boldsymbol{x}_{\mathcal{T}}^{[\boldsymbol{n}]} <_{\mathcal{T}} \boldsymbol{y}$. Then

$$\lim_{n\to\infty} \mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \mathbf{1}_{\mathcal{L}}, & x <_{\mathcal{L}} \mathbf{1}_{\mathcal{L}}, \\ \boldsymbol{I_I}(\boldsymbol{x},\boldsymbol{y}), & \boldsymbol{x} = \mathbf{1}_{\mathcal{L}}. \end{cases}$$

(iii) If, in addition, $\boldsymbol{I_I}$ satisfies NP then

$$\lim_{n\to\infty} \mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \mathbf{1}_{\mathcal{L}}, & \boldsymbol{x} <_{\mathcal{L}} \mathbf{1}_{\mathcal{L}}, \\ \boldsymbol{y}, & \boldsymbol{x} = \mathbf{1}_{\mathcal{L}}. \end{cases}$$

Proof:

(i) Since $\mathbf{I}_{\mathbf{I}} \in \mathcal{I}_{\mathcal{J}}$ satisfies the LI, $\mathbf{I}_{\mathbf{I}}^{[2]}(\mathbf{x},\mathbf{y}) = \mathbf{I}_{\mathbf{I}}(\mathbf{x}, \mathbf{I}_{\mathbf{I}}(\mathbf{x},\mathbf{y})) = \mathbf{I}_{\mathbf{I}}(\boldsymbol{\mathcal{T}}(\mathbf{x},\mathbf{x}),\mathbf{y})$. Thus $\mathbf{I}_{\mathbf{I}}^{[2]}(\mathbf{x},\mathbf{y}) = \mathbf{I}_{\mathbf{I}}(\mathbf{x}_{\mathcal{T}}^{[2]},\mathbf{y})$. By mathematical induction, we get $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}(\mathbf{x},\mathbf{y}) = \mathbf{I}_{\mathbf{I}}(\mathbf{x}_{\mathcal{T}}^{[\mathbf{n}]},\mathbf{y})$.

(ii) Take $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1, \mathbf{1}-\boldsymbol{\epsilon}_1) >_{\mathcal{L}} \mathbf{0}_{\mathcal{L}}$ and $\boldsymbol{x} <_{\mathcal{L}} \mathbf{1}_{\mathcal{L}}$. Since $\boldsymbol{\mathcal{T}}$ is Archimedean, for any $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1, \mathbf{1}-\boldsymbol{\epsilon}_1) >_{\mathcal{L}} \mathbf{0}_{\mathcal{L}}\ \exists\ \boldsymbol{m} \in \mathbb{N}$ s.t. $\mathbf{x}_{\mathcal{T}}^{[\mathbf{n}]} <_{\mathcal{L}} \boldsymbol{\epsilon}$. Thus, for any $\boldsymbol{y} \in \boldsymbol{\mathcal{L}}$, $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}(\mathbf{x},\mathbf{y}) = \mathbf{I}_{\mathbf{I}}(\mathbf{x}_{\mathcal{T}}^{[\mathbf{n}]},\mathbf{y}) \to \mathbf{I}_{\mathbf{I}}(\mathbf{0}_{\mathcal{L}},\mathbf{y})$ as $\boldsymbol{n} \to \infty$ and $\mathbf{lim}_{\boldsymbol{n}\to\infty}\ \mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{1}_{\mathcal{L}}$.
If $\boldsymbol{x} = \mathbf{1}_{\mathcal{L}}$, then $\mathbf{I}_{\mathbf{I}}^{[2]}(\mathbf{1}_{\mathcal{L}},\mathbf{y}) = \mathbf{I}_{\mathbf{I}}(\mathbf{1}_{\mathcal{L}}, \mathbf{I}_{\mathbf{I}}(\mathbf{1}_{\mathcal{L}},\mathbf{y})) = \mathbf{I}_{\mathbf{I}}(\boldsymbol{\mathcal{T}}(\mathbf{1}_{\mathcal{L}},\mathbf{1}_{\mathcal{L}}),\mathbf{y}) = \mathbf{I}_{\mathbf{I}}(\mathbf{1}_{\mathcal{L}},\boldsymbol{y})$ and in general $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}(\mathbf{1}_{\mathcal{L}},\mathbf{y}) = \mathbf{I}_{\mathbf{I}}(\mathbf{1}_{\mathcal{L}},\boldsymbol{y})$.

