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# NEIGHBORHOOD SETS AND NEIGHBORHOOD POLYNOMIAL OF A PATH

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***ABSTRACT***

***A** set S of vertices in a graph G is a neighborhood set of G if $G = \cup_{v\in S}\langle N[v]\rangle$, where is the $\langle N[v]\rangle$ subgraph of G induced by $v$ and all vertices adjacent to. The neighborhood number $n_0(G)$ of G is the minimum number of vertices in a neighborhood of G [3]. Let $P_n^i$ be the family of neighborhood sets of a Path $P_n$ with cardinality $i$. In this paper we construct family of neighborhood sets of Paths $P_n^i$ and its polynomial of a path.*

***Keywords**: Neighborhood set, neighborhood number and neighborhood polynomials.*

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## 1. INTRODUCTION

Let G be a simple graph with vertex set $V = \{v_1, v_2, v_3, .., v_n\}$ and the Edge set $E = \{v_1v_2, v_2v_3, v_3v_4, \dots, v_{n-1}v_n\}$. A neighborhood set $S \subseteq V(G)$ is a neighborhood set of $G$ if $G = \cup_{v\in S}\langle N[v]\rangle$ where $\langle N[v]\rangle$ is induced subgraph of G. The neighborhood number $n_0(G)$. Let $P_n^i$ be the family of neighborhood sets of a Paths $P_n^i$ with cardinality $i$ and $n_0(P_n, i) = |P_n^i|$ and the Polynomials are $N_0(P_n, x) = \sum_{i=n_0}^{n} n(P_n, i)x^i$ the polynomial of the path.

## 2. NEIGHBORHOOD SETS OF PATHS

Let $P_n, n \geq 4$ be the path with $n$ vertices $V(P_n) = \{1,2,3 \dots n\}$ and $E(P_n) = \{(1,2), (2,3) \dots (n-1, n)\}$. Let $P_n^i$ be the family of neighborhood sets of $P_n$ with cardinality $i$. Every path $P_n$ consist a simple path. The following lemmas and theorems are needed for the construction of family Neighborhood sets with different cardinality.

**Lemma 2.1:** For a graph $= P_n$ , $n \geq 3$ The following Properties are true
$P_n^i = \emptyset$ if and only if $i > n$ or $i < \left\lfloor\frac{n}{2}\right\rfloor$.

**Theorem 2.2:** If $X \in P_{n-3}^{i-1}$ and there exists $x \in [n]$ such that $X \cup \{x\} \in P_n^i$ then $X \in P_{n-2}^{i-1}$.

**Proof:** Suppose that $X \notin P_{n-2}^{i-1}$ since $X \in P_{n-3}^{i-1}$, $X$ contains at least one vertex label $n-4$ $or$ $n-3$. If $n-4 \in X$, then $X \in P_{n-3}^{i-1}$ a contradiction. Hence, $n-4 \in X$, but in this case, $X \cup \{x\} \notin P_n^i$ for any $x \in [n]$ also a contradiction. This $X \in P_{n-2}^{i-1}$.

**Lemma 2.3:** Let $P_n, n \geq 2$ be a path. Then
(i) $X \in P_{n-2}^{i-1} = \emptyset$ then $P_{n-1}^{i-1} = \emptyset$..
(ii) $P_{n-1}^{i-1} = P_{n-2}^{i-1} = \emptyset$ then $P_n^i = \emptyset$.

**Proof:**
(i) Let $P_{n-2}^{i-1} = \emptyset \Rightarrow i-1 < n-2 \Rightarrow i-1 < n-1$ therefore $P_{n-1}^{i-1} \neq \emptyset$ which is a contradiction. Since, By lemma 2.1. $P_{n-2}^{i-1} = \emptyset$ then $P_{n-1}^{i-1} = \emptyset$.
(ii) By the result (i), $i-1 < n-2 \Rightarrow i-1 < n$, therefore $P_n^{i-1} \neq \emptyset$ which is a contradiction. Hence $P_{n-1}^{i-1} = P_{n-2}^{i-1} = \emptyset$ then $P_n^i = \emptyset$.

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**Lemma 2.4:** If $P_n^i \neq \emptyset$ then the following properties are true

(i) $P_{n-1}^{i-1} = \emptyset$ and $P_{n-2}^{i-1} = \emptyset$ if and only if $n = 2k+1$ and $i = k = \left\lfloor \frac{n}{2} \right\rfloor$.
(ii) $P_{n-2}^{i-1} = \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ if and only if $i = n$.
(iii) $P_{n-1}^{i-1} \neq \emptyset$ and $P_{n-2}^{i-1} \neq \emptyset$ if and only if $n = 2k, k = i$.

