International Journal of Mathematical Archive-8(11), 2017, 171-178
Available online through www.ijma.info ISSN 2229 – 5046

ABOUT ISOLATE DOMINATION IN GRAPHS

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(Received On: 09-10-17; Revised & Accepted On: 12-10-17)

ABSTRACT

In this paper we further study isolate domination in graphs. In particular we study the effect of removing an isolated vertex from the graph on the isolate domination number of the graph. We prove a necessary and sufficient condition under which the isolate domination number increases when an isolated vertex is remove from the graph. Further we also prove a necessary and sufficient condition under which the isolate domination number decreases when an isolated vertex is remove from the graph. It follows that if a graph G has an isolated vertex $v \ni \gamma_0(G-v) > \gamma_0(G)$then v is the only isolated vertex of the graph .For a non isolated vertex of a graph, we prove similar results.

Keywords: *isolate dominating set, minimal isolate dominating set, minimum isolate dominating set, isolate domination number, isolate inclusive set, 1-maximal isolate inclusive set, privateneighborhood.*

AMS Subject Classification (2010): *05C69.*

1. INTRODUCTION

The concept of isolate dominating set was studied in [3]. We defined the concept of isolate inclusive set in [1]. We proved that every 1-maximal isolate inclusive set is an isolate dominating set.

In this paper we further study isolate dominating sets in graphs .We consider the operation of removing a vertex from the graph and observe the effect on isolate domination number of the graph. First we consider the operation of removing an isolated vertex from the graph and prove necessary and sufficient conditions under which the isolate domination number increases or decreases when this operation is perform. Similarly we consider the operation of removing a non isolated vertex from the graph and prove a condition under which the isolate domination number increases or decreases.

2. PRELIMINARIES AND NOTATIONS

If G is a graph then $V(G)$ denotes the vertex set of the graph G and $E(G)$ denotes the edge set of the graph G. If v is vertex of the graph G then $G-v$ is the subgraph of G induced by all the vertices different from v. If x is a vertex of G then $d(x)$ will denote the degree of the vertex x in the graph G .

We will consider only simple undirected graphs with finite vertex set.

3. DEFINITIONS AND EXAMPLES

Definition 3.1 **(Isolate Inclusive set) [1]:** *Let G be a graph and S be a nonempty subset of $V(G)$ then the S is said to be an isolate inclusive set if the $< S >$ has an isolated vertex . An isolate inclusive set will be also called isoinc set.*

An isoinc with maximum cardinality is called a miximum isoinc set and its cardinality is denoted as $\beta_{is}(G)$.

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***Definition 3.2* (1-maximal isoinc set) [1]:** *Let G be a graph and S be a isoinc set of G then S is said to be a 1-maximal isoinc set if $S \cup \{v\}$ is not an isoinc set , for every $v \in V(G) - S$.*

Definition 3.3 (Minmax set): A maximal isoinc set with minimum cardinality is called a minmax set and its cardinality is denoted as $m_{is}(G)$ and its called the minmax number of the graph.

Let G be a graph & $v \in V(G)$ such that $d(v) = \nabla(G)$.

Now $V(G) - N(v)$ is an isoinc set of G but it did not be a 1-maximal isoinc set of G. This can be observed in following example.

Example 1: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

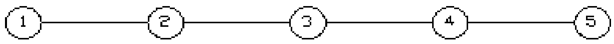

Figure-1: $G = P_5$

Consider the vertex 3 .

$d(3) = 2 = \nabla(G)$.

$N(3) = \{2, 4\}$ & $V(G) - N(3) = \{1, 3, 5\}$

This set is an isoinc set but it is not 1-maximal.

Definition 3.4 (Isolate Dominating Set) [3] *Let G be a graph and $S \subset V(G)$ then S is said to be an isolate dominating set if*

1. S is a dominating set and
2. $< S >$contains an isolated vertex.

An isolate dominating set with minimum cardinality is called a minimum isolate dominating set.

