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RESULTS ON β_M -NUMBER FOR THE GENERALIZED PETERSEN GRAPHS $P(n,k)$

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ABSTRACT

A set S of vertices of a graph G is said to be a Majority Independent set(or MI-set) if it induces a totally disconnected subgraph with $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ *and* $|pn[v,S]| > |N[S]| - \left\lceil \frac{p}{2} \right\rceil$ *for every* $v \in S$. *In this note, we investigate the Majority Independence Number* $\beta_M(G)$ *for Generalised Petersen graphs and also discussed whether it is* β_M *-excellent or not.*

Keywords: *Majority independence number-* $\beta_M(G)$, β_M *excellent graphs.*

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1. INTRODUCTION

We consider connected, undirected, finite graphs without loops. We follow the notations and terminology of Harary[2] and Haynes *et al.* [3]. Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. For every vertex $v \in V(G)$, the open neighbourhood $N(v) = \{u \in V(G) / uv \in E(G)\}$ and the closed neighbourhood $N[v] = N(v) \bigcup \{v\}$. Let S be a set of vertices, and let $u \in S$. The private neighbor set of u with respect to S is $pn[u,S] = \{v / N[v] \bigcap S = \{u\}\}$

In 2006, A subset $S \subseteq V(G)$ of vertices in a graph G is called majority dominating set if at least half of the vertices of $V(G)$ are either in S or adjacent to the vertices of S.
i.e., $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$. A majority dominating set S is minimal if no proper subset of S is a majority dominating set of G. The majority domination number $\gamma_M(G)$ of a graph G is the minimum cardinality of a minimal majority dominating set in G. The upper majority domination number $\Gamma_M(G)$ is the maximum cardinality of a minimal majority dominating set of a graph G. This parameter has been studied by Swaminathan. V and Joseline Manora. J[8].

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In 2014, A set S of vertices of a graph G is said to be a Majority Independent set(or MI-set) if it induces a totally disconnected subgraph with $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|pn[v,S]| > |N[S]| - \left\lceil \frac{p}{2} \right\rceil$ for every $v \in S$. If any vertex set S' properly containing S is not majority independent. Then S is called Maximal Majority Independent set. The minimum cardinality of a maximal majority independent set is called lower majority independence number of G and it is also called Independent Majority Domination number of G. It is denoted by $i_M(G)$. The maximum cardinality of a maximal majority independent set of G is called Majority Independence number of G and it is denoted by $\beta_M(G)$. A β_M -set is a maximum cardinality of a maximal majority independent set of G. This parameter has been highly developed by Joseline Manora. J and John. B[5].

Claude Berge in 1980, introduced B graphs. These are graphs in which every vertex in the graph is contained in a maximum independent set of the graph. Fircke *et al.* [1] in 2002 made a beginning of the study of graphs which are excellent with respect to various parameters. γ -excellent trees and total domination excellent trees have been studied in [1]. Also in 2006, N.Sridharan and Yamuna [7] made an extensive work in this area. In 2011, Swaminathan. V and Pushpalatha. A.P have defined β_o -excellent graphs, just β_o -excellent graphs and very β_o -excellent graphs and they have made a detailed study in this paper [7].

Definition: For each $n \geq 3$ and $0 < k < n$, $P(n,k)$ denotes the Generalized Petersen graph with vertex set $V(G) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and the edge set $E(G) = \{u_i u_{i+1\,(\mathrm{mod}\, n)}, u_i v_i, v_i v_{i+k\,(\mathrm{mod}\, n)}\}$, $1 \leq i \leq n$.

Definition: Let $G = (V,E)$ be a simple graph. Let $u \in V(G)$. The vertex u is said to be β_M -good if u is contained in a β_M -set of G. The vertex u is said to be β_M -bad if there exists no β_M -set of G containing u. A graph G is said to be β_M -excellent if every vertex of G is β_M -good. This parameter has been studied by Joseline Manora. J and John. B [4].