**Proof:**

(i) Let $P_{n-1}^{i-1} = \emptyset \Rightarrow i-1 > n-1$ or $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$. If $i-1 > n-1$ then $i > n$ by lemma 2.1, $P_n^i = \emptyset$ which is a contradiction. Therefore $-1 < \left\lceil \frac{n-1}{2} \right\rceil + 1$ . Since $P_n^i \neq \emptyset$ and $\left\lfloor \frac{n}{2} \right\rfloor \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil + 1$, we get $n = 2k+1$ and $i = k = \left\lfloor \frac{n}{2} \right\rfloor$. Suppose $n = 2k+1$and $i = k = \left\lfloor \frac{n}{2} \right\rfloor$ for some $\in N$ . Then by lemma 2.1 , if $P_{n-1}^{i-1} = \emptyset$ then $P_{n-2}^{i-1} \neq \emptyset$.

(ii) Let $P_{n-2}^{i-1} = \emptyset$ . By lemma 2.1, $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$ . If $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$ then $-1 < \left\lceil \frac{n-1}{2} \right\rceil$ . Therefore $P_{n-1}^{i-1} = \emptyset$ which is a contradiction. Hence, $i > n-1$. Since $P_{n-1}^{i-1} = \emptyset$, $-1 \leq n-1$. Therefore $= n$. Suppose $i = n$ then by lemma 2.1 if $P_{n-2}^{i-1} = \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$.

(iii) Let $P_{n-1}^{i-1} = \emptyset$ . Then $-1 > n-1 \Rightarrow i-1 < \left\lceil \frac{n-1}{2} \right\rceil$ . If $i-1 > n-1$ then $i-1 > n-2$. Hence by lemma 2.1, $P_{n-1}^{i-1} = P_{n-2}^{i-1} = \emptyset$, which is a contradiction. Therefore $i-1 < \left\lceil \frac{n-1}{2} \right\rceil + 1$ and $P_{n-2}^{i-1} \neq \emptyset$. Hence $\left\lceil \frac{n-2}{2} \right\rceil + 1 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil$. Therefore, $n = 2k = i$. For some $k \in N$ then by lemma 2.1 $P_{n-1}^{i-1} = P_{2k}^{k} \neq \emptyset$ for some $k \in N$.

## 3. CONSTRUCTION OF FAMILIES OF NEIGHBORHOOD SETS OF PATHS

**Theorem 3.1:** For any path $P_n^i$, $n \geq 4$ and $i \geq \left\lfloor \frac{n}{2} \right\rfloor$, the following result are true.

(i) If $P_{n-1}^{i-1} = \emptyset$ and $P_{n-2}^{i-1} \neq \emptyset$ then $P_n^i = \{2,4,\dots,n-5,n-3,n-1\}$ .
(ii) If $P_{n-2}^{i-1} = \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ then $P_n^i = \{[n]\}$.
(iii) If $P_{n-2}^{i-1} \neq \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ then $P_n^i = P_n^{n-1} = \{[n]-\{x\}/x \in [n]\}$ .
(iv) If $P_{n-2}^{i-1} \neq \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ then $P_n^i = \{X_1 \cup \{n\}/X_1 \in P_{n-1}^{i-1}\}$, $P_n^i = \{X_2 \cup \{n-1\}/1 \in X_2 \in P_{n-2}^{i-1}\}$, $P_n^i = \{X_2 \cup \{n-1\}/X_2 \in P_{n-2}^{i-1}\}$ and $P_n^i = \{X_2 \cup \{n\}/1 \notin X_2 \in P_{n-2}^{i-1}\}$.

**Proof:**

(i) Let $P_{n-1}^{i-1} = \emptyset$ and $P_{n-2}^{i-1} \neq \emptyset$ **.** By lemma 2.4 (i), $n = 2k+1$ and $i = k$ for some $k = \left\lfloor \frac{n}{2} \right\rfloor \in N$. This imply that $P_n^i = P_n^{\lceil n/2 \rceil} = \{2,4,6,8,\dots,n-1\}$ .
(ii) Let $P_{n-2}^{i-1} = \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ . By lemma 2.4 (ii) $i = n$. Hence $P_n^i = P_n^n = \{[n]\}$.
(iii) If $i = n-1$ in lemma 2.4 (iii), then we get $P_n^i = P_n^{n-1} = \{[n]-\{x\}/x \in [n]\}$ .
(iv) $P_{n-1}^{i-1} \neq \emptyset$ and $P_{n-2}^{i-1} \neq \emptyset$. Let $X_1 \in P_{n-2}^{i-1}$ then there exists at least one vertex labled in $n-3$ *or* $n-2$ is in $X_1$. If $n-3$ *or* $n-2 \in X_1$, then $X_1 \cup \{n-1\} \in P_n^i$. Let $X_2 \in P_{n-1}^{i-1}$ then there exists one vertex labled as $n-1$ is in $X_2$. If $n-1 \in X_2$ then $X_1 \cup \{n\} \in P_n^i$ .