The cardinality of a minimum isolate dominating set is called the isolate domination number of the graph G and it is denoted as $\gamma_0(\mathrm{G})$.

Obviously for any graph G, $\gamma(\mathrm{G}) \leq \gamma_0(\mathrm{G})$ where $\gamma(\mathrm{G})$ denotes the domination number of the graph G.

Remark: Note that every 1-maximal isoinc set is an isolate dominating set but converse is not true.

We introduce the following symbols:

$$V_0^+ = \{x \in V(G) \ni \gamma_0(G - x) > \gamma_0(G)\}$$
$$V_0^- = \{x \in V(G) \ni \gamma_0(G - x) < \gamma_0(G)\}$$
$$V_0^0 = \{x \in V(G) \ni \gamma_0(G - x) = \gamma_0(G)\}$$

4. MAIN RESULT

Proposition 4.1: Let G be a graph &S be a 1-maximal isoinc set of G.

(1) For each isolated vertex v of S, $N(v) = V(G) - S$.

(2) If u & v are isolates of S then $(u) = d(v)$.

Proof:

(1) Let v be an isolated vertex of Sthen $N(v) \subset V(G) - S$.
Let $x \in V(G) - S$. Since S is 1-maximal, x is adjacent to every isolated vertex of S and therefore x is adjacent to v which implies that $x \in N(v)$. Thus $(v) = V(G) - S$.

(2) Let u & v be to isolatesof S then $(u) = |N(u)| = |V(G) - S| = |N(v)| = d(v)$.
Thus $d(u) = d(v)$.

Proposition 4.2: Let$G$ be a graph and $v \in V(G)$. If M is a isoinc set of $G - v$ then M is also an isoinc set of G .

Proof: Obviously.

Proposition 4.3: Let$G$ be a graph and $v \in V(G)$ then $\beta_{is}(G - v) \leq \beta_{is}(G)$.

Proof: Let M be a maximum isoinc set of $G - v$.

By the above proposition 4.2, M is also an isoinc set of G.

Therefore $\beta_{is}(G - v) \leq \beta_{is}(G)$

Example 2: Consider the path graph P_5 with vertices $\{1, 2, 3, 4, 5\}$

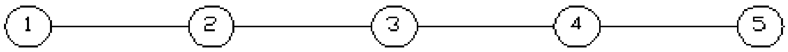

Figure-2: $G = P_5$

Here $\beta_{is}(G) = 4$.

Now consider the subgraph $G - 3$.

Figure-3

Here $\beta_{is}(G - 3) = 3$.

Therefore in this example $\beta_{is}(G - 3) < \beta_{is}(G)$.

Theorem 4.4: Let G be a graph and $v \in V(G)$ then $\beta_{is}(G - v) = \beta_{is}(G)$ if and only if there is a maximum isoinc set $M(G)$ such that $v \notin M$.

Proof: First suppose that $\beta_{is}(G - v) = \beta_{is}(G)$ that M be a maximum isoinc set of $G - v$.

Obviously, M is a isoinc set of G.

Since $\beta_{is}(G - v) = \beta_{is}(G)$.

M must be a maximum isoinc set of G.

Note that $\notin M$.

Conversely, Suppose that M is a maximum isoinc set of G such that $v \notin M$.

Now M is a subset of $G - v$ & it is also an isoinc set of $G - v$.

Therefore $\beta_{is}(G - v) \geq |M| = \beta_{is}(G)$.

It is also true that $\beta_{is}(G - v) \leq \beta_{is}(G)$.

Therefore $\beta_{is}(G - v) = \beta_{is}(G)$

Now we consider the effect of removing a vertex from a graph on the isolate domination number.

