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FIXED POINT THEOREM IN FUZZY METRIC SPACE VIA THE PROPERTY (S-B)

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ABSTRACT

Here we have to prove a fixed point theorem in favor of weakly compatible mappings with the property S-B defined by Sharma and Bamoria [13] via implicit relation.

Keywords: *Fixed point, Fuzzy metric space, S-B property, Implicit relation.*

1. INTRODUCTION

Zadeh [9] is the stand of fuzzy mathematics. The verified result becomes an asset for an applied mathematician due to its enormous applications in various sections of mathematics which includes equations like differential, integral etc. and other area of science involving mathematics especially in logic programming and electronic engineering. It was developed extensively by many other authors and used in various fields. Especially, Deng [21] Ecreg [9], and Kramosil and Michalek [7] have initiated the conceptualization of fuzzy metric spaces in various ways. Further weakened the idea of compatibility by employing the concept of weakly compatible mapping in fuzzy metric spaces by B.Singh and S.Jain [3] and displayed that every pair of compatible mappings is weakly compatible but revert is not correct. Common fixed point theorems for semi compatible mappings in fuzzy metric spaces gratifying an implicit relation validated by Singh and Jain [3] in 2005. D.Gopal *et al.* [4] defined two individualistic classes of implicit functions and produced some fixed point outcomes for two pairs of weakly compatible mappings satisfying common (E.A.) property. In 2002 the concept of property E-A in metric spaces for self-mappings which carried the class of non compatible mappings in metric spaces interpreted by M.Aamri and EL-Moutawakil. Sharma Sushil and Bamoria [13] elucidated a property (S-B) in fuzzy metric spaces for self maps.

2. PRELIMINARIES

Definition 1: [2]. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it fulfills the subsequent states:

- $*$ is associative and commutative ,
- $*$ is continuous,
- $a*1=a$ for every $a\in[0,1]$,
- $a*b\le c*d$ if $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$

Definition 2: [12] Let X be any set. A fuzzy set by domain X is a function and values in $[0,1]$.

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Definition 3: [1] If X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0,\infty)$ in a triple $(X, M, *)$ is supposed to be a fuzzy metric space, satisfying the consecutive conditions: for every $x, y, z \in X$ and $s, t > 0$

- $M(x, y, t) > 0,$
- $M(x, y, t) = 1$ if and only if $x = y$,s
- $M(x, y, t) = M(y, x, t),$
- $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s),$
- $M(x, y, \cdot) : (0, +\infty) \rightarrow (0,1]$ is continuous.

Definition 4: [1] A sequence $\{x_n\}$ in X converges to x if and only if for each $\in > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \in$ for all $n \geq n_0$.

Example: [1] Let (X, d) be a metric space. We define $a * b = ab$ for all $a, b \in [0,1]$ and let M_d be a fuzzy set on $X^2 \times (0, +\infty)$ described as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, *)$ is a fuzzy metric space and the fuzzy metric M prompted by the metric d is often mentioned to as standard fuzzy metric. The fuzzy metric space $(X, M_d, *)$ is complete if and only if the metric space (X, d) is complete.

Definition 5: If $(X, M, *)$ be a Fuzzy metric space. Then

- A sequence $\{x_n\}$ in X is supposed to be Cauchy sequence if for all $t > 0$ and $p > 0$,
  $$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$$
- A sequence $\{x_n\}$ in X is supposed to be convergent to a point $x \in X$ if for all $t > 0,$
  $$\lim_{n\to\infty} M(x_n, x, t) = 1$$

Definition 6: [8] Every fuzzy metric space is supposed to be complete if a cauchy sequence in that fuzzy metric space is convergent and reverse is also true.

Definition 7: A and S be self maps on X as $(X, M, *)$ is a fuzzy metric space. A point x in X is called a coincidence point of A and S iff $Ax = Sx$ .Thus $w = Ax = Sx$ is called a point of coincidence of A and S.

Definition 8: [17] A pair (A, S) of self-mappings of a fuzzy metric space $(X, M, *)$ is supposed to be compatible if and only if $M(ASx_n, SAx_n, t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \rightarrow z$ for some $z \in X$ as $n \rightarrow \infty$.

Definition 9: [6] A couple (A, S) of self-mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Az = Sz$ some $z \in X$, then $ASz = SAz$.

Definition 10: Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition 11: [10] A couple (A,S) of self-mappings of a fuzzy metric space $(X,M,*)$ is said to be non-compatible if and only if there exist at least one sequence $\{x_n\}$ in X $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$, but for any $t > 0, \lim_{n\to\infty} M(ASx_n, SAx_n, t)$ is either less than 1 or non-existence.

