International Journal of Mathematical Archive-8(12), 2017, 48-56
Available online through www.ijma.info ISSN 2229 – 5046

A FIXED POINT THEOREM ON PRODUCT OF METRIC SPACES

U. C. GAIROLA, DEEPAK KHANTWAL*

Department of Mathematics,
H. N. B. Garhwal University, BGR Campus, Pauri Garhwal, Uttarakhand – 246001, India.

(Received On: 23-10-17; Revised & Accepted On: 18-11-17)

ABASTRACT

J. Matkowski [19], gave an important generalization of Banach contraction principle for a finite product of metric spaces. This result has been extended and generalized by several mathematicians. Recently Pant [22] gave an important concept of reciprocal continuity for a pair of maps. In this paper, we introduced the coordinatewise reciprocal continuity and proved a fixed point theorem which extend and unify the result of Jungck [13], Matkowski [op. cit.] and some of their generalization, for non continuous systems of maps.

Subject Classifications: *47H10, 54H25.*

Key Words: *Fixed point, coincidence point, Matkowski contraction, product space, reciprocal continuous maps, coordinatewise commuting, weakly commuting, compatible and R-weakly commuting maps.*

1. INTRODUCTION

In the galaxy of contraction principles, two important generalizations of well-known Banach contraction principle were obtained by Gerald Jungck [13] and Janusz Matkowski [18]-[19]. Jungck's result being simple and elegant in nature has led to a massive growth of fixed point theorems for contractive type maps (see, [2], [3], [6]-[7], [12], [15], [24], [27]-[30], [34], [36]-[38]).Matkowski's fixed point theorem (Matkowski contraction principle) being somewhat tedious in nature could draw the attention of only a few researchers in applicable mathematics (see, [4]-[5], [8]-[11], [16], [20], [25]-[26], [31]-[33]). Singh- Gairola [31], [32] extend and unify the result of Jungck [op. cit.] and Matkowski [op. cit.] and some of their generalizations by introducing a new class of maps- coordinatewise commuting and their weaker forms (see, also [8] and [11]).

With a view to generalizing fixed point theorems for non continuous maps Pant [22], introduce reciprocal continuity. If S and T are maps on a metric space (M,d) then the pair (S,T) is said to be reciprocal continuous if and only if $\lim_{n\to\infty} STx_n = St$ and $\lim_{n\to\infty} TSx_n = Tt$ whenever $\{x_n\}$ be a sequence in M such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in M. Motivated by the work of Singh-Gairola [31] and Pant [22], we extend and generalize the Matkowski contraction for non continuous maps. We do this by introducing a new class of maps- coordinatewise reciprocal continuous maps.

Throughout this paper we shall follow the following notations and definitions. Let (a_{ik}) be a $n\times n$ square matrix with non-negative entries defined in Czerwik [4] and Matkowski [op.cit].

$$c_{ik}^{(0)} = \begin{cases} a_{ik} & i \neq k \\ 1-a_{ik} & i = k \end{cases} \qquad i,k = 1,\dots,n \tag{1.1}$$

$$c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)} c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & i \neq k \\ c_{11}^{(t)} c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & i = k \end{cases} \tag{1.2}$$

$t = 1,\dots,n-1,\ i,k = 1,\dots,n-t.$

Corresponding Author: Deepak Khantwal*, Department of Mathematics,
H. N. B. Garhwal University, BGR Campus, Pauri Garhwal, Uttarakhand – 246001, India.

Let (X_i, d_i), $i = 1,...,n,$ be metric spaces,

$$X = X_1 \times X_2 \times ...... \times X_n;$$
$$x = (x_1,...,x_n);$$
$$P_i, T_i : X \to X_i;\ i = 1,...,n \text{ and}$$
$$\{x^m\} = \{(x_1^m,...,x_n^m)\}, m \in \mathbb{N} \text{ (set of natural numbers) be a sequence in X.}$$

Definition 1.1 [31]: Two systems of maps $\{P_1,...,P_n\}$ and $\{T_1,...,T_n\}$ are coordinatewise commuting at a point $x \in X$ if and only if $P_i(T_1x,...,T_nx) = T_i(P_1x,...,P_nx)$ for all $i = 1,..,n.$ Two systems of maps $\{P_1,...,P_n\}$ and $\{T_1,...,T_n\}$ are coordinatewise commuting on X if and only if they are coordinatewise commuting at every point of X .

Definition 1.2 [31]: Two systems of maps $\{P_1,...,P_n\}$ and $\{T_1,...,T_n\}$ are coordinatewise weakly commuting at a point $x \in X$ if and only if $d_i(P_i(T_1x,...,T_nx), T_i(P_1x,...,P_nx)) \le d_i(P_ix, T_ix)$ for all $i = 1,...,n.$ Two systems of maps are coordinatewise weakly commuting on X if and only if they are coordinatewise weakly commuting at every point of X.

Remark 1.1: Evidently coordinatewise commuting systems of maps are coordinatewise weakly commuting. However, the weakly commuting systems of maps need not to be commuting (see, [31], [32]).