2. Exact β_M -number for $G = P(n,k)$

Theorem 2.1: Let G be a Generalization of Petersen graph $P(n,k)$ with $k=1$, $n \geq 3$. Then

$$\beta_M(G) = \begin{cases} \left\lceil \dfrac{p-4}{4} \right\rceil & if\ n \leq 6 \\ \left(\dfrac{p}{7} \right) & if\ n = 7 \\ \left\lfloor \dfrac{p-3}{4} \right\rfloor & if\ n \geq 8 \end{cases}$$

Proof: Let G be a Generalized Petersen graph $P(n,1)$ with $|V(G)| = 2n = p$. The graph G consists of two cycles C_1 and C_2 such that the cycle C_1 with vertex set $\{v_1, v_2, ..., v_n\}$ is nested by the another cycle C_2 with vertex set $\{u_1, u_2, ..., u_n\}$ and each u_i in C_2 is joined to exactly one v_i in C_1 and $d(v_i) = d(u_i) = 3$, $i = 1, 2, ..., n$.

Case-(i): When $n \leq 6$. The maximum majority independent sets are $\{v_i, u_{i+1\,(\mathrm{mod}\, n)}\}$, $i = 1, 2, ..., 6$. Then $\beta_M(G) = 2 = \left\lceil \dfrac{p-4}{4} \right\rceil$, if $n \leq 6$.

Case-(ii): When $n=7$. The maximum majority independent sets are $\left\{v_i, u_{i+2\,(\mathrm{mod}\,n)},\ i=1,2,...,7\right\}$.

Therefore $\beta_M(G)=2=\left(\dfrac{p}{7}\right)$.

Case-(iii): When $n\geq 8$. Let $D=\{u_1,u_2,..,u_t\}$, $t=\left\lfloor \dfrac{p-3}{4} \right\rfloor$ and $d(u_i,u_j)\geq 2, i\neq j$.

Then $|N[D]|=\sum_{i=1}^{t}(d(u_i)+1)=4t\ =4\left\lfloor \dfrac{p-3}{4}\right\rfloor \geq \left\lceil \dfrac{p}{2}\right\rceil$. Also, for every $v\in D$,

$|pn\,[v,D]|\ >|N[D]|-\left\lceil \dfrac{p}{2}\right\rceil$. Hence D is a β_M -set of G.

Therefore $\beta_M(G)\geq |D|=\left\lfloor \dfrac{p-3}{4}\right\rfloor$. Suppose $S=\{v_1,v_2,..,v_r\}$, $r=\left\lfloor \dfrac{p-3}{4}\right\rfloor+1$ with $d(v_i,v_j)\geq 2,\ i\neq j$. But $|pn\,[v,S]|\ \leq |N[S]|-\left\lceil \dfrac{p}{2}\right\rceil$, for any $v\in S$. Therefore S is not a β_M -set of G. Hence $\beta_M(G)<|S|=\left\lfloor \dfrac{p-3}{4}\right\rfloor+1.\ \Rightarrow\ \beta_M(G)\leq \left\lfloor \dfrac{p-3}{4}\right\rfloor$

Therefore $\beta_M(G)=\left\lfloor \dfrac{p-3}{4}\right\rfloor$. The maximal majority independent sets of G are

$$\left\{v_i, u_{i+1(\mathrm{mod}\,n)}, v_{i+2(\mathrm{mod}\,n)}, u_{i+3\,(\mathrm{mod}\,n)},...\right\},$$
$$\left\{u_i, v_{i+1(\mathrm{mod}\,n)}, u_{i+2(\mathrm{mod}\,n)}, v_{i+3\,(\mathrm{mod}\,n)},...\right\},\ i=1,2,...,n.$$

Proposition 2.2[4]: Let G be a Generalization of Petersen graph $P(n,k)$ with $k=1, n\geq 3$. Then $G=P(n,1)$ is β_M -excellent.

Proof: In all the cases of the above theorem [2.1], all vertices of $V(G)$ are contained in any one of the β_M -sets of G. Therefore all vertices are β_M -good vertices. Hence $G=P(n,1)$ is β_M -excellent.

