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STABILITY ANALYSIS OF MUTUALISTIC INTERACTIONS AMONG THREE SPECIES WITH UNLIMITED RESOURCES

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ABSTRACT

The aim of the present paper is to investigate mutualistic interactions among three species model in an ecosystem. The model is characterized by a system of first order non-linear ordinary differential equations. We study three cases: (1) The death rate of any one (say third) species is greater than its birth rate. (2) The death rate of any two (say second and third) species are greater than their birth rate. (3) The death rate of all the species are greater than their birth rate. Further, the local stability analysis of the system is carried out analytically as well as numerically.

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1. INTRODUCTION

In a natural ecosystem, an interaction of two or more species in closed association with the benefit of each other is found in many types of communities. Pollination is the best example for mutualism. Birds and insects visit flowering plants for a number of reasons and in the process pick up pollen, and allow these plants host a greater opportunity for genetic diversity. If enhanced out crossing leads to higher reproductive success, those plants that encouraged visitors with enticements of nectar, pollen or pseudomating opportunities naturally increase in frequency overtime. Another known mutualistic relationship is found in lichens. The lichen is made up of two organisms: a fungus and algae. The algae photosynthesize and provide nutrients to the fungus protect the algae from the often harsh environmental conditions where these organisms live, by providing moisture and minerals.

An example of a three way mutualism is found in Australian tropical forests such as there was an arboreal termite nest high up on a tree with a large, conspicuous hole (approximately 4 cm. in diameter). This hole was created by a kookaburra bird. The bird hollows out part of the termite nest to use as its own nest, where it will lay its eggs. The kookaburra lines the bottom of its nest with twigs and mud. The termites benefit from the bird nest because the hole created by the bird helps ventilate and regulate the temperature of the termite nest in the hot tropical environment. The kookaburra benefits because it has a safe nesting location where predatory animals will not disturb its young when they hatch. The third member of this mutualistic trio is a moth who lays its eggs so they will hatch at the same time as the kookaburra eggs hatch. The moth larvae eat excrement and parasites from the young kookaburra chicks. The moths benefit because they get a plentiful supply of food and a safe living location. The birds benefit because the moths keep their nest and chicks clean and free of excrement and parasites. This is a beneficial association for all three members- termites, kookaburras, and moths.

A general discussion on the multi species population model can be found in Kapur [5], wherein he made an extensive survey of the fundamental equations governing ecological interactions. Mutualism has been recognized as a significant biological interaction (Addicott [1], Boucher [2], Bronstein [3], Dhakne and Munde [4], May [6], Munde and Dhakne [7] and [8]). Despite this fact, most attention seems to have been focused on predator / prey and competition interactions (Rish and Boucher [9]). For a review and fruitful discussion on models with mutualism, we refer to Wolin [13]. However the volume of work on mutualism is significantly small compared to that of the work dealing with prey-predator and competition interactions. This motivates the authors to study three species mutualistic interactions in ecosystem. In this paper, we investigate the stability analysis of mutualistic interactions among three species with unlimited resources.

2. MATHEMATICAL MODEL

The mathematical model for three mutually interacting species with unlimited resources is given by the following system of equations:

$$\begin{aligned}\frac{dN_1}{dt} &= N_1\left(a_1+\alpha_{12}N_2+\alpha_{13}N_3\right)\\ \frac{dN_2}{dt} &= N_2\left(a_2+\alpha_{21}N_1+\alpha_{23}N_3\right)\\ \frac{dN_3}{dt} &= N_3\left(a_3+\alpha_{31}N_1+\alpha_{32}N_2\right),\end{aligned} \tag{2.1}$$

where $N_i, i=1,2,3$ represent the population density of first, second and third species respectively, a_i represent the intrinsic growth rate of first, second and third species respectively, α_{12} is the mutual coefficient of second species to first species, α_{13} is the mutual coefficient of third species to first species, α_{21} is the mutual coefficient of first species to second species, α_{23} is the mutual coefficient of third species to second species, α_{31} is the mutual coefficient of first species to third species, α_{32} is the mutual coefficient of second species to third species. Here $a_i, i=1,2,3$ and $\alpha_{12},\alpha_{13},\alpha_{21},\alpha_{23},\alpha_{31},\alpha_{32}$ are assumed to be nonnegative constants. If the death rate is greater than the birth rate for any species, we continue to use the same notation as intrinsic growth rate with negative sign for the rate of difference.

