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CHAOTIC MAPS IN TOPOLOGICAL DYNAMICAL SYSTEMS

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ABSTRACT

In this paper, we have worked out on some results about periodic points and eventually points and their orbits, which are very helpful in studying the chaos of dynamical system. This study includes the chaotic maps, the behavior of sensitive to initial conditions and sensitivity constants etc. In this paper many results have been proved regarding the dense orbit, topological transitivity and sensitivity constant. We also proved here that orbit of a periodic point are either equal or disjoint. We have proved the equality between orbit of point and orbit of a periodic point of the iterated functions. Some new concepts regarding supremum and infimum of periodic points of some functions have been introduced and also the results associated with these.

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KEYWORDS: *Chaotic map, Chaotic Dynamics, Dense orbit, Sensitivity constant, Topological Transitive, Dynamical system.*

1. INTRODUCTION

The study of chaotic dynamical system becomes popular nowadays. Although there is no universal accepted definition of chaos but it is generally believed that sensitivity dependence on initial condition is the central element of chaos [1, 15]. There are some situations/ problems where the things under consideration are vague i.e. there exists a Chaos in the situation/ problem. For a possible solution of such type of situation/ problem, first it has to be viewed as a problem in term of Mathematics and then there is need to think of a possible solution of the problem in a suitable way by applying Mathematics on it. To do this we need some mathematical tools. For this we studied some concepts/ theories of the function and functional iteration, which is going to be very helpful in studying the orbit of a point, periodic points, chaotic maps and some other dynamical systems [3, 5, 11]. The term Chaos theory means the study of instability in a periodic behavior in dynamical systems. This means that a dynamical system can in fact be generate a periodic disordered behavior that is the behavior with a hidden implicit order [2]. In Section 2 of this paper we have included some basic notation and definition like orbit of a point, periodic points, eventually periodic points, topological transitivity, sensitivity dependence to initial condition and dynamical systems which were used in many of the results. In Section 3 of this paper we cover the preliminaries parts. For completeness sake, we have included some standard results also. In view of our aim, of obtaining some results about chaotic maps in Dynamical Systems, we start with the study of periodic points, eventually periodic points and their orbits [10, 16]. The concept of start of the eventuality of an eventually periodic point introduced here helps in describing its orbit more precisely. It has been found that, unlike the case of a periodic point, the iterated images of an eventually periodic point need not always be periodic [12]. The complete information about the periodicity of iterated images of an eventually periodic point and their periods has been obtained using the term start of eventuality of an eventually periodic point. Some results on the basis of the set theory, iterated functions, orbit of function, periodic points are also included in this section. In Section 4, we come to the chaotic part in this section we cover the relations of orbits of different periodic points with regard to the denseness of the set of all periodic points [7, 15]. The minimum of the distances between iterated images of two periodic points

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helps in knowing about the distance between a point of a dynamical system (X, f) and orbits of periodic points. This minimum value, in case the map f is sensitive to initial conditions, is also found to have a relation with the sensitivity constants of the map f, where there is also obtained a description of a considerable part of the set of all sensitivity constants of f [13]. The total description of the set of all sensitivity constants of f is left as a question. It is conjectured that some variations of techniques used to find the considerable part should be helpful to cover the whole set of sensitivity constants of f.

2. NOTATION AND DEFINITIONS

We define f^0 = Identity function on X, $f^1 = f$, and for $n \in$ IN, the set of all natural numbers, $f^{n+1} = fof^n$. Let $f : X \to X$. Let $x \in X$. The set $\{f^n(x) \mid n \geq 0\}$ is called the **orbit** of x. The orbit of x is denoted by **Orb(x) or Orb(x, f).** A point x $\in$X is called **periodic** if $f^k(x) = x$ for some $k \in$ IN. The smallest k such that $f^k(x) = x$ is called the **period of x**. If $f :$ IR $\to$ IR, then we shall write, $f^0(x) = x$, $f^1(x) = f(x)$, $f^2(x) = (f \circ f)(x) = f(f(x))$, $f^3(x) = (f \circ f^2)(x) = f(f^2(x)) = f(f(f(x)))$ and similarly $f^n(x) = (f \circ f^{n-1})(x) = f(f^{n-1}(x))$, for $n \geq 3$. Also, $f^n(x)$ is called the nth iteration of f for $n \geq 0$. Let **Per(f)** be the set of all periodic points of X. **Orb(X, f)** or simply **Orb(X)**, is used to denote the union of the orbits of all periodic points of X, i.e. $Orb(X) = \cup\{Orb(x) : x \in Per(f)\}$.