Theorem 2.3: Let G be a Generalization of Petersen graph $P(n,k)$ with $k=2,\ n\geq 3$. Then

$$\beta_M(G)=\begin{cases}\left\lceil \dfrac{p}{8}\right\rceil & if\ n\leq 11\\ \left\lfloor \dfrac{p}{6}\right\rfloor & if\ n\geq 12\end{cases}$$

Proposition 2.4: Let G be a Generalization of Petersen graph $P(n,k)$ with $k=2,\ n\geq 3$. Then $G=P(n,2)$ is β_M -excellent.

Theorem 2.5: Let G be a Generalization of Petersen graph $P(n,k)$ with $k=3$, $n\geq 3$. Then

$$\beta_M(G)=\begin{cases}\left\lceil \dfrac{p-2}{6}\right\rceil & if\ n\leq 10\\ \left\lceil \dfrac{p-4}{4}\right\rceil -1 & if\ n\geq 11\end{cases}$$

Proof: Let G be a Generalized Petersen graph $P(n,3)$ with $|V(G)|=2n=p$ vertices. Then G consists of two cycles C_1 and C_2 such that the cycle C_1 with vertex set $\{v_1,v_2,...,v_n\}$ is nested by the another cycle C_2 with vertex set $\{u_1,u_2,...,u_n\}$ and each u_i in C_2 is joined to exactly one v_i in C_1, $i=1,2,...,n$ and $d(v_i)=d(u_i)=3$.

Case-(i): When $n\leq 10$. Let $n=5,6,7$. Then $p=2n=10,11,12,13,14$.

Let $D=\{u_1,v_2\}$. $|N[D]|=7\geq \left\lceil \dfrac{p}{2}\right\rceil$. Therefore $|N[D]|-\left\lceil \dfrac{p}{2}\right\rceil=2\ or\ 1\ or\ 0$.

$|pn[u_i,D]|=3>|N[D]|-\left\lceil \dfrac{p}{2}\right\rceil$, for $\forall\, u_i\in D$, $i=1,2$. Therefore D is a maximal majority independent set of $G \Rightarrow \beta_M(G)=2=\left\lceil \dfrac{p-2}{6}\right\rceil$.

Let $n=8,9,10$. Then $p=2n=16,18,20$. Let $D=\{u_1,v_2,u_3\}$. $|N[D]|=10\geq \left\lceil \dfrac{p}{2}\right\rceil$.

Then $|pn[u_i,D]|=4\,or\,3>|N[D]|-\left\lceil \dfrac{p}{2}\right\rceil$, for $\forall\, u_i\in D$, $i=1,3$ and

$|pn[v_2,D]|=3>|N[D]|-\left\lceil \dfrac{p}{2}\right\rceil$, $v_2\in D$. Therefore $\beta_M(G)=2=\left\lceil \dfrac{p-2}{6}\right\rceil$.

Case-(ii): When $n\geq 11$. Let $D=\{u_1,u_2,..,u_t\}$, $t=\left\lceil \dfrac{p-4}{4}\right\rceil -1$

and $d(u_i,u_j)\geq 2$, $i\neq j$. Then $|N[D]|=\left(\sum_{i=1}^{t} d(u_i)\right)+1=3t+1\geq \left\lceil \dfrac{p}{2}\right\rceil$.

Also, $|N[D]|-\left\lceil \dfrac{p}{2}\right\rceil=\begin{cases}0 & if\ \ n\ is\ even\\ 1 & if\ \ n\ is\ odd\end{cases}$ and $|pn[u_i,D]|=4\ or\ 3\ or\ 2$.

Therefore $|pn[u_i,D]|>|N[D]|-\left\lceil \dfrac{p}{2}\right\rceil$, for $\forall\, u_i\in D$. Therefore D is a maximal majority independent set of G. Hence $\beta_M(G)\geq |D|=\left\lceil \dfrac{p-4}{4}\right\rceil -1$.

Suppose $S=\{v_1, v_2, .., v_r\}$, $r=\left\lceil \frac{p-4}{4} \right\rceil -1+1$ with $d(v_i, v_j) \geq 2, i \neq j$.