2.1 Stability analysis. In this section we discuss the local stability analysis of the nonnegative equilibrium point of the system (2.1). There is only one nonnegative equilibrium point for the system (2.1). The equilibrium point $E_{21}=(0,0,0)$ always exists. From the variational matrix about the equilibrium point E_{21}, it is concluded that E_{21} is a unstable node with locally unstable manifold in the (N_1, N_2, N_3) space. Further, all the three species populations grow indefinitely as $t\to\infty$.

3. THE DEATH RATE OF ANY ONE (SAY THIRD) SPECIES IS GREATER THAN ITS BIRTH RATE

Under this situation, system (2.1) has the form

$$\begin{aligned}\frac{dN_1}{dt} &= N_1\left(a_1+\alpha_{12}N_2+\alpha_{13}N_3\right)\\ \frac{dN_2}{dt} &= N_2\left(a_2+\alpha_{21}N_1+\alpha_{23}N_3\right)\\ \frac{dN_3}{dt} &= N_3\left(-a_3+\alpha_{31}N_1+\alpha_{32}N_2\right),\end{aligned} \tag{3.1}$$

3.1 Stability analysis. In this section we study the local stability analysis of the nonnegative equilibrium point of the system (3.1). There is only one possible nonnegative equilibrium point for the system (3.1). The equilibrium point $E_{31}=(0,0,0)$ always exists. From the variational matrix about the equilibrium point E_{31}, it is concluded that E_{31} is a saddle point with locally stable manifold in the N_3 direction and locally unstable manifold in the (N_1, N_2) plane. Further, first and second species populations grow and third species population decline near E_{31}.

4. THE DEATH RATE OF ANY TWO SPECIES (SAY SECOND AND THIRD) ARE GREATER THAN THEIR BIRTH RATE

Under this situation, system (2.1) has the form

$$\begin{aligned}\frac{dN_1}{dt} &= N_1\left(a_1+\alpha_{12}N_2+\alpha_{13}N_3\right)\\ \frac{dN_2}{dt} &= N_2\left(-a_2+\alpha_{21}N_1+\alpha_{23}N_3\right)\\ \frac{dN_3}{dt} &= N_3\left(-a_3+\alpha_{31}N_1+\alpha_{32}N_2\right),\end{aligned} \tag{4.1}$$

4.1 Stability analysis. In this section we study the local stability analysis of the nonnegative equilibrium points of the system (4.1). There are at most two possible nonnegative equilibrium points are as follows

i. $E_{41} = (0,0,0)$ always exists.

ii. $E_{42} = \left(0, \frac{a_3}{\alpha_{32}}, \frac{a_2}{\alpha_{23}}\right)$ always exists, as the second and third species survive and first species is washed out.

From the variational matrix about the equilibrium point E_{41}, it is shown that E_{41} is saddle point with point with locally stable manifold in the (N_2, N_3) plane and locally unstable manifold in the N_1 direction. Further, second and third species populations decline near E_{41} while first species population grows indefinitely. Similarly, The eigenvalues of the variational matrix about E_{42} are given as follows

$$\lambda_1 = \frac{a_1\alpha_{23}\alpha_{32} + a_2\alpha_{13}\alpha_{32} + a_3\alpha_{12}\alpha_{23}}{\alpha_{23}\alpha_{32}}, \qquad \lambda_2 = \sqrt{a_2 a_3}, \qquad \lambda_3 = -\sqrt{a_2 a_3}.$$

Clearly $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 < 0$. Thus, the equilibrium point E_{42} is saddle point with locally stable manifold in N_3 direction and locally unstable manifold in the (N_1, N_2) plane. Moreover, third species population decline near equilibrium point E_{42} while first and second species populations grow indefinitely as $t \to \infty$.