(iii) Follows from (ii) and the fact that $\mathbf{I}_{\mathbf{I}}(\mathbf{1}_{\mathcal{L}},\boldsymbol{y}) = \boldsymbol{y}$.

8. CLOSURE OF $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}$ w.r.to THE BASIC PROPERTIES

In this section, given an $\mathbf{I}_{\mathbf{I}} \in \mathcal{I}_{\mathcal{J}}$ satisfying a particular property $\mathbf{P}$, we investigate whether all the powers $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}$ of $\mathbf{I}_{\mathbf{I}}$ satisfy the same property or not. From Theorem 9, we see that if $\mathbf{I}_{\mathbf{I}} \in \mathcal{I}_{\mathcal{J}}$ satisfies any one of the properties NP, IP, self-conjugacy and continuity then $\mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}$ satisfies the same for all $\mathbf{n} \in \mathbb{N}$. Hence it is enough to investigate the preservation of OP and EP. Below we prove that if $\mathbf{I}_{\mathbf{I}} \in \mathcal{I}_{\mathcal{J}}$ satisfies EP, then the $(\mathbf{I}_{\mathbf{I}},\ \mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]})$ satisfies ME for any $\mathbf{n} \in \mathbb{N}$.

Theorem 11: If $\boldsymbol{I_I}$ satisfies (EP) then the pair $(\mathbf{I}_{\mathbf{I}},\ \mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]})$ satisfies (ME) for all $\mathbf{n} \in \mathbb{N}$, i.e.,

$$\boldsymbol{I_I}\left(\boldsymbol{x},\ \mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}(\boldsymbol{y},\boldsymbol{z})\right) = \ \mathbf{I}_{\mathbf{I}}^{[\mathbf{n}]}\left(\boldsymbol{y}, \boldsymbol{I_I}(\boldsymbol{x},\boldsymbol{z})\right), \forall\ \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \boldsymbol{\mathcal{L}}. \tag{8.1}$$

Proof: The theorem is proved with the help of mathematical induction on n. For n = 1, $\boldsymbol{I_I}$ satisfies (8.1), from the EP of $\boldsymbol{I_I}$. Let us suppose that $\boldsymbol{I_I}$ satisfies (8.1) for n = k-1. Now, for n = k,

$$\begin{aligned} \boldsymbol{I_I}\left(\boldsymbol{x},\ \mathbf{I}_{\mathbf{I}}^{[\mathbf{k}]}(\boldsymbol{y},\boldsymbol{z})\right) &= \boldsymbol{I_I}\left(\boldsymbol{x}, \boldsymbol{I_I}\left(\boldsymbol{y},\ \mathbf{I}_{\mathbf{I}}^{[\mathbf{k-1}]}(\boldsymbol{y},\boldsymbol{z})\right)\right) = \boldsymbol{I_I}\left(\boldsymbol{x},\ \mathbf{I}_{\mathbf{I}}^{[\mathbf{k-1}]}(\mathbf{y}, \mathbf{I}_{\mathbf{I}}(\mathbf{y},\mathbf{z}))\right) = \ \mathbf{I}_{\mathbf{I}}^{[\mathbf{k-1}]}\left(\mathbf{y}, \mathbf{I}_{\mathbf{I}}(\mathbf{x}, \mathbf{I}_{\mathbf{I}}(\mathbf{y},\mathbf{z}))\right) \\ &= \mathbf{I}_{\mathbf{I}}^{[\mathbf{k-1}]}\left(\mathbf{y}, \mathbf{I}_{\mathbf{I}}(\mathbf{y}, \mathbf{I}_{\mathbf{I}}(\mathbf{x},\mathbf{z}))\right) = \mathbf{I}_{\mathbf{I}}^{[\mathbf{k}]}\left(\boldsymbol{y}, \boldsymbol{I_I}(\boldsymbol{x},\boldsymbol{z})\right), \forall\ \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \boldsymbol{\mathcal{L}}. \end{aligned}$$

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Thus the pair ($\mathbf{I_I}$, $\mathbf{I_I^{[n]}}$) satisfies (ME) for all $\mathbf{n} \in \mathbb{N}$.