Definition 1.3 [11]: Two systems of maps $\{P_1,...,P_n\}$ and $\{T_1,...,T_n\}$ are coordinatewise asymptotically commuting or, following the terminology of Jungck [15], coordinatewise compatible, if and only if

$$\lim_{m\to\infty} d_i\left(P_i\left(T_1x^m,...,T_nx^m\right), T_i\left(P_1x^m,...,P_nx^m\right)\right) = 0$$

whenever $\lim_{m\to\infty} P_ix^m = \lim_{m\to\infty} T_ix^m = u_i$ for some $u_i \in X_i$, $i = 1,...,n.$

Definition 1.4 [8]: Two systems of maps $\{P_1,...,P_n\}$ and $\{T_1,...,T_n\}$ are coordinatewise R-weakly commuting at a point $x \in X$ if and only if $d_i(P_i(T_1x,...,T_nx), T_i(P_1x,...,P_nx)) \le Rd_i(P_ix, T_ix)$, for all $i = 1,...,n$ and for any positive real number R . Two systems of maps are coordinatewise R-weakly commuting on X if and only if they are coordinatewise R-weakly commuting at every point of X .

Remark 1.2: Coordinatewise weakly commuting maps are coordinatewise R-weakly commuting. However, coordinatewise R-weakly commuting maps need not to be coordinatewise weakly commuting.
The following example shows the coordinatewise R-weak commutativity of two systems of maps and illustrates that the coordinatewise R-weak commutativity need not imply coordinatewise weak commutativity.

Example 1.1: Let $X_1 = X_2 = [0,1]$ be usual metric spaces and $P_i, T_i : X_1 \times X_2 \to X_i$, $i = 1, 2$ such that

$$P_1x = x_1^2, \qquad P_2x = 0,$$
$$T_1x = 2x_1 - 1, \quad T_2x = 0.$$

Since

$$d_1(T_1(P_1x, P_2x), P_1(T_1x, T_2x)) = d_1\left(2x_1^2 - 1, (2x_1 - 1)^2\right) = 2(x_1 - 1)^2 = 2d_1\left(2x_1 - 1, x_1^2\right) = 2d_1(T_1x, P_1x)$$

and

$$d_2(T_2(P_1x, P_2x), P_2(T_1x, T_2x)) = d_2(0, 0) = 0 \le d_2(T_2x, P_2x).$$

Then two systems of maps $\{P_1, P_2\}$ and $\{T_1, T_2\}$ are coordinatewise R-weakly commuting. However, they are not coordinatewise weakly commuting for

$$d_1(T_1(P_1x, P_2x), P_1(T_1x, T_2x)) = d_1\left(2x_1^2 - 1, (2x_1 - 1)^2\right) = 2(x_1 - 1)^2 > d_1(T_1x, P_1x), \forall\, x \in X .$$

Definition 1.5: Two systems of maps $\{P_1,...,P_n\}$ and $\{T_1,...,T_n\}$ are said to be coordinatewise reciprocal continuous if and only if $\lim_{m\to\infty} P_i(T_1x^m,...,T_nx^m) = P_iz$ and $\lim_{m\to\infty} T_i(P_1x^m,...,P_nx^m) = T_iz$, whenever there exist a sequence $\{x^m\}$ in X such that $\lim_{m\to\infty} P_ix^m = \lim_{m\to\infty} T_ix^m = z_i$ for all $i = 1,...,n.$
If each member of systems of maps is continuous then systems of maps are coordinatewise reciprocal continuous but the converse need not be true.

Remark 1.3: Notice that definitions above with $n = 1$ are standard ones for commuting, weakly commuting (see [15] and [28]), asymptotically commuting (see, [37]) (also called compatible [14]), R-weakly commuting (see [21]) and reciprocal continuous maps ([22] and see also [23]).

Remark 1.4: Asymptotically commuting (or compatible) class of maps includes commuting and weakly commuting maps. Commuting maps are necessarily weakly and asymptotically commuting both (see, for instance, [14], [28], [31], [37]).

Remark 1.5: The commutativity, weak commutativity and asymptotic commutativity (or compatibility) are equivalent at the point of coincidence of two (or two systems of) maps (see, [1], [14]).

The following example illustrates the coordinatewise reciprocal continuity of systems of maps and shows that coordinatewise reciprocal continuity of systems of maps does not imply continuity of any member of systems of maps.

