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SUPER SECURE DOMINATION IN GRAPHS

MICHAEL P. BALDADO, Jr.
Mathematics Department, Negros Oriental State University.

ENRICO L. ENRIQUEZ*
Department of Mathematics,
School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines.

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ABSTRACT

Let $G = (V(G), E(G))$ be a simple graph. A set $S \subseteq V(G)$ is called a secure dominating set of a graph G if for every vertex $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is dominating. It is a super secure dominating set if $N_G(v) \cap (V(G) \setminus S) = \{u\}$. The minimum cardinality of a super secure dominating set in G, denoted by $\gamma_{sups}(G)$, is called the super secure domination number of G. In this paper, we initiate the study of the concept and give some important results.

Mathematics Subject Classification: *05C69.*

Keywords: *domination, secure domination, super domination, super secure domination.*

1. INTRODUCTION

Graph Theory was born in 1736 with Euler's paper in which he solved the Konigsberg bridge problem [11]. Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [16]. However, it was not until following an article by Ernie Cockayne and Stephen Hedetniemi [2], that domination became an area of study by many. One type of domination parameter is the secure domination in graphs. This parameter is used to study the problem of using guards to defend the vertices of a graph G against an attacker. Several variations of this graph protection problem have been studied, including k-secure sets [1], and eternal m-secure sets [17], secure convex dominating sets [3]. Other variation of domination in graphs can be read in [4, 5, 6, 7, 8, 10, 13]. The super dominating sets in graphs was initiated by Lemanska *et al.* [12]. Motivated by these parameters, we initiate the study of super secure domination in graphs.

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v)\ N(v) = \{ u \in V(G): uv \in E(G) \}$. If $S \subseteq V(G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \cup_{v \in S} N_G(v)$. The *closed neighborhood* of S is $N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in (V(G) \setminus S)$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G.

A set $S \subseteq V(G)$ is called a secure dominating set of a graph G if for every vertex $u \in V(G) \setminus S$, there exists $v \backslash in\, S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is dominating. The minimum cardinality of a secure dominating set of G, denoted by $\gamma_s(G)$, is called the *secure domination number* of G.

A set $D \subset V(G)$ is called a super dominating set if for every vertex $u \in V(G) \setminus D$, there exists $v \in D$ such that $N_G(v) \cap (V(G) \setminus D) = \{u\}$. The super domination number of G is the minimum cardinality among all super dominating set in G. A set $S \subseteq V(G)$ is called a secure dominating set of a graph G if for every vertex $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is dominating.

A secure dominating set S is called a *super secure dominating set* of a graph G if for every vertex $u \in V(G) \setminus S$, there exists $v \in S$ such that $N_G(v) \cap (V(G) \setminus S) = \{u\}$. The minimum cardinality of a super secure dominating set of G, denoted by $\gamma_{sups}(G)$, is called the super secure domination number of G. For general concepts we refer the reader to [9].

Corresponding Author: Enrico L. Enriquez*, Department of Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines.

2. RESULTS

From the definitions, the following remarks are immediate.

Remark 2.1: Let G be a nontrivial connected graph of order $n \geq 2$. Then
(i) $1 \leq \gamma_{sups}(G) \leq n-1$, and
(ii) $\gamma(G) \leq \gamma_s(G) \leq \gamma_{sups}(G)$.

Remark 2.2: The super secure dominating set is a super dominating set and a secure dominating set.
It is worth mentioning that the upper bound in Remark 2.1(i) is sharp. For example, $\gamma_{sups}(K_n) = n-1$ for all $n \geq 2$. The lower bound is also attainable as the following result shows.

Remark 2.3: The $\gamma_{sups}(G) = 1$ if and only if $G \cong K_2$.

The next result says that the value of the parameter $\gamma_{sups}(G)$ ranges over all positive integers.