Example 3: Consider the cycle graph C_7 with 7 vertices $\{1, 2, 3, 4, 5, 6, 7\}$

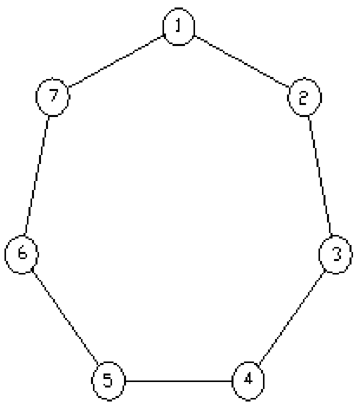

Figure-4: $G = C_7$

In this graph the set $\{1, 2, 5\}$ is a minimum isolate dominating set and therefore the isolate domination number is 3.

Now consider the graph $G - 7$ which is the path graph with 6 vertices $\{1, 2, 3, 4, 5, 6\}$ the isolate domination number is 2.

Thus the isolate domination number decreases in this graph.

Example 4: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

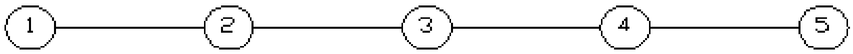

Figure-5: $\boldsymbol{G = P_5}$

The isolated domination number of this graph is 2. If we remove any vertex from the graph the isolate domination number of the resulting graph remains unchanged.

Example 5: Consider the following graph with 8 vertices $\{1, 2, 3, 4, 5, 6, 7, 8\}$

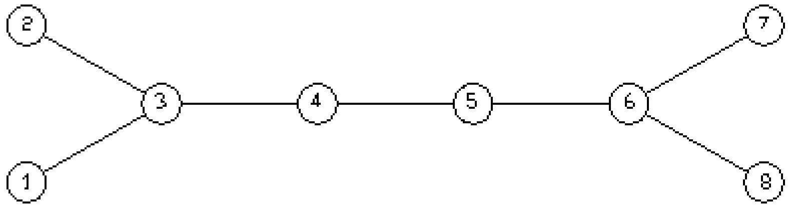

Figure-6

The isolate domination number of this graph is 2. If we remove the vertex 6 from the graph the isolate domination number of the resulting graph will be 4.

Thus the isolate domination number increases in this graph.

Now we state and prove an necessary and sufficient condition under which the removal of an isolated vertex increases the isolate domination number of a graph.

Theorem 4.5: Let G be a graph and v be an isolated vertex in G then $\gamma_0(G) < \gamma_0(G - v)$ if and only if for any minimum isolate dominating set S. The following two conditions are satisfied.

(1) $v \in S$ and
(2) v is the only isolate in the $< S >$.

Proof: suppose $\gamma_0(G - v) > \gamma_0(G)$.

If $v \notin S$ then v is not adjacent to any vertex of S which implies that S is not a dominating set.

Which is a contradiction.

Thus $v \in S$.

Hence condition (1) is satisfied.

Suppose u is anther vertex in S. Which is an isolate in the $< S >$.

Let $S_1 = S - \{v\}$.

Note that $u \in S_1$ and u is an isolate in $< S_1 >$.

Thus S_1 is an isolate dominating set in $G - v$.

Therefore $\gamma_0(G - v) \leq |S_1| < |S| = \gamma_0(G)$.

This is a contradiction and therefore (2) hold.

Conversely, Suppose (1) and (2) hold.

Let T be a set of vertices of $G - v$ such that $|T| < \gamma_0(G)$ and $< T >$ contains an isolate.

If T is an isolate dominating set in $G - v$ then $T_1 = T \cup \{v\}$ is an isolate dominating set of G.

Note that T is a minimum isolate dominating set in $G - v$ then T_1 is also a minimum isolate dominating set in G. Then T_1 is a minimum isolate dominating set of G containing atleast two distinct isolate in G.

This contradict condition (2).

Therefore any set T with $|T| < \gamma_0(G)$ cannot be a minimum isolate dominating set of $G - v$.

Suppose T is a set of vertices of $G - v$ such that $|T| = \gamma_0(G)$ and suppose T is a minimum isolate dominating set of $G - v$.

Now $T_1 = T \cup \{v\}$ is an isolate dominating set of G with $|T_1| = \gamma_0(G) + 1$. Also T is a proper subset of T_1 and therefore T_1 is not a minimal isolate dominating set of G.