Example 1.2: Let $X_1 = [0,1]$, $X_2 = [0,1]$ be metric spaces with usual metrics and mappings $P_i, T_i : X_1 \times X_2 \to X_i$, for $i = 1, 2$ such that

$$P_1(x_1, x_2) = \begin{cases} 0 & if \quad x_1 = 0 \\ \dfrac{1}{2} & if \quad x_1 > 0 \end{cases} \qquad P_2(x_1, x_2) = \begin{cases} 0 & if \quad x_2 = 0 \\ \dfrac{1}{2} & if \quad x_2 > 0 \end{cases}$$

$$T_1(x_1, x_2) = \begin{cases} 0 & if \quad x_1 = 0 \\ 1 & if \quad x_1 > 0 \end{cases} \qquad T_2(x_1, x_2) = \begin{cases} 0 & if \quad x_2 = 0 \\ 1 & if \quad x_2 > 0 \end{cases}.$$

Suppose $\{x^m\}$ be a sequence in $X_1 \times X_2$ such that $P_i x^m \to z_i$ and $T_i x^m \to z_i$ for some z_i, $i = 1, 2,$ as $m \to \infty$ .Then for $z = (0,0)$ and $\{x^m\} = \{(0,0)\} \in X$ for each m, $P_i(T_1 x^m, T_2 x^m) \to 0 = P_i z$ and $T_i(P_1 x^m, P_2 x^m) \to 0 = T_i z$ as $m \to \infty$. Hence systems of maps $\{P_1, P_2\}$ and $\{T_1, T_2\}$ are coordinatewise reciprocal continuous at $z = (0,0)$ but they are not continuous at $z = (0,0)$. To see this, let $\{x^m\} = \left\{\left(\frac{1}{m}, \frac{1}{m}\right)\right\}$ be a sequence in $X_1 \times X_2$ .Since $\{x^m\} \to (0,0)$ as $m \to \infty$ but then $P_i x^m \to \frac{1}{2} \neq P_i(0,0)$ and $T_i x^m \to 1 \neq T_i(0,0)$ for $i = 1, 2,$ as $m \to \infty.$

The following Lemma is due to Matkowski [19] (see also [4], [33]).

Lemma 1.1: Let $c_{i,k}^{(0)} \geq 0,\ i, k = 1, ..., n,\ n \geq 2,$ then the system of inequalities

$$\sum_{k=1}^{n} a_{ik} r_k < r_i\ , \quad i = 1, ..., n \tag{1.3}$$

has a positive solution $r_1, ..., r_n$ if and only if the following inequalities hold:

$$c_{ii}^{(t)} > 0, \quad i = 1, ..., n - t\ ;\ t = 0, 1, ..., n-1,\ n \geq 2. \tag{1.4}$$

Moreover, there exists a positive number $h < 1$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \leq h r_i\ ,\ i = 1, ..., n, \tag{1.5}$$

for some positive number $r_1, .., r_n$. Indeed such an h may be found by

$$h = \max_i \left\{ r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \right\}. \tag{1.6}$$

Now we will state our main results.

2. RESULTS

Theorem 2.1: Let $(X_i, d_i), i = 1, ..., n,$ be complete metric spaces and $P_i, Q_i, S_i, T_i : X \to X_i$, for all $i = 1, ..., n,$ be such that

$$P_i(X) \subset T_i(X),\ \ Q_i(X) \subset S_i(X), \quad i = 1, . \quad n. \tag{2.1}$$

The system $(P_1, ..., P_n)$ coordinatewise R-weakly commutes with the system $(S_1, ..., S_n)$ and the system $(Q_1, ..., Q_n)$ coordinatewise R-weakly commutes with the system $(T_1, ..., T_n)$. (2.2)

The system of maps $(P_1, ..., P_n)$ coordinatewise reciprocal continuous with the system $(S_1, ..., S_n)$ or the system of maps $(Q_1, ..., Q_n)$ coordinatewise reciprocal continuous with the system $(T_1, ..., T_n)$. (2.3)

If there exist non- negative numbers b and a_{ik} , $i,k=1,....,n$ such that (1.1), (1.2), (1.4) and the following hold:
$0 \le b < 1-h,$ where h is defined in (1.6) and (2.4)

$$d_i\left(P_i x, Q_i y\right) \le \max_i\left\{\sum_{k=1}^{n} a_{ik} d_k\left(S_k x, T_k y\right), b\max\left\{\begin{matrix} d_i(S_i x, P_i x),\ d_i(T_i y, Q_i y), \\ \dfrac{d_i(S_i x, Q_i y)+d_i(T_i y, P_i x)}{2}\end{matrix}\right\}\right\} \quad (2.5)$$

for all $x, y \in X$, then the system of equations

$$P_i x = Q_i x = x_i = S_i\ x = T_i x \quad (2.6)$$

has a unique common solution $(x_1,...,x_n)$ such that $x_i \in X_i$, $i=1,...,n$.

Proof: First we note that in view of the system of inequality (1.5), h defined in (1.6) exists and $0<h<1$(c.f. Czerwik [4]). From the Lemma 1.1 and (1.6) we may choose positive numbers $r_1,..,r_n$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \le h r_i\ ,\ i=1,...,n.$$

Pick $x_i^0 \in X_i$, $i=1,...,n.$ We in view of (2.1), construct sequences $\{x_i^m\}$ and $\{y_i^m\}$ in X_i such that

$$y_i^{2m+1} = P_i x^{2m} = T_i x^{2m+1}$$
$$y_i^{2m+2} = Q_i x^{2m+1} = S_i x^{2m+2},\ m=0,1,2,... .$$