Theorem 2.4 (Realization Problem 1): Given positive integers k and n such that $n \geq 2$, $1 \leq k \leq n-1$, there exists a connected graph G with $|V(G)| = n$ and $\gamma_{sups}(G) = k$.

***Proof*:** Consider the following cases:

Case-1: Suppose $k = 1$.

Let $G \cong K_2$. Clearly, $|V(G)| = 2 = n$ and $\gamma_{sups}(G) = 1$.

Case-2: Suppose $2 \leq k < n-1$.

Let $G \cong H \circ P_1$ where H is a nontrivial connected graph. Let $V(H) = \{a_1, a_2, \ldots, a_k\}$ and $n = 2k$. Then $V(H)$ is a γ_{sups}-set in G (with $2 \leq |V(H)| = k$ and $k < n-1$). Thus, $\gamma_{sups}(G) = k$ and

$$|V(G)| = |V(H \circ P_1)| = |V(H) \cup (\cup_{x \in V(H)} V(P_1^x))| = |V(H)| + |\cup_{x \in V(H)} V(P_1^x)| = k + k = 2k = n.$$

Case-3: Suppose $k = n-1$.

Let $G = K_n$. Then $\gamma_{sups}(G) = n-1 = k$ and $|V(G)| = n$. This proves the assertion.

Theorem 2.5 (Realization Problem 2): Given positive integers k, m and $n \geq 6$ such that $1 \leq k \leq m \leq n-1$, there exists a connected graph G with $|V(G)| = n$, $\gamma_{sups}(G) = m$, and $\gamma(G) = k$.

Proof: Consider the following cases:

Case-1: Suppose $m = n-1$.

Let $k = 1$ and consider the graphs $G = K_n$. Let $x \in V(G)$. The set $A = V(G) \setminus \{x\}$ is a γ_{sups}-set and $B = \{x\}$ is a γ-set in G. Thus, $\gamma_{sups}(G) = |A| = |V(G) \setminus \{x\}| = |V(G)| - 1 = n-1 = m$ and $\gamma(G) = |B| = 1 = k$. Further, $|V(G)| = n$.

Case-2: Suppose $m < n-1$.

If $k = m$, then let $n = 2m$ and consider the graph $G \cong H\backslash circ\, P_1$ where H is a nontrivial connected graph. Let $V(H) = \{a_1, a_2, \ldots, a_k\}$. Then the set $A = V(H)$ is a γ_{sups}-set and a γ-set in G. Thus, $\gamma(G) = k = m = \gamma_{sup}(G)$. Further, $|V(G)| = |V(H \circ P_1)| = |V(H)| + |\cup_{x \in V(H)} V(P_1^x)| = k + k = 2k = 2m = n$.

If $k < m$, then let $k = m - i$, $n = 2m$, $m = 3i$ $(\forall\, i = 1,2,\ldots)$\$ and consider the graph $G = P_n$, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. The set $A = \left\{v_{4j-1}: j = 1,2,\ldots,\frac{n}{4}\right\} \cup \left\{v_{4j}: j = 1,2,\ldots,\frac{n}{4}\right\}$ is a γ_{sups}-set in G, whenever $\frac{n}{12}$ is an integer, otherwise $A = \left\{v_{4j-3}: j = 1,2,\ldots,\frac{n+2}{4}\right\} \cup \left\{v_{4j}: j = 1,2,\ldots,\frac{n-2}{4}\right\}$ is a γ_{sups}-set in G. The set $B = \left\{v_{3j-1}: j = 1,2,\ldots,\frac{n}{3}\right\}$ is a γ-set in G. This implies that $\gamma_{sups}(G) = \frac{n}{4} + \frac{n}{4} = \frac{n+2}{4} + \frac{n-2}{4} = \frac{n}{2} = \frac{2m}{2} = m$ and $\gamma(G) = \frac{n}{3} = \frac{2m}{3} = \frac{6i}{3} = 2i = 3i - i = m - i = k. Further, |V(G)| = n$.
This proves the assertion.