5. THE DEATH RATE OF ALL THE SPECIES ARE GREATER THAN THEIR BIRTH RATE

Under this situation, system (2.1) has the form

$$\begin{aligned}
\frac{dN_1}{dt} &= N_1\left(-a_1 + \alpha_{12}N_2 + \alpha_{13}N_3\right)\\
\frac{dN_2}{dt} &= N_2\left(-a_2 + \alpha_{21}N_1 + \alpha_{23}N_3\right)\\
\frac{dN_3}{dt} &= N_3\left(-a_3 + \alpha_{31}N_1 + \alpha_{32}N_2\right),
\end{aligned} \tag{5.1}$$

5.1 Stability analysis. In this section, the existence of the equilibrium points of system (5.1) and local stability analysis of each one are investigated. At most there are five possible nonnegative equilibrium points for system (5.1), the existence conditions of them are given as the following.

1) The equilibrium point $E_{51}(0,0,0)$ always exists.
2) The equilibrium point $E_{52}\left(0, \frac{a_3}{\alpha_{32}}, \frac{a_2}{\alpha_{23}}\right)$ always exists in the (N_2, N_3) plane, as the first species is washed out and second and third species survive.
3) The equilibrium point $E_{53}\left(\frac{a_3}{\alpha_{31}}, 0, \frac{a_1}{\alpha_{13}}\right)$ always exists in the (N_1, N_3) plane, as the second species is washed out and first and third species survive.
4) The equilibrium point $E_{54}\left(\frac{a_2}{\alpha_{21}}, \frac{a_1}{\alpha_{12}}, 0\right)$ always exists in the (N_1, N_2) plane, as the third species is washed out and first and second species survive.
5) The positive equilibrium point $E_{55}\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ exists, if and if there is unique a positive solution to the following set of equations

$$\begin{aligned}
\alpha_{12}N_2 + \alpha_{13}N_3 &= a_1\\
\alpha_{21}N_1 + \alpha_{23}N_3 &= a_2\\
\alpha_{31}N_1 + \alpha_{32}N_3 &= a_3,
\end{aligned}$$

provided that the three conditions

$(C_1)\; a_2\alpha_{13}\alpha_{32} + a_3\alpha_{12}\alpha_{23} > a_1\alpha_{23}\alpha_{32}$,

$(C_2)\; a_1\alpha_{23}\alpha_{31} + a_3\alpha_{13}\alpha_{21} > a_2\alpha_{13}\alpha_{31}$,

$(C_3)\; a_1\alpha_{21}\alpha_{32} + a_2\alpha_{12}\alpha_{31} > a_3\alpha_{12}\alpha_{21}$, hold, where

$$\overline{N_1} = \frac{-a_1\alpha_{23}\alpha_{32} + a_2\alpha_{13}\alpha_{32} + a_3\alpha_{12}\alpha_{23}}{\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32}},$$

$$\overline{N_2} = \frac{a_1\alpha_{23}\alpha_{31} - a_2\alpha_{13}\alpha_{31} + a_3\alpha_{13}\alpha_{21}}{\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32}},$$

$$\overline{N_3} = \frac{a_1\alpha_{21}\alpha_{32} + a_2\alpha_{12}\alpha_{31} - a_3\alpha_{12}\alpha_{21}}{\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32}},$$

Computing the variational matrices corresponding to each equilibrium point and then using the Routh-Hurwitz criteria the following dynamical behaviour results are observed.

(1) The equilibrium point E_{51} is locally asymptotically stable in the (N_1, N_2, N_3) space and the entire three species populations decline near E_{51}.

(2) If $a_2\alpha_{13}\alpha_{32} + a_3\alpha_{12}\alpha_{23} > a_1\alpha_{23}\alpha_{32}$, the equilibrium point E_{52} is saddle point with locally stable manifold in the N_3 direction and locally unstable manifold in (N_1, N_2) plane. Further, third species population decline near E_{52} while first and second species populations grow indefinitely. If $a_2\alpha_{13}\alpha_{32} + a_3\alpha_{12}\alpha_{23} < a_1\alpha_{23}\alpha_{32}$, the equilibrium point E_{52} is saddle point with locally stable manifold in (N_1, N_3) plane and locally unstable manifold in the N_2 direction. Further, first and third species populations decline near E_{52} while second species population grows indefinitely as $t \to \infty$.