We may assume that $d_i(y_i^2, y_i^1) \le r_i$, $i=1,...,n.$ From condition (2.5), we have

$$\begin{aligned} d_i(y_i^3, y_i^2) &= d_i(P_i x^2, Q_i x^1) \\ &\le \max\left\{\sum_{k=1}^{n} a_{ik} d_k(S_k x^2, T_k x^1), b\max\left\{\begin{matrix} d_i(S_i x^2, P_i x^2),\ d_i(T_i x^1, Q_i x^1), \\ \dfrac{d_i(S_i x^2, Q_i x^1)+d_i(T_i x^1, P_i x^2)}{2}\end{matrix}\right\}\right\} \\ &= \max\left\{\sum_{k=1}^{n} a_{ik} d_k(y_k^2, y_k^1), b\max\left\{d_i(y_i^2, y_i^3),\ d_i(y_i^1, y_i^2), \frac{d_i(y_i^2, y_i^2)+d_i(y_i^1, y_i^3)}{2}\right\}\right\} \\ &= \max\left\{\sum_{k=1}^{n} a_{ik} d_k(y_k^2, y_k^1), b\max\left\{d_i(y_i^2, y_i^3),\ d_i(y_i^1, y_i^2), \frac{d_i(y_i^1, y_i^3)}{2}\right\}\right\}, \end{aligned}$$

Since $\dfrac{d_i(y_i^1, y_i^3)}{2} \le \dfrac{d_i(y_i^1, y_i^2)+d_i(y_i^2, y_i^3)}{2}$ implies that

$$d_i(y_i^3, y_i^2) \le \max\left\{\sum_{k=1}^{n} a_{ik} d_k(y_k^2, y_k^1), b\max\left\{d_i(y_i^2, y_i^3),\ d_i(y_i^1, y_i^2)\right\}\right\}.$$

If $d_i(y_i^2, y_i^3) > d_i(y_i^1, y_i^2)$, then

$$\begin{aligned} d_i(y_i^3, y_i^2) &\le \max\left\{\sum_{k=1}^{n} a_{ik} d_k(y_k^2, y_k^1),\ \ b d_i(y_i^2, y_i^3)\right\} \\ &\le \max\left\{\sum_{k=1}^{n} a_{ik} r_k, b d_i(y_i^2, y_i^3)\right\} \le \max\left\{h r_i, b d_i(y_i^2, y_i^3)\right\} \le h r_i . \end{aligned}$$

Since otherwise, we get a contradiction.

If $d_i(y_i^2, y_i^3) < d_i(y_i^1, y_i^2)$, then

$$\begin{aligned} d_i(y_i^3, y_i^2) &\le \max\left\{\sum_{k=1}^{n} a_{ik} d_k(y_k^2, y_k^1),\ \ b d_i(y_i^1, y_i^2)\right\} \\ &\le \max\left\{\sum_{k=1}^{n} a_{ik} r_k, b d_i(y_i^1, y_i^2)\right\}, \\ &\le \max\left\{h r_i, b r_i\right\} = c r_i,\ i=1,...,n, \text{ where } c=\max\{h,b\}. \end{aligned}$$

Again from condition (2.5), we get

$$d_i(y_i^4, y_i^3) = d_i(P_i x^2, Q_i x^3)$$
$$\leq \max\left\{\sum_{k=1}^{n} a_{ik} d_k(S_k x^2, T_k x^3), b \max\left\{d_i(S_i x^2, P_i x^2), d_i(T_i x^3, Q_i x^3), \frac{d_i(S_i x^2, Q_i x^3) + d_i(T_i x^3, P_i x^2)}{2}\right\}\right\}$$
$$= \max\left\{\sum_{k=1}^{n} a_{ik} d_k(y_k^2, y_k^3), b \max\left\{d_i(y_i^2, y_i^3), d_i(y_i^3, y_i^4), \frac{d_i(y_i^2, y_i^4) + d_i(y_i^3, y_i^3)}{2}\right\}\right\}$$
$$= \max\left\{\sum_{k=1}^{n} a_{ik} d_k(y_k^2, y_k^3), b \max\left\{d_i(y_i^2, y_i^3), d_i(y_i^3, y_i^4), \frac{d_i(y_i^2, y_i^4)}{2}\right\}\right\}.$$

Since $\frac{d_i(y_i^2, y_i^4)}{2} \leq \frac{d_i(y_i^2, y_i^3) + d_i(y_i^3, y_i^4)}{2}$ implies that

$$d_i(y_i^4, y_i^3) \leq \max\left\{\sum_{k=1}^{n} a_{ik} d_k(y_k^2, y_k^3), b \max\left\{d_i(y_i^2, y_i^3), d_i(y_i^3, y_i^4)\right\}\right\}$$
$$\leq \max\left\{\sum_{k=1}^{n} a_{ik} c r_k, b \max\left\{d_i(y_i^3, y_i^4), c r_i\right\}\right\}$$

and arguing same as before this implies

$$d_i(y_i^4, y_i^3) \leq \max\left\{\sum_{k=1}^{n} a_{ik} c r_k, b c r_i\right\} \leq \max\left\{h c r_i, b c r_i\right\}$$
$$= c r_i \max\left\{h, b\right\} = c^2 r_i, \text{ where } c = \max\{h, b\}.$$

