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SOME RESULTS ON FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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ABSTRACT

This paper is an attempt to prove some critically analysed results on the fixed point theorems, which are unique and well established in fuzzy metric spaces.

Keywords: *fuzzy set, fuzzy metric space, continuous t-norm, fixed point*

AMS Subject Classifications: *54H25; 47H10; 47S40; 54A40.*

1. INTRODUCTION

Zadeh invented the theory of fuzzy sets in 1965[8]. The concept of fuzzy metric space was introduced initially by Kramosil and Michalek [5]. Later on, George and Veeramani [1] gives the modified notion of fuzzy metric spaces due to Kramosil and Michalek [5] and analysed a Hausdorff topology of fuzzy metric spaces. Recently, Gregori *et al.* [4] gave many interesting examples of fuzzy metrics in the sense of George and Veeramani [1] and have also applied these fuzzy metrics to color image processing. In 1988, Grabiec [3] proved an analog of the Banach contraction theorem in fuzzy metric spaces. In his proof, he used a fuzzy version of Cauchy sequence.

In this paper, we establish some results on existence and uniqueness of fixed point theorems in fuzzy metric spaces. Here some examples are given to explicate our main theorem.

2. PRELIMINARIES

To initiate the concept of a fuzzy metric space, which was introduced by Kramosil and Michalek [5] in 1975 is recalled here.

Definition 2.1: A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu A(x)) : x \in A, \mu A(x) \in [0,1]\}$.

In the pair$(x, \mu A(x))$, the first element x belongs to the classical set A, the second element $\mu A(x)$ belongs to the interval$[0,1]$, is called the membership function.

Definition 2.2: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$

Examples 2.3:

i. Lukasievicz t-norm: $a * b = max\{a + b - 1, 0\}$
ii. Product t-norm: $a * b = a.b$
iii. Minimum t-norm: $a * b = min(a, b)$

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Definition 2.4: A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty) \to [0,1]$ satisfies the following conditions:

$\forall\, x, y, z \in X \; and \; s, t > 0$
(KM1) $M(x, y, 0) = 0, \forall\, t > 0$
(KM2) $M(x, y, t) = 1$ if and only if $x = y, t > 0$
(KM3) $M(x, y, t) = M(y, x, t)$
(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$
(KM5) $M(x, y, .)$: $[0, \infty) \to [0,1]$ is left-continuous.
Then M is called a fuzzy metric on X.

Definition 2.5: A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty) \to [0,1]$ satisfies the following conditions:

$\forall\, x, y, z \in X \; and \; s, t > 0$
(GV1) $M(x, y, t) > 0, \forall\, t > 0$
(GV2) $M(x, y, t) = 1$ if and only if $x = y, t > 0$
(GV3) $M(x, y, t) = M(y, x, t)$
(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$
(GV5) $M(x, y, .)$: $(0, \infty) \to [0,1]$ is continuous.
Then M is called a fuzzy metric on X.

Definition 2.6: Let $(X, M, *)$ be a fuzzy metric space, for $t > 0$ the open ball $B(x, r, t)$ with a centre $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : \mathrm{M(x, y, t)} > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of$X$. Then τ is topology on X, called the topology induced by the fuzzy metric M.

Example 2.7: Let (X, d) be a metric space. Define $a * b = ab(a * b = min\{a, b\})$ for all $a, b \in [0,1]$, and define $M: X \times X \times (0, \infty) \to [0,1]$ as

$$\mathrm{M(x, y, t)} = \frac{t}{t + d(x, y)}$$

$\forall\, x, y, z \in X \; and \; t > 0$. Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric induced by the metric d the standard fuzzy metric.

Definition 2.8: Let $(X, M, *)$ be a fuzzy metric space

(i) A sequence $\{x_n\}$in X is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} \mathrm{M(x, y, t)} = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$in X is called a Cauchy sequence, if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
(iv) A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Lemma 2.8: Let $(X, M, *)$ be a fuzzy metric space. For all $\mathrm{u}, v \in X, M(u, v, \,.)$ is non-decreasing function.

Proof: If $M(u, v, t) > M(u, v, s)$ for some $0 < t < s$.

Then $M(u, v, t) * M(v, v, s - t) \leq M(u, v, s) < M(u, v, t)$,
Thus,$M(u, v, t) < M(u, v, t) < M(u, v, t)$, (since $M(v, v, s - t) = 1$)
which is a contradiction.

3. MAIN RESULTS

Theorem 3.1: Let $(X, M, *)$ be a complete fuzzy metric space and $T: X \to X$ is continuous function and satisfied the condition:

$$M(Tu, Tv, t) \geq min\{M(Tu, v, t), M(u, Tv, t), M(u, v, t)\} \quad (1)$$

Moreover, the fuzzy metric $M(u, v, t)$ satisfies the condition

$$\lim_{t \to \infty} \mathrm{M(u, v, t)} = 1$$

Where $u, v \in X$and $u \neq v$, then T has a fixed point in X.