Let M be a non empty set. A function d: M x M $\longrightarrow [0, \infty)$ is called **Metric Space** if d holds the following properties (i) $d(x, y) \geq 0$ for all $x, y \in M$. (ii) $d(x, y) = d(y, x)$ for all $x, y \in M$. (iii) $d(x, y) = 0$ iff $x = y$ for all $x, y \in M$. (iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in M$. Let (M, d) be a Metric Space and let $A \subset M$ be non empty sub set of M. Define a function $f : M \to$ IR for all $x \in A$ by $f(x) = \inf\{d(x, y) ; y \in$ IR$\}$.

Let (X, d) be a metric space. For $A \subset X$, **diam(A)** denotes the diameter of A. For a point z of X and $r > 0$, **S(z, r)** (resp. **S[z, r]**) denotes the open sphere (closed sphere) with centre at z and radius r. Let (M, d) be a Metric Space and $A \subset M$. Then A is said to be dense in M if for every $z \in M$ and for every $r > 0$, we have $S(z, r) \cap A \neq \phi$ i.e. every open ball in (M, d) contains a point of A. A **Topological Space** is a set X and a collection of subsets of X, τ called the topology defined on X which we together denote by (X, τ) such that (i) the empty set ϕ and the whole set X are contained in τ. (ii) If $U_i \in \tau$ for all i in τ then any arbitrary union of subsets in τ is contained in τ. (iii) If $U_1, U_2, U_3, \ldots\ldots\ldots U_n \in \tau$ then any finite intersection of n subsets of X in τ is contained in τ. Let X is a topological space. Let $f : X \longrightarrow X$. Then (X, f) is called a **dynamical system** if f is continuous. A map $f : X \to X$ is called **topologically transitive (TT)** if for every pair of non empty open sets G, V in X, there exists some $m \in$ IN such that $f^m(G) \cap V \neq \phi$. If f^n is transitive for each n $\in$ N, then f is called **totally transitive**. A point $x \in X$ is called **eventually periodic** if for some nonnegative integer t, $f^t(x)$ is periodic. Let (X, f) be a dynamical system where X is a metric space. f is said to be **sensitive to initial conditions (SIC)** if there exists a $\delta > 0$ such that for a given $x \in X$ and a neighborhood N(x) of x, there exists some $y \in N(x)$ and some $v \in$ IN such that $d(f^v(x), f^v(y)) \geq \delta$; such a δ is called a **sensitivity constant for f**. Let $x \in X$ be eventually periodic with eventual period k. Let t be the smallest nonnegative integer such that $f^t(x)$ is periodic with period k. Then we say that t is the **start of eventuality of x.** Let $p, q \in Per(f)$ be such that $q \in X - Orb(p)$. Let k and m be the periods of p and q respectively. For $0 \leq j \leq m-1$, let $\delta^j = d(f^j(q), Orb(p))$. Let $\boldsymbol{\delta^\vee} = \sup\{\delta^j \mid 0 \leq j \leq m\text{-}1\}$. $\boldsymbol{\delta^\vee(p, q)}$ is used for $\delta^\vee$ if dependence on p, q is to be shown. Since $\delta^\vee = d(f^\vee(q), f^i(p))$ for some i, $0 \leq i \leq k-1$, we write $\boldsymbol{\delta^\vee = d(f^\vee(q), f^{i\vee}(p))}$. For $0 \leq i \leq k-1$, let $\boldsymbol{\varsigma^i} = d(f^i(p), Orb(q))$. Let $\sup\{\varsigma^i \mid 0 \leq i \leq k\text{-}1\} = \varsigma^\vee(q, p)$ or simply $\varsigma^\vee$. For periodic elements, x and y of X with periods k and m respectively such that $y \in X - Orb(p)$, let $\delta^{min} = \inf\{d(f^r(x)), d(f^s(y)) \mid 0 \leq r \leq k\text{-}1, 0 \leq s \leq m\text{-}1\}$.

3. PRELIMINARIES

We shall need the following Remarks from our previous published papers.

Remark 1.1: (i) Let $n \in$ IN, then $f^{n+1} = f^n \circ f$. (ii) Let $m, n \in$ IN, then $f^{n+m} = f^n \circ f^m$. (iii) Let $x \in X$. For an integer $t \geq 0$, let $f^t(x)$ be periodic with period k. (iv) For every $n \in$ IN, $f^n(x) \in \{f^0(x), f^1(x), f^2(x), \ldots f^{t+k-1}(x)\}$. (v) A point $x \in X$ is eventually periodic iff its orbit is finite.