Inductively

$$d_i(y_i^{m+2}, y_i^{m+1}) \leq c^m r_i, \; m = 0, 1, 2, \ldots$$

Since $0 \leq c < 1$, hence $\left\{y_i^m\right\}_{m=1}^{\infty}$ is a Cauchy sequence in X_i for all $i = 1, \ldots, n$. As X_i is a complete metric space, there exists a point u_i (say) in X_i such that $\left\{y_i^m\right\} \to u_i$ as $m \to \infty$. Moreover $y_i^{2m+1} = P_i x^{2m} = T_i x^{2m+1} \to u_i$ and $y_i^{2m+2} = Q_i x^{2m+1} = S_i x^{2m+2} \to u_i$ as $m \to \infty$.

If systems of maps $(P_1, \ldots, P_n)$ and $(S_1, \ldots, S_n)$ are coordinatewise reciprocal continuous then

$$\lim_{m\to\infty} P_i(S_1 x^{2m}, \ldots, S_n x^{2m}) = P_i u \text{ and } \lim_{m\to\infty} S_i(P_1 x^{2m}, \ldots, P_n x^{2m}) = S_i u, \quad i = 1, .., n. \tag{2.7}$$

Now, coordinatewise R- weak commutativity of systems of maps $(P_1, \ldots, P_n)$ and $(S_1, \ldots, S_n)$ yields

$$d_i\left(P_i(S_1 x^{2m}, \ldots, S_n x^{2m}), S_i(P_1 x^{2m}, \ldots, P_n x^{2m})\right) \leq R d_i\left(P_i x^{2m}, S_i x^{2m}\right).$$

Taking $\lim m \to \infty$ and using condition (2.7), we have

$$d_i\left(P_i u, S_i u\right) = 0, \; i = 1, \ldots, n$$

and therefore

$$P_i u = S_i u, \; i = 1, \ldots, n. \tag{2.8}$$

Since $P_i(X) \subset T_i(X)$, so there exists a point $w = (w_1, \ldots, w_n) \in X$ such that $P_i u = T_i w$ for $i = 1, \ldots, n$.

Now from condition (2.5),

$$d_i\left(P_i u, Q_i w\right) \leq \max\left\{\sum_{k=1}^{n} a_{ik} d_k\left(S_k u, T_k w\right), b \max\left\{d_i(S_i u, P_i u), d_i(T_i w, Q_i w), \frac{d_i(S_i u, Q_i w) + d_i(T_i w, P_i u)}{2}\right\}\right\}.$$

Using condition (2.8), we get

$$d_i\left(P_i u, Q_i w\right) \leq \max\left\{\sum_{k=1}^{n} a_{ik} d_k\left(P_k u, T_k w\right), b \max\left\{\begin{matrix} d_i(P_i u, P_i u), d_i(P_i u, Q_i w), \\ \frac{d_i(P_i u, Q_i w) + d_i(T_i w, P_i u)}{2} \end{matrix}\right\}\right\}$$
$$\leq b d_i\left(P_i u, Q_i w\right).$$

So,

$$P_i u = Q_i w,\ i = 1,\ldots,n\ .$$

By the above we then have

$$P_i u = S_i\, u = T_i w = Q_i w,\ \text{for}\ i = 1,\ldots,n\ . \tag{2.9}$$

Coordinatewise R-weak commutativity of systems of maps $(P_1,\ldots,P_n)$ and $(S_1,\ldots,S_n)$ implies that there exists $R > 0$ such that

$$d_i\left(P_i(S_1u,\ldots,S_nu),\, S_i(P_1u,\ldots,P_nu)\right) \le R d_i\left(P_iu, S_iu\right).$$

Using condition (2.9), we get

$$d_i\left(P_i(S_1u,\ldots,S_nu),\, S_i(P_1u,\ldots,P_nu)\right) = 0$$

and therefore

$$P_i(S_1u,\ldots,S_nu) = S_i(P_1u,\ldots,P_nu),\quad i = 1,\ldots,n.$$

Again using condition (2.9), we have

$$P_i(S_1u,\ldots,S_nu) = P_i(P_1u,\ldots,P_nu) = S_i(P_1u,\ldots,P_nu) = S_i(S_1u,\ldots,S_nu),\ i = 1,\ldots,n.$$

Similarly, R-weak commutativity of systems of maps $(Q_1,\ldots,Q_n)$ and $(T_1,\ldots,T_n)$ implies

$$Q_i(T_1w,\ldots,T_nw) = Q_i(Q_1w,\ldots,Q_nw) = T_i(Q_1w,\ldots,Q_nw) = T_i(T_1w,\ldots,T_nw),\ i = 1,\ldots,n.$$