(3) If $a_1\alpha_{23}\alpha_{31} + a_3\alpha_{13}\alpha_{21} > a_2\alpha_{13}\alpha_{31}$, the equilibrium point E_{53} is saddle point with locally stable manifold in the N_3 direction and locally unstable manifold in (N_1, N_2) plane. Further, third species population decline near E_{53} while first and second species populations grow indefinitely. If $a_1\alpha_{23}\alpha_{31} + a_3\alpha_{13}\alpha_{21} < a_2\alpha_{13}\alpha_{31}$, the equilibrium point E_{53} is saddle point with locally stable manifold in (N_1, N_3) plane and locally unstable manifold in the N_2 direction. Further, first and third species populations decline near E_{53} while second species population grows indefinitely as $t \to \infty$.

(4) The stability behaviour of the equilibrium point E_{54} is similar to the equilibrium points E_{52} or E_{53} and hence we omit the details.

Theorem 5.1: *The positive equilibrium point* $E_{55}\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ *is not stable.*

Proof: Assume $N_1 = \overline{N_1} + u_1$, $N_2 = \overline{N_2} + u_2$, $N_3 = \overline{N_3} + u_3$ where u_1, u_2, u_3 are small perturbations. The variational matrix about equilibrium point E_{55} is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{12}\overline{N_1} & \alpha_{13}\overline{N_1} \\ \alpha_{21}\overline{N_2} & 0 & \alpha_{23}\overline{N_2} \\ \alpha_{31}\overline{N_3} & \alpha_{32}\overline{N_3} & 0 \end{bmatrix}.$$

The characteristic equation of the above variational matrix about equilibrium point E_{55} is

$$\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3 = 0,$$

$$k_1 = 0; \quad k_2 = -\left(\alpha_{12}\alpha_{21}\overline{N_1}\,\overline{N_2} + \alpha_{23}\alpha_{32}\overline{N_2}\,\overline{N_3} + \alpha_{13}\alpha_{31}\overline{N_1}\,\overline{N_3}\right); \quad k_3 = -\left(\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32}\right)\overline{N_1 N_2 N_3}.$$

According to Routh-Hurwitz criterion the necessary and sufficient conditions for stability are

$$k_1 > 0, \qquad k_3 > 0 \qquad k_1k_2 > k_3.$$

We observe that $k_1 = 0$ and $k_3 < 0$ and therefore, by Routh-Hurwitz criterion, the equilibrium point E_{55} is unstable.

6. NUMERICAL SIMULATION

In this section, we carry numerical simulations by using fourth order Runge-Kutta method with the help of MATLAB software.

Consider the parametric values

$$a_1 = 0.9, \quad a_2 = 0.6, \quad a_3 = 1.3, \quad \alpha_{12} = 1.2, \quad \alpha_{13} = 0.6, \quad \alpha_{21} = 0.5,$$
$$\alpha_{23} = 0.9, \quad \alpha_{31} = 1.2, \quad \alpha_{32} = 2. \tag{6.1}$$

For this set of parametric values, the system (5.1) has an equilibrium point at (0.2658, 0.4905, 0.5190). It is saddle point as the eigenvalues of the variational matrix of the system (5.1) about the equilibrium point (0.2658, 0.4905, 0.5190) are approximately 0.8837, -0.2181, -0.6656. Furthermore, all the three species being extinct due to death rate are greater than their birth rate (See Fig.(1)). However also we find the solution from the initial state (0.6, 0.4, 0.4) tends rapidly to (∞, ∞, ∞) Thus we observe that the system (5.1) is not globally stable (See Fig.(2)).

Figs. (3) and (4) are drawn for the following choice of parameters:

$$a_1 = 0.01, \quad a_2 = 0.5, \quad a_3 = 0.6, \quad \alpha_{12} = 0.4, \quad \alpha_{13} = 0.2, \quad \alpha_{21} = 0.3,$$
$$\alpha_{23} = 1, \quad \alpha_{31} = 0.2, \quad \alpha_{32} = 1. \tag{6.2}$$

For this set of parametric values, the system (4.1) has an equilibrium point at (0, 0.6, 0.5) which is saddle point as the eigenvalues of the variational matrix about the equilibrium point (0, 0.6, 0.5) are approximately 0.3500, 0.5477, -0.5477. Furthermore, second and third species being extinct due to death rate are greater than their birth rate while first species grows indefinitely (See Fig.(3)). However also we find the solution from the initial state (0.4, 0.3, 0.4) tends rapidly to (∞, ∞, ∞) Thus we observe that the system (4.1) is not globally stable (See Fig.(4)).