Definition 2.12: (Implicit Relation) Let ϕ_4 be the set of real and continuous function from $(R^+)^4 \to R$ so that

2.12.1 ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument and

2.12.2 For $u,v \geq 0$ $\phi(u,v,v,v) \geq 0 \Rightarrow u \geq v$

Definition 2.13: Let S and T be two self mappings of a fuzzy metric space $(X,M,*)$.We read that S and T satisfy the property(S-B) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Example 2.13.1[G]: Let $X = [0,+\infty)$. Define $S,T : X \to X$ by $Tx = x/4$ and $Sx = 3x/4, \forall x \in X$.

examine the sequence $x_n = 1/n$, clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$. Then S and T satisfy theS-B property.

Lemma 1: Let $\{u_n\}$ be a sequence in a fuzzy metric space $(X,M,*)$. If $\exists$ a constant $k \in (0,1)$ such that $M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t)$ for all $t > 0$ and $n = 1,2,3.....$ then $\{u_n\}$ is a Cauchy sequence in X.

Lemma 2: [17] Let $(X,M,*)$ be a fuzzy metric space and $\forall$ $x,y \in X, t > 0$ and if for a number $k \in (0,1)$, $M(x,y,kt) \geq M(x,y,t)$ then $x = y$.

Lemma 3: [6] Let X be a set, f and g be occasionally weakly compatible self maps of X. If f and g have a onliest point of coincidence, $w = fx = gx$, then w is the onliest common fixed point of f and g.

3. MAIN RESULT

Theorem 3.1: Let $(X,M,*)$ be a fuzzy metric space and let A,B,S and T be mappings of X into itself satisfying following conditions:

3.1.1) $AX \subset TX$ and $BX \subset SX$

3.1.2) $\{A,S\}$ or $\{B,T\}$ satisfy the S-B property

3.1.3) there exists a constant $q \in (0,1)$ such that

$$\phi\left(M(Ax,By,qt), \frac{M(Sx,Ty,t) + M(Ax,Sx,t)}{2}, \frac{M(By,Ty,t) + M(Ax,Ty,t)}{2} \right) \geq 0 \tag{1}$$

For all $x,y \in X$

3.1.4) If the pairs $\{A,S\}$ or $\{B,T\}$ are weakly compatible

3.1.5) One of $AX, BX, SX,$ or TX is closed subset of X, then A,B,S and T have a unique common fixed point in X.

Proof: Suppose that $\{B,T\}$ satisfies the property S-B. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Since $BX \subset SX$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sx_n = z$. Let us show that $\lim_{n\to\infty} Ay_n = z$.

Now by inequality (1), we have

$$\phi\left(M\left(Ay_n, Bx_n, qt\right), \frac{M\left(Sy_n, Tx_n, t\right) + M\left(Ay_n, Sy_n, t\right)}{2}, \frac{M\left(Bx_n, Tx_n, t\right) + M\left(Ay_n, Tx_n, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(Ay_n, Bx_n, qt\right), \frac{M\left(Bx_n, Tx_n, t\right) + M\left(Ay_n, Bx_n, t\right)}{2}, \frac{M\left(Bx_n, Tx_n, t\right) + M\left(Ay_n, Tx_n, t\right)}{2} \right) \geq 0$$

Taking $\lim n \to \infty$

$$\phi\left(M\left(Ay_n, Bx_n, qt\right), \frac{1 + M\left(Ay_n, Bx_n, t\right)}{2}, \frac{1 + M\left(Ay_n, Bx_n, t\right)}{2} \right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\phi\left(M\left(Ay_n, Bx_n, qt\right), M\left(Ay_n, Bx_n, t\right), M\left(Ay_n, Bx_n, t\right)\right) \geq 0$$

By 2.12

$$M\left(Ay_n, Bx_n, qt\right) \geq M\left(Ay_n, Bx_n, t\right)$$

Since M is continuous function

$$\lim_{n\to\infty} M\left(Ay_n, Bx_n, qt\right) \geq \lim_{n\to\infty} M\left(Ay_n, Bx_n, t\right)$$

By lemma 2

$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n$ and we deduce that

$$\lim_{n\to\infty} Ay_n = z$$

Suppose SX is a closed subset of X. Then $z = Su$ for some $u \in X$. Subsequently we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Su.$$