Remark 1.2: Let $x \in X$ be periodic with period k. Then (i) $f^{mk}(x) = x$ for all m in IN, (ii) If $f^m(x) = x$, then $m = kq$, for some q in IN, (iii) For $n \in$ IN, $f^n(x) = f^r(x)$, for some integer r with $0 \leq r < k$. (iv) $Orb(x) = \{x, f(x)\ldots f^{k-1}(x)\}$. (v) The elements of $\{f^0(x), f^1(x), f^2(x), \ldots f^{k-1}(x)\}$ are distinct.

Lemma 1.3: Let $g : X \to X$.(i) If $g^r(X) = X$ for some $r \in$ IN, then $g^n(X) = X$ for every $n \in$ IN. (ii) For $G \subset X$ and $m \in$ IN, if $k \leq m$, then $g^m((g^k)^{-1}(G)) \subset g^{m-k}(G)$, the equality holds if g^r is onto for some $r \in$ IN. (iii) For $G \subset X$ and $m \in$ IN, if $k > m$, then $g^m((g^k)^{-1}(G) \subset (g^{(k-m)})^{-1}(G)$, the equality holds if g^r is onto for some $r \in$ IN.

Proof: (i) It is sufficient to prove the result for n = 1, then apply induction. $X = g^r(X) = g(g^{r-1}(X)) \subset g(X) \subset X$. So $g(X) = X$. $g^{n+1}(X) = g(g^n(X) = g(X) = X$. (ii) Let $k \leq m$. Since $g^k((g^k)^{-1}(G)) \subset G$, $g^{m-k}(g^k((g^k)^{-1}(G)) \subset g^{m-k}(G)$. Thus $g^m((g^k)^{-1}(G)) \subset g^{m-k}(G)$. If g^r is onto for some $r \in$ IN, then, by (i), g^k is onto. So the equality holds. (iii) Let $k > m$.

Since $(g^k)^{-1}(G) = (g^m)^{-1}((g^{(k-m)})^{-1}(G))$, $g^m((g^k)^{-1}(G)) = g^m((g^m)^{-1}((g^{(k-m)})^{-1}(G))) \subset (g^{(k-m)})^{-1}(G)$. The equality holds if g^r is onto for some $r \in$ IN, as, then, using (i), g^m is onto.

Remark 1.4: Let $g : X \to X$ and G, V $\subset$ X. (a) (i) If $g(G)\cap V \neq \phi$, then $G\cap g^{-1}(V) \neq \phi$. (ii) If $G\cap g^{-1}(V) \neq \phi$, then $g^{-1}(g(G)\cap V)) \neq \phi$, and so $g(G)\cap V \neq \phi$. (b) Let m$\in$ IN and g is onto. (i) If $k \le m$, then $g^m((g^k)^{-1}(G))\cap V \neq \phi$ iff $g^{m-k}(G)\cap V \neq \phi$. (ii) If k > m, then $g^m((g^k)^{-1}(G))\cap V \neq \phi$ iff $(g^{(k-m)})^{-1}(G)\cap V \neq \phi$.

Proof: (a) (i) Let $z \in g(G)\cap V$. Then $z \in V$ and there exists $x\in G$ such that g(x) = z. Thus $x \in G\cap g^{-1}(V)$.
(ii) $G\cap g^{-1}(V) \subset g^{-1}(g(G)\cap g^{-1}(V) = g^{-1}(g(G)\cap V))$. Thus $g^{-1}(g(G)\cap V) \neq \phi$. (b) It follows using (ii) and (iii) of Lemma 1.3.

In view of Remark 1.4(a), we have the following Remark.

Remark 1.5: A map $f : X \to X$ is Topological transitive (TT) iff for every pair of non empty open sets G, V in X, there exists some $m \in$ IN such that $G\cap(f^m)^{-1}(V) \neq \phi$.

Lemma 1.6: Let $f : X \longrightarrow X$ be Topological transitive (TT). Then f is onto.

Proof: In view of Remark 1.5, there exists some $m \in$ IN such that $(f^m)^{-1}(X)\cap X \neq \phi$. This implies that $f^m(X) = X$. The result now follows by (i) of Lemma 1.3.

Remark 1.7: Let $x \in X$. If x is periodic with period k, then the elements of $\{x, f(x),\ldots f^{k-1}(x)\}$, which is Orb(x), are distinct. Suppose that x is eventually periodic with eventual period k. There exists an integer $t \ge 0$ such that $f^t(x)$ is periodic with period k. Using Remark 1.2 (iv), Orb (x) = $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+t-1}(x)\}$. Let s > t. Since $f^s(x)$ is periodic with period k and, therefore, by Remark 1.2 (iv), $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+s-1}(x)\}$ = Orb(x) = $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+t-1}(x)\}$.