The following result follows from Theorem 2.4.

Corollary 2.6: The difference $\gamma_{sups} - \gamma$ can be made arbitrarily large.

Proof: Let n be a positive integer. By Theorem 2.4, there exists a connected graph G such that $\gamma_{sups}(G) = n + 1$ and $\gamma(G) = 1$. Thus, $\gamma_{sups}(G) - \gamma(G) = n$, showing that $\gamma_{sups} - \gamma$ can be made arbitrarily large.

Remark 2.7: If $S^* \subseteq S$ and S^* is a super secure dominating set in G, then S is a super secure dominating set in G.

We need the following results for the characterization of the super dominating set in the corona of two graphs.

Lemma 2.8: Let G be a nontrivial connected graph and H be a complete graph. Then a proper subset S of $V(G \circ H)$ is a super secure dominating set in $G \circ H$ if $S = V(G) \cup (\bigcup_{v\in V(G)}(V(H^v) \setminus \{x\})$ for any $x \in V(H^v)$.

Proof: Suppose that $S = V(G) \cup (\bigcup_{v\in V(G)}(V(H^v) \setminus \{x\})$ for any $x \in V(H^v)$. Then

$$S = V(G) \cup (\bigcup_{v\in V(G)}(V(H^v) \setminus \{x\}) = \bigcup_{v\in V(G)}(V(H^v + v) \setminus \{x\})$$

for any $x \in V(H^v)$. Since H is a complete graph, $H^v + v$ is a complete graph for each $v \in V(G)$. Let $x \in V(H)$. For each $v \in V(G), V(H^v + v) \setminus\{x\}$ is a super secure dominating set in $H^v + v$ by case3 of Theorem 2.4. This implies that $S = \bigcup_{v\in V(G)}(V(H^v + v)\setminus \{x\})$ is a super secure dominating set in $G \circ H$.

Lemma 2.9: Let G be a nontrivial connected graph and H be a complete graph. Then a proper subset S of $V(G \circ H)$ is a super secure dominating set in $G \circ H$ if $S = \bigcup_{v\in V(G)}(V(H^v)$.

Proof: Suppose that $S = \bigcup_{v\in V(G)}(V(H^v)$. Since H is a complete graph, $H^v + v$ is a complete graph for each $v \in V(G)$. This implies that $V(H^v) = V(H^v + v) \setminus \{v\}$ is a super secure dominating set in $H^v + v$. Thus, $S = \bigcup_{v\in V(G)}(V(H^v)$ is a super secure dominating set in $G \circ H$.

The next result is the characterization of the super secure dominating set in the corona of two graphs.

Theorem 2.10: Let G be a nontrivial connected graph and H be a complete graph. Then a proper subset S of $V(G \circ H)$ is a super secure dominating set in $G \circ H$ if and only if one of the following statements holds:

(i) $S = V(G) \cup (\bigcup_{y\in V(G)\setminus C}\left(V(H^y)\right) \cup (\bigcup_{v\in C}(V(H^v)\setminus\{x\})$ for any $C \subseteq V(G)$ and $x \in V(H^v)$.

(ii) $S = C \cup (\bigcup_{u\in V(G)\setminus C}\left(V(H^u)\right) \cup (\bigcup_{v\in C}(V(H^v)\setminus\{x\})$ for any $C \subseteq V(G)$ and $x \in V(H^v)$.

Proof: Suppose that a proper subset S of $V(G \circ H)$ is a super secure dominating set in $G \circ H$. Since H is a complete graph, $H^v + v$ is a complete graph for each $v \in V(G)$. Let $x \in V(H)$. For each $v \in V(G)$, $V(H^v + v)\setminus\{x\}$ is a super secure dominating set in $H^v + v$ by case3 of Theorem 2.4. This implies that $\bigcup_{v\in V(G)}(V(H^v + v)\setminus\{x\}$ is a super secure dominating set in $G \circ H$. Thus, $S = \bigcup_{v\in V(G)}(V(H^v + v)\setminus\{x\} = V(G) \cup (\bigcup_{v\in V(G)}(V(H^v)\setminus\{x\})$ for any $x \in V(H^v)$.