Proof: Take $\{u_n\}$ be a sequence in X such that $u_{n+1} = Tu_n$

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If $u_n = u_{n+1}$ then $u_{n+1} = Tu_n = u_n$ implies u_n is a fixed point of T.

Suppose $u_n \neq u_{n+1}$,

To prove that $\{u_n\}$ is a Cauchy's sequence in X

Put $u = u_{n-1}$ and $v = u_n$ in (1) we get

$$M(u_n, u_{n+1}, t) = M(Tu_{n-1}, Tu_n, t) \geq min\{M(Tu_{n-1}, u_n, t), M(u_{n-1}, Tu_n, t), M(u_{n-1}, u_n, t)\}$$
$$M(u_n, u_{n+1}, t) \geq min\{M(u_n, u_n, t), M(u_{n-1}, u_{n+1}, t), M(u_{n-1}, u_n, t)\}$$
$$M(u_n, u_{n+1}, t) \geq min\{1, M(u_{n-1}, u_{n+1}, t), M(u_{n-1}, u_n, t)\}$$
$$M(u_n, u_{n+1}, t) \geq min\{M(u_{n-1}, u_{n+1}, t), M(u_{n-1}, u_n, t)\}$$
$$M(u_n, u_{n+1}, t) \geq M(u_{n-1}, u_n, t)$$

Therefore

$$M(u_{n+1}, u_n, t) \geq M(u_n, u_{n-1}, t), \text{ for all n} \quad (2)$$

Put $u = u_{n-1}$ and $v = u_{n-2}$ in (1) we get

$$M(Tu_{n-1}, Tu_{n-2}, t) \geq min\{M(Tu_{n-1}, u_{n-2}, t), M(u_{n-1}, Tu_{n-2}, t), M(u_{n-1}, u_{n-2}, t)\}$$
$$M(u_n, u_{n-1}, t) \geq min\{M(u_n, u_{n-2}, t), M(u_{n-1}, u_{n-1}, t), M(u_{n-1}, u_{n-2}, t)\}$$
$$M(u_n, u_{n-1}, t) \geq min\{M(u_n, u_{n-2}, t), 1, M(u_{n-1}, u_{n-2}, t)\}$$
$$M(u_n, u_{n-1}, t) \geq min\{M(u_n, u_{n-2}, t), M(u_{n-1}, u_{n-2}, t)\}$$
$$M(u_n, u_{n-1}, t) \geq M(u_{n-1}, u_{n-2}, t)$$

Therefore

$$M(u_{n-1}, u_n, t) \geq M(u_{n-2}, u_{n-1}, t), \text{ for all n} \quad (3)$$

Let us assume that $\{u_n\}$ is not a Cauchy's sequence, then for $0 < \epsilon < 1,\ t > 0$, then there exists sequence $\{u_{n_k}\}$ and $\{u_{m_k}\}$ where $n_k, m_k \geq n$ and $n_k, m_k \in \mathbb{N},\ (n_k > m_k)$ such that

$$M(u_{n_k}, u_{m_k}, t) \leq 1 - \epsilon$$
$$M(u_{n_k-1}, u_{m_k-1}, t) > 1 - \epsilon,\ M(u_{n_k-1}, u_{m_k}, t) > 1 - \epsilon \quad (4)$$

Consider, $1 - \epsilon \geq M(u_{n_k}, u_{m_k}, t)$

$$\geq min\left\{M\left(u_{n_k}, u_{n_k-1}, \frac{t}{2}\right), M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)\right\}$$
$$\geq min\left\{M\left(u_{n_k}, u_{n_k-2}, \frac{t}{4}\right), M\left(u_{n_k-2}, u_{n_k-1}, \frac{t}{4}\right), M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)\right\}$$
$$\geq min\left\{M\left(u_{n_k}, u_{n_k-2}, \frac{t}{4}\right), M\left(u_{n_k-2}, u_{n_k}, \frac{t}{4}\right), M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)\right\} \text{ (By using (2))}$$
$$\geq min\left\{M\left(u_{n_k}, u_{n_k-2}, \frac{t}{4}\right), M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)\right\}$$
$$\geq min\left\{M\left(u_{n_k}, u_{n_k-1}, \frac{t}{4}\right), M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)\right\} \text{(By using (2))}$$
$$\geq min\left\{M\left(u_{n_k-1}, u_{n_k-2}, \frac{t}{4}\right), M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)\right\} \quad \text{(By using (3))}$$
$$\geq min\left\{M\left(u_{n_k-1}, u_{n_k-1}, \frac{t}{4}\right), M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)\right\} \quad \text{(By using (2))}$$
$$\geq min\left\{1, M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)\right\}$$
$$= M\left(u_{n_k-1}, u_{m_k}, \frac{t}{2}\right)$$
$$> 1 - \epsilon \quad \text{which is a contradiction.}$$

Therefore $\{u_n\}$ is a Cauchy's sequence.