We suppose that $k \geq 2$. For $0 \leq r \leq k-1$, let s = t+k-r. k+s-1 = 2k+t-(r+1). Since $r+1 \leq k$, $f^{2k+t-(r+1)}(x) = f^{k+t-(r+1)}(x)$. Thus $f^{2k+t-(r+1)}(x)$ and $f^{k+t-(r+1)}(x)$ are not distinct elements of $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+s-1}(x)\}$, where s = t+k-r. The elements of even $\{f^0(x), f^1(x),f^2(x),\ldots f^{k+t-1}(x)\}$ need not be distinct. If we take s < t, if $f^s(x)$ is periodic then its period is k. In this case also, the elements of $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+s-1}(x)\}$ need not be distinct.

Proposition 1.8: Let $x \in X$ be periodic with period k. Then the following hold.
(a) (i) For integers r and s with $0 \le r, s < k$, if $f^s(x) = f^r(x)$, then s = r. (b) (i) For an integer r with $0 \le r < k$, $f^r(x)$ is periodic with period k and $Orb(f^r(x)) = Orb(x)$. (ii) For $n \in$ IN, $f^n(x)$ is periodic with period k and $Orb(f^n(x)) = Orb(x)$.

Proof: (a) (i) We suppose that $r \le s$. Sin ce k-r > 0, using Remark 1.1(ii), $f^{k-r+s}(x) = f^{k-r}(f^s(x)) = f^{k-r}(f^r(x) = f^k(x) = x$. Therefore, r = s, otherwise k-r+s < k, which is not possible as $f^{k-r+s}(x) = x$. (b) (i) $f^r(x) = f^r(f^k(x)) = f^k(f^r(x))$. So $f^r(x)$ is periodic. Let m be the period of $f^r(x)$. Since $f^r(x) = f^r(f^k(x)) = f^k(f^r(x))$, by Remark 1.2(ii), k = mq for some $q \in$ IN. $f^m(f^r(x)) = f^r(x)$, so $f^{m+r}(x) = f^r(x)$. Therefore $f^{m+r+k-r}(x) = f^{r+k-r}(x) = f^k(x) = x$. So $f^{m+k}(x) = x$. Thus using Remark 1.2.(ii), m+k = kq* for some $q \in$ IN. So m = k(q*-1). Since k = mq, we have k = m. To prove that $Orb(f^r(x)) = Orb(x)$, let $y = f^r(x)$. For $n \in$ IN, $f^n(y) = f^{n+r}(x)$. Since $f^{n+r}(x) \in Orb(x)$, $f^n(y) \in Orb(x)$. So $Orb(f^r(x)) \subset Orb(x)$. For $f^s(x) \in Orb(x)$, $0 \leq s< k$. If $r \leq s$, then $f^s(x) = f^{s-r}(f^r(x)) = f^{s-r}(y)$. Therefore $f^s(x) \in Orb(f^r(x))$. Suppose s < r. s+k-r > 0 as r < k. $f^{s+k-r}(y) = f^{s+k-r}(f^r(x)) = f^{s+k}(x) = f^s(x)$. Since $f^{s+k-r}(y) \in Orb(y)$, $f^s(x) \in Orb(y)$. (ii) Orb(x) = $\{x, f(x),\ldots f^{k-1}(x)\}$. So $f^n(x) = f^r(x)$ for some integer r, with $0 \leq r < k$. Now the result follows from (i).

Theorem 1.9: Let x, y $\in$ X be periodic. Then either Orb(x) = Orb(y) or $Orb(x) \cap Orb(y) = \phi$.

Proof: Suppose that $Orb(x) \cap Orb(y) \neq \phi$. Then $f^\nu(y) = f^s(x)$ for some nonnegative integers ν and s. By Proposition 1.8(b) (ii), $Orb(f^s(x)) = Orb(x)$ and $Orb(f^\nu(y)) = Orb(y)$. Therefore Orb(x) = Orb(y). If $x \in X$ is periodic, then $f^n(x)$ is periodic for every $n \in$ IN, but if $f^n(x)$ is periodic, the x need not be periodic.

Proposition 1.10: Let x and y be two periodic elements of X with periods k and m respectly such that $y \in X - Orb(p)$. Let $z \in X$. Then (a) $\delta^{min} \leq d(z, Orb(x)) + d(z, Orb(y))$. (b) Either $d(z, Orb(x)) \geq \delta^{min}/2$ or $d(z, Orb(y)) \geq \delta^{min}/2$. (c) If $z \in Orb(y)$, then $\delta^{min} \leq d(z, Orb(x))$.