Let $C \subseteq V(G)$. Then

$$\begin{aligned} S^* &= V(G) \cup (\bigcup_{v\in V(G)}(V(H^v) \setminus \{x\})) \\ &= V(G) \cup \left(\bigcup_{v\in V(G)\setminus C}(V(H^v) \setminus \{x\})\right) \cup \left(\bigcup_{x\in C}(V(H^v) \setminus \{x\})\right) \\ &\subset V(G) \cup \left(\bigcup_{v\in V(G)\setminus C}\left(V(H^v)\right)\right) \cup \left(\bigcup_{x\in C}(V(H^v) \setminus \{x\})\right) = S \end{aligned}$$

for any $x \in V(H^v)$. Thus, $S^* \subset S$. By Remark 2.7, S is a super secure dominating set in $G \circ H$. This proves statement (i).

To show statement (ii). Similarly, for each $v \in V(G)$, $V(H^v + v) \setminus \{v\}$ is a super secure dominating set in $H^v + v$. Let $C \subseteq V(G)$. Then,

$$\begin{aligned} S^* &= \bigcup_{v\in V(G)}(V(H^v + v) \setminus \{v\}) \\ &= \bigcup_{v\in V(G)}(V(H^v + v)) \end{aligned}$$

$$S^* = (\bigcup_{v\in V(G)\setminus C}\left(V(H^v)\right)) \cup (\bigcup_{v\in C}\left(V(H^v)\right))$$

Observed that $|(\bigcup_{v\in C}V(H^v))| = |(\bigcup_{v\in C}(V(H^v + v) \setminus \{x\}))|$ with both $(\bigcup_{v\in C}V(H^v))$ and $(\bigcup_{v\in C}(V(H^v + v) \setminus \{x\}))$ are super secure dominating set in the subgraph $\langle\bigcup_{v\in C}(V(H^v + v))\rangle$ induced by $\bigcup_{v\in C}(V(H^v + v))$. Thus,

$$\begin{aligned} S^{**} &= (\bigcup_{v\in V(G)\setminus C}\left(V(H^v)\right)) \cup (\bigcup_{v\in C}(V(H^v + v) \setminus \{x\})) \\ &= (\bigcup_{v\in V(G)\setminus C}\left(V(H^v)\right)) \cup C \cup (\bigcup_{v\in C}(V(H^v) \setminus \{x\})) \\ &= C \cup (\bigcup_{v\in V(G)\setminus C}\left(V(H^v)\right)) \cup (\bigcup_{v\in C}(V(H^v) \setminus \{x\})) \end{aligned}$$

Therefore, S^{**} is a super secure dominating set in $G \circ H$. This proves statement (ii).

For the converse, suppose that statement (i) or (ii) holds. Consider first that (i) holds and consider the following cases:

Case-1: Suppose that $C = \emptyset$. Then $S = V(G) \cup (\bigcup_{y \in V(G)} (V(H^y)) = V(G \circ H)$ is a super secure dominating set in $G \circ H$.