Since X is complete, there exists an element $z \in X$ such that $\lim_{n\to\infty} u_n = z$

That is $M(u_n, z, t) = 1$ as $n \to \infty$

Next prove that limit z is unique.

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Suppose $\lim_{n\to\infty} u_n = w$ for some $w \in X$ where $w \neq z$,

Then $M(w, z, t) \geq M(w, u_n, \frac{t}{2}) * M(u_n, z, \frac{t}{2})$ $\quad (\because a * b = \min(a, b))$

Taking limit $n \to \infty$ on both sides

$M(w, z, t) \geq M(w, w, \frac{t}{2}) * M(z, z, \frac{t}{2})$

$M(w, z, t) \geq 1$, which is a contradiction

∴limit z is unique.

To show that z is a fixed point of T:

$\because T$ is continuous, $u_n \to z \Longrightarrow Tu_n \longrightarrow Tz$

Consider (i), $M(u_n, u_{n+1}, t) \geq M(u_n, u_{n-1}, t)$

$M(u_n, Tu_n, t) \geq M(u_n, u_{n-1}, t)$

Taking limit $n \to \infty$, we get

$M(z, Tz, t) \geq M(z, z, t)$ $\quad (\because \{u_n\}$ is a Cauchy's sequence)

$M(z, Tz, t) \geq 1$

Hence $M(z, Tz, t) = 1 \Longrightarrow Tz = z$

Thus z is a fixed point of T.

To prove Uniqueness,

Suppose that q is another fixed point of T, that is $Tq = q$ where $q \neq z$

Now we show that $q = z$,

$$
\begin{aligned}
& M(z, q, t) < 1 \\
\Longrightarrow\ & 1 > M(z, q, t) \\
& \geq min\left\{M\left(z, z, \frac{t}{2}\right), M(z, q, \frac{t}{2})\right\} \\
& \geq min\left\{1, M\left(z, z, \frac{t}{4}\right), M(z, q, \frac{t}{4})\right\} \\
& \geq min\left\{1,1, M\left(z, z, \frac{t}{8}\right), M(z, q, \frac{t}{8})\right\} \\
& \geq min\left\{1,1,1, M\left(z, z, \frac{t}{16}\right), M(z, q, \frac{t}{16})\right\} \\
& .. \\
& \geq min\left\{1,1,1,1, \dots, M(z, q, \frac{t}{2^k})\right\} \\
& = 1 \text{ as } k \to \infty
\end{aligned}
$$

This implies that $z = q \Longrightarrow z$ is the unique fixed point of T.

Theorem 3.2: Let $(X, M,*)$ be a complete fuzzy metric space and $T: X \to X$ is continuous function and satisfied the condition:

$$M(Tu, Tv, t) \geq min\{M(u, Tu, t), M(v, Tv, t), M(u, v, t)\} \tag{5}$$

Moreover, the fuzzy metric $M(u, v, t)$ satisfies the condition

$$\lim_{t\to\infty} \mathrm{M(u, v, t)} = 1$$

Where $u, v \in X$ and $u \neq v$, then T has a fixed point in X.

Proof: The proof of the above theorem 3.1 considering the fuzzy contraction on the fuzzy metric space $(X, M,*)$, where $M(Tu, Tv, t) \geq min\{M(u, Tu, t), M(v, Tv, t), M(u, v, t)\}$.

Example 3.3: Let $X = [0, 1]$ and fuzzy metric is defined by $M(u, v, t) = \frac{t}{t+d(u,v)}$, $(t > 0)$

Where $d(u, v) = |u - v|$ and $*$ be the continuous t −norm defined by $a * b = \min(a, b)$

explicitly, $(X, M,*)$ is a complete fuzzy metric space and T be a self map on X is given by $Tu = \frac{u}{2}$

Here, (1) of theorem 1 are satisfied by above defined fuzzy metric and T.

Thus T has a unique fixed point in X, That is at $u = 1$.

Example 3.4: Let $X = [0, 2]$ and fuzzy metric is defined by $M(u, v, t) = \frac{t}{t+d(u,v)}$, $(t > 0)$
where $d(u, v) = |u - v|$ and $*$ be the continuous t −norm defined by $a * b = \min(a, b)$

Clearly, $(X, M,*)$ is a complete fuzzy metric space and T be a self map on X is given by $Tu = 2 - u$

Now, (1) of theorem 1 are satisfied by above defined fuzzy metric and T.

Thus T has a unique fixed point in X, That is at $u = 0$.

CONCLUSION

This present paper tried to prove some fixed point theorems in fuzzy metric spaces with illustrative examples. The finding of this research paper is to assert fixed point theory in establishing an existence and uniqueness theorem for a self-mapping in fuzzy metric spaces.

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