Proof: $\delta^{min} > 0$, as by Theorem 1.9, $Orb(x) \cap Orb(y) = \phi$. There exit some i and j, with $0 \leq i \leq k-1$ and $0 \leq j \leq m-1$, such that $d(z, Orb(x)) = d(z, f^i(x))$ and $d(z, Orb(y)) = d(z, f^j(y))$. (a) We have $\delta^{min} \leq d(f^i(x)), d(f^j(y)) \leq d(z, f^i(x)) + d(z, f^j(y))$.

So $\delta^{min} \leq d(z, Orb(x)) + d(z, Orb(y))$. (b) Because of (a) it is not possible that $d(z, Orb(x)) < \delta^{min}/2$ and $d(z, Orb(y)) < \delta^{min}/2$. Therefore (b) holds. (c) If $z \in Orb(y)$, then $d(z, Orb(y)) = 0$. Now by (a) $\delta^{min} \leq d(z, Orb(x))$.
It should be interesting to think about corresponding results for eventually periodic points.

4. CHAOTIC MAPS

Let (X, d) be a metric space. For $A \subset X$, diam(A) denotes the diameter of A. Let (X, f) be a dynamical system, where X is a metric space. It is proved as Theorem in [7] that if X is infinite, f has a dense orbit and Per(f) is dense, then f is TT and SIC. From their proof it follows that f is TT and SIC. But their observation (used after the proof of Theorem, to obtain 'possibly largest' sensitivity constant for f) that d/4 is a sensitivity constant for f, does not follow. In the proof, $p \in Per(f)$, $q \in X - Orb(p)$ and $d = d(q, Orb(p))$. After assuming that there exists an x and a neighborhood N(x) of x such that for all $n \geq 0$, $diam(f^n(N(x)) < d/4$, a contradiction is obtained. If we assume that d/4 is not a sensitivity constant for f, then $diam(f^n(N(x)) < d/2$. Therefore, from the arguments given in the proof of Theorem, it follows that d/8 is a sensitivity constant for f. We obtain below some results about sensitivity constant for f.

Lemma 2.1: Let X be a metric space X. Let $f : X \to X$ be TT. Let $p \in Per(f)$ and $q \in X - Orb(p)$. Let $\gamma > 0$ be given. Let $z \in X$ and N(z) a neighborhood of z such that $N(z) \cap Per(f) \neq \varphi$ and $diam(f^n(N(z)) \leq 2\gamma$ for every $n \in IN$. Then given $\eta > 0$ there exists a nonnegative integer v such that $d(f^v(p), q) < 4\gamma + \eta$.

Proof: Let N(q) be a neighborhood of q with $D(N(q)) < \eta/2$. Let $y \in N(z) \cap Per(f)$. Let m be the period of y. f is continuous at p, therefore there exist a neighborhood N(p) of p such that for every r with $0 \leq r \leq m-1$, $d(f^r(x), f^r(p)) < \eta/2$ for every $x \in N(p)$. f is TT, so there exists $i, j \in IN$ such that $N(z) \cap (f^i)^{-1}(N(p)) \neq \varphi$ and $N(z) \cap (f^j)^{-1}(N(q)) \neq \varphi$. Let $p^* \in N(z) \cap (f^i)^{-1}(N(p))$ and $q^* \in N(z) \cap (f^j)^{-1}(N(q))$. Let $0 \leq r \leq m-1$. We have, $d(f^r(f^i(p^*)), f^r(p)) < \eta/2$ as $f^i(p^*) \in N(p)$. Since $y, p^* \in N(z)$, $d(f^{r+i}(y)), f^{r+i}(p^*)) \leq 2\gamma$. Thus $d(f^{r+i}(y)), f^r(p)) \leq d(f^{r+i}(y)), f^{r+i}(p^*)) + d(f^r(f^i(p^*)), f^r(p)) < 2\gamma + \eta/2$. Since $f^j(q^*) \in N(q)$, $d(f^j(q^*), q) < \eta/2$. Now $d(f^j(y), q) \leq d(f^j(y), f^j(q^*) + d(f^j(q^*), q) < \gamma + \eta/2$. Using that $\{f^i(y), f^{i+1}(y), \ldots, f^{i+m-1}(y)\} = Orb(y)$, $f^j(y) = f^{i+v}(y)$ for some v, $0 \leq v \leq m-1$. $d(f^v(p), q) \leq d(f^v(p)), f^{v+i}(y)) + d(f^j(y), q) < 2\gamma + \eta$.