Now from condition (2.5) and assuming

$$d_i\left(P_i(P_1u,\ldots,P_nu),\ P_iu\right) \le r_i,\ i = 1,\ldots,n.$$

We have

$$\begin{aligned}
d_i\left(P_i(P_1u,\ldots,P_nu),\ P_iu\right) &= d_i\left(P_i(P_1u,\ldots,P_nu), Q_iw\right)\\
&\le \max\left\{\begin{array}{l}\sum_{k=1}^{n} a_{ik} d_k\left(S_k(P_1u,\ldots,P_nu), T_kw\right),\\
b\max\left\{\begin{array}{l} d_i\left(S_i(P_1u,\ldots,P_nu), P_i(P_1u,\ldots,P_nu)\right), d_i(T_iw, Q_iw),\\ \dfrac{d_i\left(S_i(P_1u,\ldots,P_nu), Q_iw\right) + d_i\left(T_iw, P_i(P_1u,\ldots,P_nu)\right)}{2}\end{array}\right\}\end{array}\right\}\\
&\le \max\left\{\begin{array}{l}\sum_{k=1}^{n} a_{ik} d_k\left(P_k(P_1u,\ldots,P_nu), P_ku\right),\\
b\max\left\{\begin{array}{l} d_i\left(P_i(P_1u,\ldots,P_nu), P_i(P_1u,\ldots,P_nu)\right), d_i(T_iw, Q_iw),\\ \dfrac{d_i\left(P_i(P_1u,\ldots,P_nu), P_iu\right) + d_i\left(P_iu, P_i(P_1u,\ldots,P_nu)\right)}{2}\end{array}\right\}\end{array}\right\}\\
&\le \max\left\{\sum_{k=1}^{n} a_{ik} r_k, bd_i\left(P_i(P_1u,\ldots,P_nu),\ P_iu\right)\right\}\\
&\le \max\left\{hr_i, br_i\right\} \le cr_i,\ \text{where}\ c = \max\{h,b\}.
\end{aligned}$$

Inductively, we can prove that

$$d_i\left(P_i(P_1u,\ldots,P_nu),\ P_iu\right) \le c^m r_i\ ,\ m = 0,1,2,\ldots$$

and consequently

$$P_i(P_1u,\ldots,P_nu) = P_iu,\ i = 1,\ldots,n\ .$$

By the above we then have

$$P_i(P_1u,\ldots,P_nu) = S_i(P_1u,\ldots,P_nu) = P_iu\ . \tag{2.10}$$

Similarly, by using the condition (2.5), we can obtain

$$Q_i(Q_1w,\ldots,Q_nw) = T_i(Q_1w,\ldots,Q_nw) = Q_iw\ . \tag{2.11}$$

Since

$$P_iu = Q_iw = v_i\ \text{(say) for}\ i = 1,\ldots,n\ , \tag{2.12}$$

then condition (2.10), (2.11) and (2.12) together implies that

$$P_i v = Q_i v = v_i = S_i v = T_i v,\ i = 1,...,n.$$

This proves that the system of equation (2.6) has a common solution when systems of maps $(P_1,...,P_n)$ and $(S_1,...,S_n)$ are coordinatewise reciprocal continuous. In case, when systems of maps $(Q_1,...,Q_n)$ and $(T_1,...,T_n)$ are coordinatewise reciprocal continuous, the proof may be accomplished in an analogous manner.

Now, to prove the uniqueness of the solution, we assume that $v_i, \overline{v_i} \in X_i$ such that $v_i \neq \overline{v_i}$ and $P_i\overline{v} = Q_i\overline{v} = S_i\overline{v} = T_i\overline{v} = \overline{v}$. We can assume that

$$d_i\left(v_i, \overline{v_i}\right) \leq r_i,\ i = 1,...,n.$$

From condition (2.5), we get

$$d_i(v_i, \overline{v_i}) = d_i(P_i v, Q_i\overline{v}) \leq \max\left\{\sum_{k=1}^{n} a_{ik} d_k(S_k v, T_k\overline{v}), b\max\left\{\begin{array}{l} d_i(S_i v, P_i v),\ d_i(T_i\overline{v}, Q_i\overline{v}), \\ \dfrac{d_i(S_i v, Q_i\overline{v}) + d_i(T_i\overline{v}, P_i v)}{2}\end{array}\right\}\right\}$$

$$\leq \max\left\{\sum_{k=1}^{n} a_{ik} d_k(v_i, \overline{v_i}), b\max\left\{\begin{array}{l} d_i(v_i, v_i),\ d_i(\overline{v_i}, \overline{v_i}), \\ \dfrac{d_i(v_i, \overline{v_i}) + d_i(\overline{v_i}, v_i)}{2}\end{array}\right\}\right\}$$

$$\leq \max\left\{\sum_{k=1}^{n} a_{ik} r_k, bd_i(v_i, \overline{v_i})\right\} = \max\left\{hr_i, br_i\right\} = cr_i \text{, where } c = \max\{h, b\}.$$

Inductively

$$d_i(v_i, \overline{v_i}) \leq c^m r_i,\ m = 0,1,2,...$$

and consequently

$$v_i = \overline{v_i},\ i = 1,...,n.$$

This completes the proof.