### 3.1 NEIGHBORHOOD POLYNOMIAL OF PATHS

**Definition 3.1.1:** Let $P_n^i$ be the family of neighborhood sets of a path $P_n$ with cardinality *i* and let $n_0(P_n^i) = |P_n^i|$. Then the **neighborhood polynomial of** $P_n$ is defined as

$$N(P_n, x) = \sum_{n_0 = \left\lfloor \frac{p}{2} \right\rfloor}^{n} n(P_n, x) x^i$$

We obtain a neighborhood polynomial of $P_7$ . $N(P_7, x) = x^3 + 10x^4 + 15x^5 + 7x^6 + x^7$.

**Theorem 3.1.2:** Let $= P_n$ , $n \geq 3$ be a path. Then the following properties are true.

(i) If $P_n^i$ is the family of neighborhood sets with cardinality *i* of $P_n$ then $|P_n^i| = |P_{n-1}^{i-1}| + |P_{n-2}^{i-1}|$.
(ii) For every $n \geq 4$, $N(P_n, x) = x[N(P_{n-1}, x) + N(P_{n-2}, x)]$ with the initial values,

$$N(P_1, x) = x, N(P_2, x) = x^2 + 2x, N(P_3, x) = x^3 + 3x^2 + 3x$$

## 4. COEFFICIENTS OF NEIGHBORHOOD POLYNOMIAL OF PATHS

The coefficients of $N(P_n, x)$ is determined for $1 \leq n \leq 12$. Let $n\,(P_n, i) = |P_n^i|$. Also there are some relationship exist between the coefficients $n(P_n, i)$ where $\frac{n}{2} \leq i \leq n$.

 

**Theorem 4.1:** Let $= P_n, n \geq 3$ be a path and Let $N(P_n, n)$ be the total number of neighborhood sets with size $n$. Then The following properties hold for coefficients of $N(P_n, x)$.

(i) For every $n \in N$, $n\ (P_{2n+1}, n) = 1$

(ii) For every $n \geq 4 i \geq \left\lfloor \frac{n}{2} \right\rfloor$, $n(P_n, i) = n\ (P_{n-1}, i-1) + n\ (P_{n-2}, i-1)$

(iii) For every $n \in N$, $n\ (P_{2n+1}, n+1) = \frac{(n+1)(n+2)}{2}$

(iv) For every $n \in N$, $n\ (P_{2n+2}, n+1) = (n+2)$

(v) For every $n \in N$, $n(P_n, n) = 1$

(vi) For every $n \in N$, $n(P_n, n-1) = n$

(vii) For every $n \in N$, $n\ (P_{2n}, n+1) = \frac{n(n+1)(n+2)}{6}$

(viii) For every $n \in N$, $n(P_{2n}, n) = 1 + n, n \geq 2$

(ix) For every $n \in N$, $n\ (P_n, n-2) = \frac{(n-1)(n-2)}{2}$

(x) If $S_n = \sum_{i=\left\lfloor \frac{n}{2} \right\rfloor}^{n} n(P_n, i)$ then for every $n \geq 4$, $S_n = S_{n-1} + S_{n-2}$.

(xi) For every $n \in N$ and $k = 0,1,2, \dots, 2n-1$, then
$n\ (P_{n+1}, i+1) - n\ (P_n, i+1) = n(P_n, i) - n(P_{n-2}, i)$.

**Proof:**

(i) Since, $P_{2n+1}^n = \{\{2,4,6,8, \dots, n-1\}\}$, $|P_{2n+1}^n| = 1$, $P_{2n+1}^n = \{2,4,6,8, \dots, n-1\}$ therefore $P_{2n+1}^n = 1$, $n = 1,2,3, \dots$

(ii) From the theorem (3.8.2) $|P_n^i| = |P_{n-1}^{i-1}| + |P_{n-2}^{i-1}|$, $n(P_n, i) = n\ (P_{n-1}, i-1) + n\ (P_{n-2}, i-1)$.