Lemma 2.2: For a metric space X, let $f : X \to X$ be TT. Let $p \in Per(f)$ and $q \in X - Orb(p)$. Let $\gamma > 0$ be given. Let $z \in X$ and N(z) a neighborhood of z such that $N(z) \cap Per(f) \neq \varphi$ and $D(f^n(N(z)) \leq 2\gamma$ for every $n \in IN$. Then (a) $d(f^v(p), q) \leq 4\gamma$ for some nonnegative integer v. (b) Either there exists a nonnegative integer v such that $d(f^v(p), q) < 4\gamma$, or there exists a nonnegative integer v such that $d(f^v(p), q) = 4\gamma$ and $d(f^s(p), q) \geq 4\gamma$ for every nonnegative integer s.

Proof: (a) Suppose $d(f^s(p), q) > 4\gamma$ for every nonnegative integer s. Let k be the period of p. $Orb(p) = \{f^r(p): 0 \leq r \leq k-1\}$. Let $\delta^0 = \min\{d(f^r(p), q) : 0 \leq r \leq k-1\}$. Then $\delta^0 > 4\gamma$. By Lemma 2.1, for $\eta = \delta^0 - 4\gamma$, there exists a nonnegative integer v such that $d(f^v(p), q) < 4\gamma + \eta = \delta^0$. $f^v(p) = f^r(p)$ for some r, $1 \leq r \leq k-1$. Therefore $d(f^r(p), q) < \delta^0$. This contradicts the definition of δ^0. This proves (a). (b) follows from (a).

Lemma 2.3: Let X be a metric space X, let $f : X \to X$ be TT and Per(f) is dense. Let $p \in Per(f)$ and $q \in X - Orb(p)$. Let $\delta^0 = d(q, Orb(p))$. If δ is not a sensitivity constant for f, then $\delta^0 \leq 4\delta$.

Proof: Since δ is not a sensitivity constant for f, there exists some $z \in X$ and a neighborhood N(z) of z such that for every $z^* \in N(z)$ and every $n \in IN$, $d(f^n(z), f^n(z^*)) < \delta$. This implies $D(f^n(N(z)) \leq 2\delta$ for every $n \in IN$. $N(z) \cap Per(f) \neq \varphi$, as Per(f) is dense. In view of Lemma 2.2, there exists $f^v(p) \in Orb(p)$ such that $d(f^v(p), q) \leq 4\delta$. Therefore, $\delta^0 \leq 4\delta$.
It will be seen later that $\delta^0/4$ need not be an upper bound for sensitivity factors for f.

Proposition 2.4: Let X be a metric space X, let $f : X \to X$ be TT and Per(f) is dense. Let $p \in Per(f)$ and $q \in X - Orb(p)$. Let $\delta^0 = d(q, Orb(p))$. For each ξ, $0 < \xi < \delta^0/4$, $\delta^0/4 - \xi$ is a a sensitivity constant for f.

Proof: It follows by Lemma 2.3.
The following is immediate.

Corollary 2.5: Let $f : X \to X$ be TT and Per(f) is dense. Let $p \in Per(f)$ and $q \in X - Orb(p)$. Let $\delta^0 = d(q, Orb(p))$. $\delta^0/4$ is a limit point of the set of sensitivity constant for f.
Let $p, q \in Per(f)$ be such that $q \in X - Orb(p)$. Let k and m be the periods of p and q respectively. For $0 \leq j \leq m-1$, let $\boldsymbol{\eta^j = d(f^j(q), Orb(p))}$.

Proposition 2.6: Let $f : X \to X$ be TT and Per(f) is dense. Let $p, q \in Per(f)$ be such that $q \in X - Orb(p)$. Let m be the period of q. Let $0 \leq j \leq m-1$. For each ξ, $0 < \xi < \delta^j/4$, $\delta^j/4 - \xi$ is a sensitivity constant for f.

Proof: In view of Theorem 1.9, $f^j(q) \in X - Orb(p)$. Now the result follows by Proposition 2.4.

Remark: Let $p, q \in Per(f)$ be such that $q \in X - Orb(p)$. Let k and m be the periods of p and q respectively. For $0 \leq j \leq m-1$, if we denote η^j by $\eta^j(q,p)$, then, since $f^j(q) \in X - Orb(p)$, we have $\eta^0(f^j(q), p) = \eta^j(q,p)$.

Remark 2.7: Let f : X → X be TT and Per(f) is dense. Let p, q ∈ Per(f) be such that q ∈ X – Orb(p). Let m be the period of q. Let $0 \le j \le m-1$. In view of Theorem 1.9, Lemma 2.3 holds with δ^0 replaced by δ^j. We can rewrite δ^j as γ^j such that $\gamma^0 \le \gamma^1 \le \ldots \le \gamma^{m-1}$. Thus we see that $\delta^j/4$ need not be an upper bound for sensitivity constants for f, also the converse of Lemma 2.3 need not be true. Also, $\eta^j/4$ need not be an upper bound for sensitivity constants for f.