Case-2: Suppose that $C \neq \emptyset$. If $C = V(G)$, then $S = V(G) \cup (\bigcup_{v \in V(G)} (V(H^v \setminus \{x\}))$ for any $x \in V(H^v)$. Thus, S is a super secure dominating set in $G \circ H$ by Lemma 2.8. If $C \neq V(G)$, then $S = V(G) \cup (\bigcup_{y \in V(G) \setminus C} (V(H^y))) \cup (\bigcup_{v \in C} (V(H^v) \setminus \{x\}))$ for any nonempty $C \subset V(G)$ and $x \in V(H^v)$. Since $V(G) \subset S$, S is a dominating set in $G \circ H$. Now, $x \notin S$ implies that $x \in V(G \circ H) \setminus S$ and there exists $v \in S$ such that $vx \in E(G \circ H)$. Let $z \in V(H^v + v)$. Since $H^v + v$ is complete and $x \in V(H^v + v)$, $zx \in E(G \circ H)$. Consequently, $(S \setminus \{v\}) \cup \{x\}$ is a dominating set, that is, S is a secure dominating set in $G \circ H$. To show that S is a super dominating set, let $v \in C$ and $A = N_G(v) \cap V(G)$. Since $v \in C \subset V(G)$, $A \subseteq V(G)$ (equality occur if G is complete) and $N_{G \circ H}(v) = A \cup V(H^v)$. Let $S_v = \{x\} \subseteq V(H^v)$ for all $v \in C$ (equality occur if H is trivial). Then $V(G \circ H)\ S = \bigcup_{v \in C} S_v$. Since $x \in V(H^v)$ and $x \in \bigcup_{v \in C} S_v$, it follows that $x \in N_{G \circ H}(v) \cap (V(G \circ H) \setminus S)$. Suppose there exists x' distinct from x such that $x' \in N_{G \circ H}(v) \cap (V(G \circ H) \setminus S)$. Then $x' \in N_{G \circ H}(v)$ implies that $x' \in V(G)$ or $x' \in V(H^v)$. If $x' \in V(G)$, then $x' \notin V(H^v)$. Since $S_v \subseteq V(H^v), x' \notin S_v$ for all $v \in C$, that is, $x' \notin \bigcup_{v \in C} S_v$. If $x' \in V(H^v)$, then $x' \notin S_v$ for all $v \in C$ since $x' \neq x$. Hence, $x' \notin \bigcup_{v \in C} S_v$. In either case, $x' \notin N_{G \circ H}(v) \cap (V(G \circ H) \setminus S)$ contrary to our assumption. This implies that x is the only element of $N_{G \circ H}(v) \cap (V(G \circ H) \setminus S)$. Thus, $N_{G \circ}(v) \cap (V(G \circ H) \setminus S) = \{x\}$. Accordingly, S is a super secure dominating set in $G \circ H$.

Next suppose that (ii) holds. Then consider the following cases:

***Case*-1:** Suppose that $C = \emptyset$. Then $S = (\bigcup_{u \in V(G)} V(H^u))$. Thus, S is a super secure dominating set in $G \circ H$ by Lemma 2.9.