$|N[S]| = \left(\sum_{i=1}^{r} d(v_i) \right) + 1 = 3r+1 \geq \left\lceil \frac{p}{2} \right\rceil$. But $|pn[v_i, S]| \leq |N[S]| - \left\lceil \frac{p}{2} \right\rceil$, for any $v_i \in S$.

S is not a β_M -set of G . Therefore $\beta_M(G) \leq |S| = \left\lceil \frac{p-4}{4} \right\rceil \Rightarrow \beta_M(G) \leq \left\lceil \frac{p-4}{4} \right\rceil -1.$

Hence $\beta_M(G) = \left\lceil \frac{p-4}{4} \right\rceil -1.$

The maximal majority independent sets of $G=P(n,3)$ are

$\left\{u_i, v_{i+1(\bmod n)}, u_{i+2(\bmod n)}, v_{i+3(\bmod n)};\ i=1,2,3,...,n\right\}$, if $n=11,12$,...

$\left\{u_i, v_{i+1(\bmod n)}, u_{i+2(\bmod n)}, v_{i+3(\bmod n)}, u_{i+4(\bmod n)};\ i=1,2,3,...,n\right\}$,
if $n=13,14$,...

In general, the maximal majority independent sets of G are

$\left\{u_i, v_{i+1(\bmod n)}, u_{i+2(\bmod n)}, v_{i+3(\bmod n)};\ i=1,2,3,...,n\right\}$,
when $n=3k-1, 3k, 3k+3, 3k+4, 3k+7, 3k+8, ,...,$ if $k=4$.
When $n=3k+1, 3k+2, 3k+5, 3k+6, 3k+9, 3k+10, ...,$ if $k=4$,
then the maximal majority independent sets of G are

$\left\{u_i, v_{i+1(\bmod n)}, u_{i+2(\bmod n)}, v_{i+3(\bmod n)}, u_{i+4(\bmod n)};\ i=1,2,3,...,n\right\}$.

Proposition 2.6: Let G be a Generalization of Petersen graph $P(n,k)$ with $k=3, n \geq 3$. Then $G=P(n,3)$ is β_M -excellent.

Proof: All vertices of $V(G)$ are contained in any one of the β_M -sets of G by theorem (2.5). Therefore all vertices of $G=P(n,3)$ are β_M -good vertices. Hence $G=P(n,3)$ is β_M -excellent.

CONCLUSION

In this paper we surveyed the β_M -number for the Generalised Petersen graphs $G=P(n,k)$ where $k=1,2,3,\ n \geq 3$ and also discussed β_M -excellent. Further we extend this idea to find β_M -excellent and β_M -number for $G=P(n,k)$ where $k>3,\ n \geq 3$ and also extend this idea for the some more interesting different types of graphs.

REFERENCES

1. Fricke. G.H, Teresa W. Haynes, Hadetniemi .S. T, Hedetniemi. S. M and Laskar. R. C, Excellent trees, Bull. Inst. Combin. Appl. Vol 34(2002), pp 27-38.

2. Harary .F, Graph Theory, Adison-Wesley, Reading, Mass., 1972.
3. Haynes. T. W, Hedetniemi S. T and P. J. Slater, Fundamentals of domination in Graphs, Marcel Dekkar, New York, 1998.
4. Joseline Manora. J, John. B, $\beta_M(G)$ -Excellent Graphs-Proceedings of the International Conference on Jamal Research Journal: (Special Issue) ISSN 0973-0303(2015), pp 513-517.
5. Joseline Manora. J, John. B, Majority Independence Number of a Graph, International Journal of Mathematical Research, Vol-6, No.1 (2014), 65-74.
6. Pushpalatha. A.P, Jothilakshmi. G, Suganthi. S, Swaminathan. V, β_o -excellent graphs, WSEAS Transaction on Mathematics, Issue 2, Vol. 10, February 2011.
7. Sridharan. N and Yamuna. M, A Note on Excellent graphs, ARS COMBINATORIA 78(2006), pp. 267-276.
8. Swaminathan. V, Joseline Manora. J, Majority Dominating Sets of a graph, Jamal Academic Research Journal, Vol-3, No.2 (2006), 75-82.

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