Remark: We note that, for every j, $0 \le j \le m-1$, $\eta^j \le \delta^{max}$, and, for every i, $0 \le i \le k-1$, $\varsigma^i \le \delta^{max}$. δ^{max} may not be equal to any η^j or, ς^i.

Theorem 2.8: Let X be a metric space X, let f : X → X be TT and Per(f) is dense. Let p, q ∈ Per(f) be such that q ∈ X – Orb(p). If $\delta < \delta(p, q)/4$, then δ is a sensitivity constant for f, or equivalently for each ξ, $0 < \xi < \delta/4$, $\delta/4 - \xi$ is a a sensitivity factor for f.

Proof: Let m be the period of q. $\delta(p, q) = \delta^v$, for some v, $0 \le v \le m-1$. In view of Remark 4.7, we have the result by Lemma 2.3.

Remark 2.9: Let f: X → X be TT and Per(f) is dense. Let p, q ∈ Per(f). In view of Theorem 1.9, q ∈ X – Orb(p) iff p ∈ X – Orb(q). So we can interchange the role of p and q in the above considerations i.e. Proposition 2.6, Remark 2.7 and Theorem 2.8. Let k and m be the periods of p and q respectively. Proposition 2.6 holds with p and q interchanged and δ^j replaced by ς^i. Every $\delta < \delta(q, p)/4$ is a sensitivity constant for f, i.e. Theorem 2.8 holds with p and q interchanged. In fact, we have the following.

Theorem 2.10: Let f : X → X be TT and Per(f) is dense. Let p, q ∈ Per(f) be such that q ∈ X – Orb(p). Let $\delta^*(p,q) = \max\{\delta(p, q), \delta(q, p)\}$. If $\delta < \delta^*(p, q)/4$, then δ is a sensitivity constant for f.

Proof: In view of Remark 2.9, we can suppose that $\delta^*(p, q) = \delta(p, q)$. Now apply Theorem 2.8.

We note below that δ^{min} involved in Proposition 1.10, has a relation with sensitivity constants for f.

Proposition 2.11: Let X be a metric space X, let f : X → X be a function. Let p, q ∈ Per(f) with periods of p and q respectively, be such that q ∈ X – Orb(p). Let $\delta^{min} = \inf\{d(f^r(p)), d(f^s(q)) \mid 0 \le r \le k-1, 0 \le s \le m-1\}$.Then (a) for every j, $0 \le j \le m-1$, $\delta^{min} \le \delta^j$, and there exists some j^*, $0 \le j^* \le m-1$, such that $\delta^{min} = \delta^{j^*}$, (b) for every i, $0 \le i \le m-1$, $\delta^{min} \le \varsigma^i$, and there exists some i^*, $0 \le i^* \le k-1$, such that $\delta^{min} = \varsigma^{i^*}$.

Proof: For every j, $0 \le j \le m-1$, by Proposition 1.10(c), $\delta^{min} \le \delta^j$. There exit some i and j, with $0 \le i \le k-1$ and $0 \le j \le m-1$, such that $\delta^{min} = d(f^i(p)), d(f^j(q))$. $\delta^j = d(f^j(q),Orb(p)) \le d(f^j(q),f^i(p))) = \delta^{min}$. The other part follows in view of Remark 2.9.

Remark 2.12: We note δ^{min} appearing in Proposition 2.11 is $\min\{\delta^j \mid 0 \le j \le m-1\}$ and $\min\{\varsigma^i \mid 0 \le i \le k-1\}$.

Proposition 2.13: Let X be a metric space X, let f: X → X be a function. Let p, q∈ Per(f) with periods of p and q respectively, be such that q ∈ X – Orb(p). Let k and m be the periods of p and q respectively. (a) For every j, $0 \le j \le m-1$, there exists some i, $0 \le i \le k-1$ such that $\varsigma^i \le \delta^j$. (b) for every i, $0 \le i \le k-1$, there exists some j, $0 \le j \le m-1$ such that $\delta^j \le \varsigma^i$.

Proof: (a) $\delta^j = d(f^j(q),Orb(p)) = d(f^j(q),f^i(p))$ for some i, $0 \le i \le k-1$. $\varsigma^i = d(f^i(p),Orb(q)) \le d(f^i(p), f^j(q))$. Therefore $\varsigma^i \le \delta^j$. (b) follows similarly in view of Remark 2.9.

Let **Senc(f)** denote the set of all sensitivity constants for f.