Theorem 12: If $\mathrm{I_I}$ satisfies EP then $\mathrm{I_I^{[n]}}$ satisfies EP for all n ∈ ℕ.

Proof: A direct verification provides the proof.

Theorem 13: Let $\mathrm{I_I} \in \mathcal{I}_{\mathcal{J}}$ satisfy OP. Then the following statements are equivalent:

(i) $I_I^{[2]}$ *satisfies OP.*

(ii) $\mathrm{I_I}$ *satisfies the following for all* $x, y \in \mathcal{L}$*:*
$x >_{\mathcal{L}} I_I(x,y)$, whenever $x >_{\mathcal{L}} y$.

(iii) $I_I^{[n]}$ *satisfies OP for all* n ∈ ℕ.

Proof: The same as Theorem10

9. CLOSURE OF $I_I^{[n]}$ w.r.t. FUNCTIONAL EQUATIONS

Theorem 14: *If I_I satisfies LI w.r.t. t-norm $\mathcal{T}$*, then $I_I^{[n]}$ *also satisfies LI w.r.t. t-norm $\mathcal{T}$.*

Proof: This theorem is proved with the help of mathematical induction on n. For n = 1, $I_I^{[n]}$ satisfies LI. Let us suppose that $I_I^{[n]}$ satisfies $(\widetilde{\mathrm{LI}})$ w.r.t. the t-norm $\mathcal{T}$ for n = k-1. Then

$$I_I^{[k-1]}(\mathcal{T}(x,y),z) = I_I^{[k-1]}\left(x,\ I_I^{[k-1]}(y,z)\right), \forall\, x,y,z \in \mathcal{L}. \tag{9.1}$$

From Theorem 10(i), recall that if I_I satisfies (LI) w.r.to a t-norm $\mathcal{T}$, then $I_I^{[k]}(x,y) = I_I(x^{[k]})$. Now, for n = k,

$$\begin{aligned}
I_I^{[k]}(\mathcal{T}(x,y),z) &= I_I\left(\left(\mathcal{T}(x,y)\right)_{\mathcal{T}}^{[k]}, z\right) = I_I\left(\mathcal{T}(\mathcal{T}(x,y), \left(\mathcal{T}(x,y)\right)_{\mathcal{T}}^{[k-1]}), z\right) = I_I\left(\mathcal{T}(x,y), I_I(\left(\mathcal{T}(x,y)\right)_{\mathcal{T}}^{[k-1]}, z)\right) \\
&= I_I\left(\mathcal{T}(x,y), I_I^{[k-1]}(\mathcal{T}(x,y),z)\right) = I_I\left(x, I_I\left(y, I_I^{[k-1]}(\mathcal{T}(x,y),z)\right)\right) \\
&= I_I\left(x, I_I\left(y, I_I^{[k-1]}\left(x, I_I^{[k-1]}(y,z)\right)\right)\right) = I_I\left(x, I_I^{[k-1]}\left(x, I_I\left(y, I_I^{[k-1]}(y,z)\right)\right)\right) \\
&= I_I^{[k]}\left(x,\ I_I^{[k]}(y,z)\right).
\end{aligned}$$

Thus $I_I^{[n]}$ satisfies LI for all $n \in \mathbb{N}$.

10. CONCLUSIONS AND FUTURE SCOPE

In this paper, we have investigated ⊛-composition between between the pair of IFIs, the pair of n th powers of self IFIs and the pair of a IFI and its n th power, which is a generalization of ⊛-composition in FIs [15, 16, 17]. Specifically, we have studied the algebraic structures and shown that some other properties like NP, IP, continuity are preserved but fail to preserve OP and EP. Finally, we have also proposed a new concept of ME, a generalization of EP to a pair of IFIs and n th power of self IFIs. In future, based on the ⊛ composition, we plan to work on interval valued IFIs w.r.to two different as well as n th order self.

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