Corollary 2.1: Let $(X_i, d_i), i = 1, 2....,n,$ be complete metric spaces and $P_i, T_i : X \to X_i,\ i = 1,...,n,$ be such that

$$P_i(X) \subset T_i(X),\ i = 1,...,n; \qquad (2.13)$$

Systems of maps $(P_1,...,P_n)$ and $(T_1,...,T_n)$ are coordinatewise R- weakly commuting; (2.14)

Systems of maps $(P_1,...,P_n)$ and $(T_1,...,T_n)$ are coordinatewise reciprocal continuous; (2.15)

If there exist non- negative numbers b and a_{ik}, $i, k = 1,....,n$ such that (1.1), (1.2), (1.4) and the following hold:

$0 \leq b < 1 - h$, where h is defined in (1.6) and (2.16)

$$d_i\left(P_i x, P_i y\right) \leq \max_i\left\{\sum_{k=1}^{n} a_{ik} d_k\left(T_k x, T_k y\right), b\max\left\{d_i(P_i x, T_i x),\ d_i(P_i y, T_i y), \frac{d_i(P_i x, T_i y) + d_i(P_i y, T_i x)}{2}\right\}\right\} \qquad (2.17)$$

for all $x, y \in X$, then the system of equations

$$P_i x = T_i x = x_i, \qquad (2.18)$$

has a unique common solution $(x_1,...,x_n)$ such that $x_i \in X_i,\ i = 1,..,n$.

Proof: Proof may be completed by putting $P_i = Q_i$ and $S_i = T_i, i = 1,...n,$ in the proof of Theorem 2.1.

Example 2.1: Let $X_1 = X_2 = [2, 20]$ be usual metric spaces and $P_i, T_i : X_1 \times X_2 \to X_i,\ i = 1, 2,$ such that

$$P_1(x_1, x_2) = \begin{cases} 2 & x_1 = 2 \\ 6 & 2 < x_1 \leq 6 \\ 2 & x_1 > 6 \end{cases}, \qquad P_2(x_1, x_2) = \begin{cases} 2 & x_2 = 2 \\ 6 & 2 < x_2 \leq 6 \\ 2 & x_2 > 6 \end{cases},$$

$$T_1(x_1, x_2) = \begin{cases} 2 & x_1 = 2 \\ 12 & 2 < x_1 \leq 6, \\ x_1 - 3 & x_1 > 6 \end{cases} \qquad T_2(x_1, x_2) = \begin{cases} 2 & x_2 = 2 \\ 12 & 2 < x_2 \leq 6 \\ x_2 - 3 & x_2 > 6 \end{cases},$$

for all $(x_1, x_2) \in X_1 \times X_2$. This example satisfy all the conditions of our Corollary 2.1 and have a unique common solution at $x = (2,2)$ .It is notable that two systems of maps $\{P_1, P_2\}$ and $\{T_1, T_2\}$ are coordinatewise reciprocal continuous but none of P_1, P_2, T_1, T_2 is continuous, not even at the point $x = (2,2)$.

Remark 2.1: If we assume $(M, d) = (X_i, d_i)$, $P = P_i$, $Q = Q_i$, $S = S_i$, $T = T_i$, $i = 1, ..., n$, and $n = 1$ then (2.5) with $a_{11} = k$ may be written as:

$$P, Q, S, T : M \to M,\ 0 < k < 1,$$

$$(*)\ d(Px, Qy) \le \beta \max\left\{ d(Sx, Ty),\ d(Sx, Px),\ d(Ty, Qy), \frac{d(Sx, Qy) + d(Ty, Px)}{2} \right\}, x, y \in X,$$

where $0 < \beta = \max\{k, b\} < 1$.

Remark 2.2: A multitude of fixed point theorems generalizing the Jungck contraction principle have been establish under the condition (*) and its particular cases (see for [2], [3], [7], [12], [15], [17], [27], [29]-[30], [34]-[38]). It may also be mentioned that Kubiak main result proved exactly under the condition (*) uses the commutativity of P with S and that of Q with T .

ACKNOWLEDGEMENT

The second author is thankful to CSIR, New Delhi for financial Assistance vide file no. 09/386(0052)/2015- EMR-I.