Case-2: Suppose that $C \neq \emptyset$. If $C = V(G)$, then $S = V(G) \cup (\bigcup_{v \in V(G)} V(H^v) \setminus \{x\})$ for any $x \in V(H^v)$. Thus, S is a super secure dominating set in $G \circ H$ by Lemma 2.8. If $C \neq V(G)$, then $S = C \cup (\bigcup_{u \in V(G) \setminus C} (V(H^u))) \cup (\bigcup_{v \in C} (V(H^v) \setminus \{x\})$ for any nonempty $C \subset V(G)$ and $x \in V(Hv)$. Let $w \in VH \cap S$. Since H is complete, $\{w\}$ is a dominating set in H and hence $\{w\}$ is a dominating set in $H^z + z$ for each $z \in V(G)$. This implies that $N_H[w] = V(H^z + z)$ for all $z \in V(G)$. Thus, $\bigcup_{w \in V(H) \cap S} N_H[w] = \bigcup_{z \in V(G)} V(H^z + z) = V(G \circ H)$. Since $w \in S$, it follows that $N_{G \circ H}[S] = V(G \circ H)$, that is, S is a dominating set in $G \circ H$. Now, consider $u \notin S$. Then $u \in V(G \circ H) \setminus S$ and there exists $v \in S$ such that $uv \in E(G \circ H)$. If $u \in V(H)$, then let $x = u$ and let $z \in V(H^v + v)$ for all $v \in C$. Since $H^v + v$ is complete and $x \in V(H^v + v)$, $zx \in E(G \circ H)$. Consequently, $(S \setminus \{v\}) \cup \{x\}$ is a dominating set in $\bigcup_{v \in C} V(H^v + v)$. If $u \notin V(H)$, then $u \in V(G)$. Since $u \notin S$, $u \notin C$, that is, $u \in V(G) \setminus C$. Choose $v \in V(H^u)$. Since H^u is complete for all $u \in V(G) \setminus C$, it follows that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set in $\bigcup_{u \in V(G) \setminus C} V(H^u + u)$. This implies that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set in $\left(\bigcup_{u \in V(G) \setminus C} V(H^u + u)\right) \cup (\bigcup_{u \in C} V(H^u + u)) = V(G \circ H)$. Therefore, S is a secure dominating set in $G \circ H$. To show that S is a super dominating set, let $z \in S$. Suppose that $z \in C$ and $A = N_G(z) \cap V(G)$. Since $z \in C \subset V(G)$, $A \subseteq V(G)$ (equality occur if G is complete) and $N_{G \circ H}(z) = A \cup V(H^z)$. Let $S_z = \{x\} \subseteq V(H^z)$ for all $z \in C$ (equality occur if H is trivial). Then $(\bigcup_{z \in C} S_z) \subset V(G \circ H) \setminus S$. Since $x \in V(H^z)$ and $x \in \bigcup_{z \in C} S_z$, it follows that $x \in N_{G \circ H}(z) \cap (V(G \circ H) \setminus S)$. If $z \notin C$, then either $z \in V(H^v) \setminus \{x\}$ for all $v \in C \; or \; z \in V(H^u)$ for all $u \in V(G) \setminus C$. Consider that $z \in V(H^v) \setminus \{x\}$ for all $v \in C$. Then $B = N_H(z) \cap V(G) = \{v\}$, that is, $B \subseteq C \subset V(G)$ and $N_{G \circ H}(z) = B \cup (V(H^v) \setminus \{z\})$. Let $S_v = \{x\} \subseteq V(H^v)$ for all $v \in C$ (equality occur if H is trivial). Then $(\bigcup_{v \in C} S_v) \subset V(G \circ H) \setminus S$. Since $x \in V(H^v)$ and $x \in \bigcup_{v \in C} S_v$, it follows that $x \in N_{G \circ H}(z) \cap (V(G \circ H) \setminus S)$. Similarly, if $z \in V(H^u)$ for all $u \in V(G) \setminus C$, then $x \in N_{G \circ H}(z) \cap (V(G \circ H) \setminus S)$. In either cases, $x \in N_{G \circ H}(z) \cap (V(G \circ H) \setminus S)$. Suppose there exists x' distinct from x such that $x' \in N_{G \circ H}(v) \cap (V(G \circ H) \setminus S)$. Using similar arguments used above, $N_{G \circ H}(v) \cap (V(G \circ H) \setminus S) = \{x\}$. Accordingly, S is a super secure dominating set in $G \circ H$.

The following result is an immediate consequence of Theorem 2.10.

Corollary 2.11: Let G be a nontrivial connected graph and H be a complete graph. Then

$$\gamma_{sups}(G \circ H) = |V(G)||V(H)|.$$

Proof: Let $S = C \cup \setminus \left(\bigcup_{u \in V(G) \setminus C} V(H^u)\right) \cup (\bigcup_{v \in C} (V(H^v) \setminus \{x\}))$for any $C \subseteq V(G)$ and $x \in V(H^v)$. Then by Theorem 2.10(ii), S is a super secure dominating set in $G \circ H$. Thus $|S| \geq \gamma_{sups}(G \circ H)$. Consider the following cases:

***Case*1.** Suppose that $C = \emptyset$. Then $S = (\bigcup_{u \in V(G)} V(H^u))$. This implies that $|S| = \left|\bigcup_{u \in V(G)} V(H^u)\right| = |V(G)||V(H)| \geq \gamma_{sups}(V(G \circ H))$. Let $v \in S$ and $S^* = S \setminus \{v\}$. Then $v \in (\bigcup_{u \in V(G)} V(H^u))$ implies that $v \in V(H^u)$ for some $u \in V(G)$, that is, $u \in V(G \circ H) \setminus S$. Thus, $u, v \notin S^*$. If H is trivial, then $vz \notin E(G \circ H)$ for any $z \in S^*$. This means that S^* is not a dominating set in $G \circ H$. If H is nontrivial, then let $w \in V(H^u) \setminus \{v\}$ for some $v \in V(H^u)$ and

$u \in V(G)$. Clearly, $N_{G\circ H}(z) = (V(H^u) \setminus \{z\}) \cup \{u\}$ and $V(G \circ H) \setminus S^* = V(G) \cup \{v\}$. Since $u \in V(G)$ and $v \in V(H^u)$, it follows that $u, v \in N_{G\circ H}(z) \cap (V(G \circ H) \setminus S^*)$. Thus, S^* is not a super secure dominating set in $G \circ H$. This implies that $|S| = \gamma_{sups}(G \circ H)$. Therefore, $\gamma_{sups}(G \circ H) = |V(G)||V(H)|$.

***Case*2.** Suppose that $C \neq \emptyset$. If $C = V(G)$, then $S = V(G) \cup (\cup_{v\in V(G)}(V(H^v) \setminus \{x\}))$ for any $C \subset V(G)$ and $x \in V(H^v)$. Thus, $|S| = \left|V(G) \cup \left(\cup_{v\in V(G)}(V(H^v) \setminus \{x\})\right)\right| = |V(G)| + |V(G)|(|V(H)| - 1) = |V(G)||V(H)| \geq \gamma_{sups}(G \circ H)$. Similarly, $if\ v \in S\ and\ S^* = S \setminus \{v\}, then\ S^*$ is not a super secure dominating set in $G \circ H$ and hence $\gamma_{sups}(G \circ H) = |V(G)||V(H)|$. If the nonempty set $C \neq V(G)$, then for any $C \subset V(G)$ and $x \in V(H^v), S = C \cup \left(\cup_{u\in V(G)\setminus C} V(H^u)\right) \cup (\cup_{v\in C}(V(H^v) \setminus \{x\}))$. Thus,

$$\begin{aligned} |S| &= \left|C \cup \left(\cup_{u\in V(G)\setminus C} V(H^u)\right) \cup (\cup_{v\in C}(V(H^v) \setminus \{x\}))\right| \\ &= |C| + |(\cup_{u\in V(G)\setminus C} V(H^u))| + |(\cup_{v\in C}(V(H^v) \setminus \{x\}))| \\ &= |C| + |V(G) \setminus C||V(H)| + |C|(|(V(H)| - 1) \\ &= |V(G)||V(H)| \geq \gamma_{sups}(G \circ H). \end{aligned}$$

Similarly, if $v \in S$ and $S^v = S \setminus \{v\}$, then S^* is not a super secure dominating set in $G \circ H$ and hence $\gamma_{sups}(G \circ H) = |V(G)||V(H)|$.

Finally, if $S = V(G) \cup \left(\cup_{y\in V(G)\setminus C} V(H^y)\right) \cup (\cup_{v\in C}(V(H^v) \setminus \{x\}))$ for any $C \subseteq V(G)$ and $x \in V(H^v)$. Then by Theorem 2.10(i), S is a super secure dominating set in $G \circ H$. Thus $|S| \geq \gamma_{sups}(G \circ H)$. It can be shown similarly that $\gamma_{sups}(G \circ H) = |V(G)||V(H)|$.

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