(iii) This property is proved for $P_{2n+1}$by induction on . The result is true for $n = 1$ and $i = 1$ Similarly for $n = 3$ and $i = 2$, we get $n(P_3, 2) = n(P_2, 1) + n(P_1, 1) = 2 + 1 = 3$, therefore $P_3^2 = \{\{1,2\}, \{1,3\}, \{2,3\}\}$. For $n = 2$ in $P_{2n+1} = P_5$ and $i = 2$ the Neighborhood sets are $n(P_5, 3) = \{\{1,2,4\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}\} = 6$ For $n = 3,4,5, \dots, n-1$ this result is true. Then by (i) and (ii), it is true for $n$ in $P_{2n+1}$.
By property (ii),

$$n\ (P_{2n+1}, n+1) = n\ (P_{2n}, n) + n\ (P_{2n-1}, n)$$
$$= (n+1) + \frac{n(n+1)}{2}$$
$$n\ (P_{2n+1}, n+1) = \frac{(n+1)(n+2)}{2}$$

(iv) By induction on $n$. When $n = 1$ $P_{2n+2}$ and $i = 2$ the neighborhood sets are $n(P_4, 2) = n(P_3, 1) + n(P_2, 1) = 2 + 1 = 3$ $P_4^2 = \{\{1,3\}, \{2,3\}, \{2,4\}\}$, therefore $n(P_4, 2) = 3$. This result is true for $n = 2,3,4, \dots, n-1$. Then by the results (i), (ii) & (iii), this result is true for $n$

$$n\ (P_{2n+2}, n+1) = n\ (P_{2n+1}, n) + n\ (P_{2n}, n)$$
$$= (n+1) + 1$$
$$n\ (P_{2n+2}, n+1) = n + 2.$$

(v) For any Path $P_n$ with $n$ vertices the number of neighborhood sets of $P_n$ of size $i = n$ is $n(P_n, n) = 1$.

(vi) For any Path $P_n$ with $n$ vertices the number of neighborhood sets of $P_n$ of size $i = n-1$ is $(P_n, n-1) = n$.

(vii) By induction on $n$, the result is true for$n = 1$ in $P_{2n}$ and $i = 2$ ie). $n(P_2, 2) = 3$. Then the result is true for all $n = 2,3, \dots, n-1$ in and $= n+1$. Therefore it is true for $n$. By the results (iii) and (iv) and the induction hypothesis, the number of neighborhood sets of $P_{2n}$ and $i = n+1$ is

$$n\ (P_{2n}, n+1) = n\ (P_{2n-1}, n) + n\ (P_{2n-2}, n)$$
$$= \frac{n(n+1)}{2} + \frac{(n-1)n(n+1)}{6}$$
$$n(P_{2n}, n+1) = \frac{n(n+1)(n+2)}{6}$$

(viii) It is proved by induction on $\geq 2$ . The result is true for $n = 2.$ Then $(P_4, 2) = 3$. The result is true for all $n = 3, \dots, n-1$. and it is true for $n$ by the result (v) and (vi), for $n$ in $P_{2n}$

$$n(P_{2n}, n) = n(P_{2n-1}, n-1) + n(P_{2n-2}, n-1)$$
$$n(P_{2n}, n) = 1 + n$$

(ix) It is proved by induction on $n \geq 4$. The result is true $n = 4$ then $(P_4, 2) = 3$. The result is true for all $n = 4,5, .., n-1$.
In $P_n$ with $i = n-2$. Then it is true for $n$ in $P_n$ with $i = n-2$

$$n(P_n, n-2) = n(P_{n-1}, n-3) + n(P_{n-2}, n-3)$$
$$= \frac{(n-2)(n-3)}{2} + n - 2$$
$$n(P_n, n-2) = \frac{(n-1)(n-2)}{2}$$

(x) It is proved from the result theorem 3.1.2 (i)

$$S_n = \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n} n(P_n, i)$$

$$= \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n} n(P_{n-1}, i-1) + n(P_{n-2}, i-1)$$

$$= \sum_{i=\lfloor \frac{n}{2} \rfloor -1}^{n-1} n(P_{n-1}, i) + \sum_{i=\lfloor \frac{n}{2} \rfloor -1}^{n-2} n(P_{n-2}, i-1)$$

$$S_n = S_{n-1} + S_{n-2}.$$

(xi) From the result theorem 3.1.2 (i) for every $n \in N$ and $k = 0,1,2, \dots, 2n-1$ then

$$n(P_{n+1}, i+1) - n(P_n, i+1) = \big((n(P_n, i) + n(P_{n-1}, i))\big) - ((n(P_{n-1}, i) + n(P_{n-2}, i)))$$

$$n(P_{n+1}, i+1) + n(P_n, i+1) = n(P_n, i) - n(P_{n-2}, i)$$

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