REFERENCES

1. J. B. Baillon and S. L. Singh, Nonlinear hybrid contractions on product spaces, Far East J. Math. Sci., 1(1993), 117-127.
2. Y.J. Cho and S.L. Singh, A coincidence theorem and fixed point theorems in Saks spaces, Kobe J. Math., 3 (1986), 1-6.
3. V. Conserva, Common fixed point theorems for commuting maps on a metric space, Publ. Inst. Math. (Beograd), 32 (46) (1982), 37-43.
4. S. Czerwik, A fixed point theorem for a system of multi-valued transformations, Proc. Amer. Math. Soc., 55 (1976), 136-139.
5. S. Czerwik, Generalization of Edelstein’s fixed point theorem, Demonstratio Math., 9 (1976), 281-285.
6. K. M. Das and K.V. Naik, Common fixed point theorems for commuting maps on a metric space, Proc. Amer. Math. Soc., 77 (1979), 369-373.
7. B. Fisher, Mappings with a common fixed point, Math. Sem. Notes Kobe Univ., 7 (1979), 81-84; addendum 8 (1980).
8. U.C. Gairola and P.S. Jangwan, Coordinatewise R-weakly commuting maps and fixed point theorem on product spaces, Demonstratio Math., 36 (4) (2003), 939-949.
9. U. C. Gairola and P.S. Jangwan, Coincidence theorem for multi-valued and single-valued systems of transformations, Demonstratio Math., 41(1) (2008), 129-136.
10. U. C. Gairola, S. N. Mishra and S. L. Singh, Coincidence and fixed point theorems on product spaces, Demonstratio Math., 30 (1997), 15-24.
11. U. C. Gairola, S. L. Singh and J. H. M. Whitfield, Fixed point theorems on product of compact metric spaces, Demonstratio Math., 28 (1995), 541-548.
12. O. Hadzic, Common fixed point theorems for family of mappings in complete metric spaces, Math. Japon., 29 (1984), 127-134.
13. G. Jungck, Commuting maps and fixed points, Amer. Math. Monthly, 83 (1976), 261-263.
14. G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci., 9 (4) (1986), 771-779.
15. G. Jungck, Common fixed points for commuting and compatible maps on compacta, Proc. Amer. Math. Soc. 103 (1988), 977-983.
16. Z. Kominek, A generalization of K. Goeble’s and J. Matkowski’s theorems, Univ. Slaskiw Katowicach PraceNauk- Prace. Mat., 12 (1982), 30-33.
17. T. Kubiak, Common fixed points of pairwise commuting mappings, Math. Nachr., 118 (1984), 123-127.
18. J. Matkowski, Some inequalities and a generalization of Banach’s Principle, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 21(1973), 323-324.
19. J. Matkowski, Integrable solutions of functional equations, Dissert. Math. (Rozprawy Math.), 127, Warszawa, 1975.
20. J. Matkowski and S.L. Singh, Banach type fixed point theorems on product of spaces, Indian Journal of Mathematics, 38 (1) (1996), 73-78.
21. R .P. Pant, Common fixed points of non-commuting mappings, J. Math. Anal. Appl., 188(1994), 436-440.

22. R.P. Pant, Common fixed points of four mappings, Bull. Cal. Math. Soc., 90(1998), 281-286.
23. R.P. Pant and S. Padaliya, Reciprocal continuity and fixed point, Jnanabha, 29(1999), 137-143.
24. S. Park, fixed points of f-contractive maps, Rocky Mountain J. Math., 8 (1978), 743-750.
25. K. B. Reddy and P. V. Subrahmanyam, Extensions of Krasnoselskii's and Matkowski's fixed point theorems, FunkcialajEkv., 24 (1981), 67-83.
26. K. B. Reddy and P. V. Subrahmanyam, Altman's contractors and fixed points of multi-valued mappings, Pacific J. Math., 99 (1) (1982), 127-136.
27. B.E. Rhoades and S. Sessa, Common fixed point theorems for three maps under a weak commutativity condition, Indian J. Pure. Appl. Math. 17 (1986), 47-57.
28. S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math., 32 (46) (1982), 149-153.
29. S. Sessa and B. Fisher, Common fixed points of weakly commuting mappings, Bull. Polish. Acad. Sci. Math., 35 (1987), 341-349.
30. S. Sessa, R.N. Mukherjee and T. Som, A common fixed point theorem for weakly commuting mappings, Math. Japon., 31 (1986), 235-245.
31. S. L. Singh and U. C. Gairola, A general fixed point theorem, Math. Japon, 36(1991), 791-801.
32. S. L. Singh and U. C. Gairola, Coordinatewise commuting and weakly commuting maps and extension of Jungck and Matkowski contraction principles, J. Math. Phy. Sci., 25(4) (1991), 305-318.
33. S. L. Singh and C. Kulshrestha, A common fixed point theorem for two systems of transformations, Pusan. Kyo. Math. J., 2 (1986), 1-8.
34. S.L. Singh and B.D. Pant, Coincidence and fixed point theorems for a family of mappings on Menger spaces and extension to uniform spaces, Math. Japon., 33 (1988), 957-973.
35. S.L. Singh and S.P. Singh, A fixed point theorem, Indian J. Pure Appl. Math., 11 (1980), 1584-1586.
36. S.L. Singh and B.M.L. Tivari, Common fixed points of mappings in complete metric spaces, Proc. Nat. Acad. Sci. India, Sect. A, 51 (1981), 41-44.
37. B.M.L. Tivari and S.L. Singh, A note on recent generalizations of Jungck contraction principle, J. UPGC. Acad. Soc., 3 (1986), 13-18.
38. C.C. Yeh, On common fixed point theorem of continuous mappings, India J. Pure Appl. Math. 10(1979), 415-420.

Source of support: CSIR, New Delhi, India. Conflict of interest: None Declared.