By (3.1.3), we have

$$\phi\left(M\left(Au, Bx_n, qt\right), \frac{M\left(Su, Tx_n, t\right) + M\left(Au, Su, t\right)}{2}, \frac{M\left(Bx_n, Tx_n, t\right) + M\left(Au, Tx_n, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(Au, Bx_n, qt\right), \frac{M\left(Su, Tx_n, t\right) + M\left(Au, Su, t\right)}{2}, \frac{M\left(Bx_n, Tx_n, t\right) + M\left(Au, Tx_n, t\right)}{2} \right) \geq 0$$

Taking $\lim n \to \infty$, we have

$$\phi\left(M\left(Au, Su, qt\right), \frac{M\left(Su, Su, t\right) + M\left(Au, Su, t\right)}{2}, \frac{M\left(Su, Su, t\right) + M\left(Au, Su, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(Au, Su, qt\right), \frac{1 + M\left(Au, Su, t\right)}{2}, \frac{1 + M\left(Au, Su, t\right)}{2} \right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\phi\left(M\left(Au, Su, qt\right), M\left(Au, Su, qt\right), M\left(Au, Su, qt\right)\right) \geq 0$$

By 2.12

$$M\left(Au, Su, qt\right) \geq M\left(Au, Su, t\right)$$

Thus by lemma 2

We have $Au = Su$ .The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$. On the other hand ; since $AX \subseteq TX$, there exists a point $v \in X$ such that $Au = Tv$. We claim that $Au = Bv$ using 3.1.3; we have

$$\phi\left(M\left(Au, Bv, qt\right), \frac{M\left(Su, Tv, t\right) + M\left(Au, Su, t\right)}{2}, \frac{M\left(Bv, Tv, t\right) + M\left(Au, Tv, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(Au, Bv, qt\right), \frac{M\left(Su, Au, t\right) + M\left(Au, Su, t\right)}{2}, \frac{M\left(Bv, Au, t\right) + M\left(Au, Au, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(Au, Bv, qt\right), 1, \frac{1 + M\left(Au, Bv, t\right)}{2} \right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\phi\left(M\left(Au, Bv, qt\right), M\left(Au, Bv, t\right), M\left(Au, Bv, t\right) \right) \geq 0$$

By 2.12

$$M\left(Au, Bv, qt\right) \geq M\left(Au, Bv, t\right)$$

Therefore by lemma, we have

$$Au = Bv$$

Thus $Au = Su = Tv = Bv.$ The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv.$ Let us show that Au is a common fixed point of A, B, S and T . In view of (3.1.3) we have

$$\phi\left(M\left(AAu, Bv, qt\right), \frac{M\left(SAu, Tv, t\right) + M\left(AAu, SAu, t\right)}{2}, \frac{M\left(Bv, Tv, t\right) + M\left(AAu, Tv, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(AAu, Au, qt\right), \frac{M\left(AAu, Au, t\right) + M\left(AAu, AAu, t\right)}{2}, \frac{M\left(Au, Au, t\right) + M\left(AAu, Au, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(AAu, Au, qt\right), \frac{1 + M\left(AAu, Au, t\right)}{2}, \frac{1 + M\left(AAu, Au, t\right)}{2} \right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\phi\left(M\left(AAu, Au, qt\right), M\left(AAu, Au, t\right), M\left(AAu, Au, t\right) \right) \geq 0$$

By 2.12

$$M\left(AAu, Au, qt\right) \geq M\left(AAu, Au, t\right)$$

Therefore by lemma, we have $Au = AAu = SAu$ and Au is a common fixed point of A and S . Similarly, we can validate that Bv is a common fixed point of B and T . Since $Au = Bv$, we achieve that Au is point of A, B, S and T which is called common fixed point..

If $Au = Bu = Su = Tu = u$ and $Av = Bv = Sv = Tv = v$. Then by 3.1.3, we have

$$\phi\left(M\left(Au, Bv, qt\right), \frac{M\left(Su, Tv, t\right) + M\left(Au, Su, t\right)}{2}, \frac{M\left(Bv, Tv, t\right) + M\left(Au, Tv, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(u, v, qt\right), \frac{M\left(u, v, t\right) + M\left(u, u, t\right)}{2}, \frac{M\left(v, v, t\right) + M\left(u, v, t\right)}{2} \right) \geq 0$$

$$\phi\left(M\left(u, v, qt\right), \frac{1 + M\left(u, v, t\right)}{2}, \frac{1 + M\left(u, v, t\right)}{2} \right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\phi\left(M\left(u,v,qt\right), M\left(u,v,t\right), M\left(u,v,t\right)\right) \geq 0$$

By 2.12

$$M\left(u,v,qt\right) \geq M\left(u,v,t\right)$$

Therefore by lemma, we have $u = v$ and the common fixed point is a unique. Hence this explanation is verified the theorem.

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