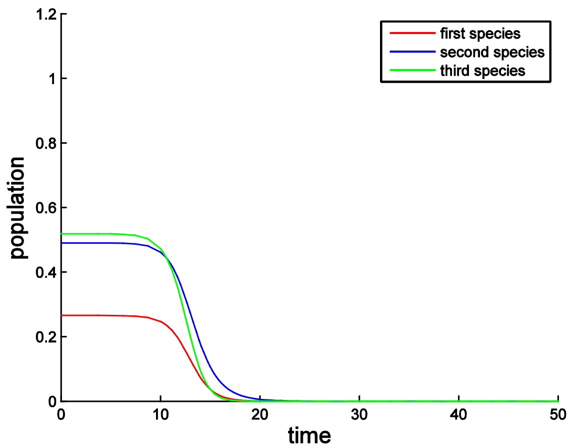

Figure-1

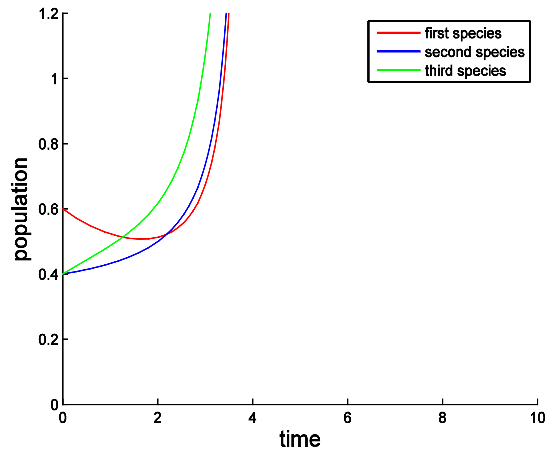

Figure-2

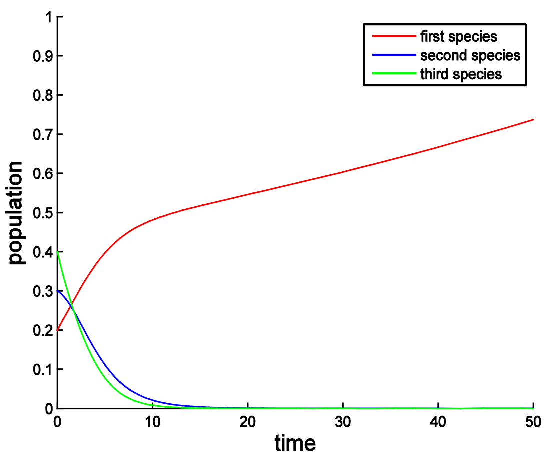

Figure-3

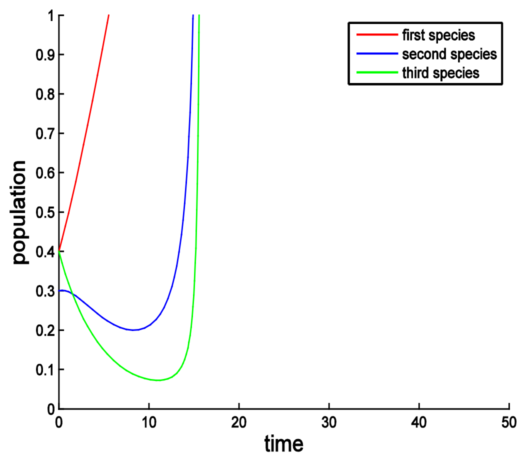

Figure-4

7. DISCUSSION

In this paper, we have considered a mathematical model for three mutually interacting species with unlimited resources. We carried out the numerical simulations and observed the effect of death rates in the system (4.1) and (5.1). There is still a lot of work to do in this area. For example, it would be interesting to see what the behaviour of model 2.1 would be when the limited resources are in all the three species with harvesting. However, less attention has been given to the study of mutualism model as compared to the prey-predator and competition. Thus the present article contributes a few more results on the stability of mutualistic interactions among three species system.

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