Proposition 2.14: Let X be a metric space X, let f: X → X be TT and Per(f) is dense. Let p, q ∈ Per(f) with periods of p and q respectively, be such that q ∈ X – Orb(p). Let k and m be the periods of p and q respectively. There exists α such that $(-\infty,\alpha) \subset$ Senc(f).

Proof: $\sup\{\delta^j \mid 0 \le j \le m-1\} = \delta^v$, for some v, $0 \le v \le m-1$. In view of Remark 2.9, we can suppose that $\varsigma^i \le \delta^v$ for all i, $0 \le i \le k-1$. Since every real number not exceeding a sensitivity constant for f is a sensitivity constant for f, in view of Theorem 2.8, $(-\infty, \delta^v/4) \subset$ Senc(f), Take $\alpha = \delta^v/4$.

Remark 2.15: In Proposition 2.14, for given p, q ∈ Per(f) with q ∈ X – Orb(p), $(-\infty, \delta^v/4) \subset$ Senc(f). Writing $(-\infty, \delta^v/4)$ as S_{pq}, $\cup\{ S_{pq}$: p, q ∈ Per(f) with q ∈ X – Orb(p)} ⊂ Senc(f). There arises a question. Given a sensitivity constant δ for f , does there exist p, q ∈ Per(f) with q ∈ X – Orb(p) such that $\delta \in S_{pq}$? Or, is Senc(f) = $\cup\{S_{pq}$: p, q ∈ Per(f) with q ∈ X – Orb(p)}?

The following is a variation of Proposition 1.10.

Proposition 2.16: Let X be a metric space X, let $f: X \rightarrow X$ be TT and Per(f) is dense. Let p, q $\in$ Per(f) with periods of p and q respectively, be such that q $\in$ X – Orb(p). (a) For z $\in$ X, there exists some j^*, $0 \leq j^* \leq m-1$ such that $\delta^{j^*} \leq d(z,Orb(p)) + d(z,Orb(q))$ (b) If $z = f^j(q)$, where $0 \leq j \leq m-1$, then $j^* = j$.

Proof: (a) Let $0 \leq j \leq m-1$. For z $\in$ X, $d((f^j(q),f^i(p)) \leq d(z, (f^j(q)) + d(z, f^i(p))$ for every i, $0 \leq i \leq k-1$. We have $\inf\{d((f^j(q), f^i(p)) : 0 \leq i \leq k-1\} \leq \inf\{d(z,f^i(p)) : 0 \leq i \leq k-1\} + d(z,f^j(q))$. So $\delta^j = d(f^j(q),Orb(p)) \leq d(z, Orb(p)) + d(z,f^j(q))$. $d(z,Orb(q)) = d(z,f^{j^*}(q))$ for some j^*, $0 \leq j^* \leq m-1$. This implies that $\delta^{j^*} \leq d(z,Orb(p)) + d(z,Orb(q))$. (b) We note that in (a), for z $\in$ X, j^* is such that $d(z,Orb(q)) = d(z,f^{j^*}(q))$. For $z = f^j(q)$, $d(z,Orb(q)) = 0$, therefore, $d(f^j(q), f^{j^*}(q)) = 0$. Now by Lemma 1.1(i), $j^* = j$.

Remark 2.17: Comparing δ^{min} of Proposition 1.10 with δ^{j^*}`s of Proposition 2.13, note that δ^{min} of Proposition 1.10 and Proposition 2.11 is same for all z $\in$ X. But, in Proposition 2.13, there is δ^{j^*} for a given z $\in$ X. By Remark 2.12, δ^{min} is the smallest of δ^j, $0 \leq j \leq m-1$. There is no comparison of δ^{j^*} with δ^j, except when $z = f^j(q) \in Orb(q)$, in which case $\delta^{j^*} = \delta^j$. It may be added that, if we define $\delta^{max} = \max\{d(f^r(p)), d(f^s(q)) \mid 0 \leq r \leq k-1, 0 \leq s \leq m-1\}$, then, for every j, $0 \leq j \leq m-1$ $\delta^j \leq \delta^{max}$, and, for every i, $0 \leq i \leq k-1$, $\varsigma^i \leq \delta^{max}$. δ^{max} may not be equal to any δ^j or, ς^i.

5. CONCLUSION

The results regarding topological transitivity, dense orbits and sensitivity to initial conditions were obtained and should be used for further study of cases in Dynamical systems. The results obtained above are expected to be used for further study of orbits of points and transitivity of function in some dynamical systems. For continuous self maps of compact metric spaces, we initiated a preliminary study of stronger form of sensitivity. We have constructed the relation between topological transitivity, orbit of a point and sensitivity conditions.

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