Hence there is a vertex u in T_1 such that $T_1 - u$ is an isolate dominating set of G.

Also $|T_1 - u| = |T| = \gamma_0(G)$.

Further note that u cannot be an isolate in T_1 because otherwise $T_1 - u$ would not be an dominating set.

Thus $T_1 - u$ is a minimum isolate dominating set of G containing atleast two isolates.

Which is a contradiction.

Thus there is no set of T of vertices of $G - v$ such that $|T| = \gamma_0(G)$ and T is an isolate dominating set of $G - v$.

Therefore any isolate dominating set of $G - v$ must have cardinality $> \gamma_0(G)$

Therefore $\gamma_0(G - v) > \gamma_0(G)$

Theorem 4.6: Let G be a graph and v be an isolated vertex in G then $\gamma_0(G - v) < \gamma_0(G)$ if and only if there is a minimum isolate dominating set S which contains v and it also contains some other isolate.

Proof: Suppose $\gamma_0(G - v) < \gamma_0(G)$.

Let S_1 be a minimum isolate dominating set of $G - v$ then S_1contain an isolate.

Let $= S_1 \cup \{v\}$.

Then S is a minimum isolate dominating set of G and $v \in S$.

Further S contains two isolates one of them is v.

Conversely suppose that condition is satisfied.

Let $S_1 = S - \{v\}$ then by assumption S_1 contains an isolate. It is also isolate dominating set of $G - v$.

Therefore $\gamma_0(G - v) \leq |S_1| < |S| = \gamma_0(G)$

Corollary 4.7: Let G be a graph and $v_{1,}v_2, \dots \dots, v_k$ be all the isolated vertices of G $(k \geq 2)$ then $\gamma_0(G - v_i) < \gamma_0(G)$; for all $i = 1,2, \dots \dots \dots, k$

Proof: Let S be a minimum isolate dominating set of G then $v_i \in S$ for every $i = 1,2, \dots \dots \dots, k$ then by above theorem 4.6, $\gamma_0(G - v_i) < \gamma_0(G)$; for all $i = 1,2, \dots \dots \dots, k$

In view of the above corollary 4.7 and theorem 4.5

Corollary 4.8: If there is an isolated vertex $v \ni \gamma_0(G - v) > \gamma_0(G)$ then the graph has only one isolated vertex namely.

Proof: Obvious.

Now we consider the operation of removing a non isolated vertex from the graph on the isolate domination number of the graph.

Theorem 4.9: Let G be a graph and v be a non isolated vertex in G then $\gamma_0(G-v) > \gamma_0(G)$ if and only if the following two conditions are satisfied.

(1) $v \in S$, for every minimum isolate dominating set S of G.

(2) There is no subset S of $G-v$ such that $|S| \leq \gamma_0(G)$, S is a subset of $V(G)-N[v]$ and S is an isolate dominating set of $G-v$.

Proof: suppose $\gamma_0(G-v) > \gamma_0(G)$.

(1) Suppose there is a minimum isolate dominating set S of G such that $v \notin S$ then S is an isolate dominating set of $G-v$.
Therefore $\gamma_0(G-v) \leq |S| = \gamma_0(G)$.
That is $\gamma_0(G-v) \leq \gamma_0(G)$
Which is a contradiction.
Therefore $v \in S$, for every minimum isolate dominating set S of G.

(2) Suppose there is a subset S of $G-v$ such that $|S| \leq \gamma_0(G)$, S is a subset of $V(G)-N[v]$ and S is an isolate dominating set of $G-v$.
Then $\gamma_0(G-v) \leq |S| = \gamma_0(G)$ and therefore $\gamma_0(G-v) \leq \gamma_0(G)$.
Which is a contradiction.
Therefore condition (2) is also satisfied.

Conversely, Suppose condition (1) and (2) are satisfied.

First suppose that $\gamma_0(G-v) = \gamma_0(G)$.

Let S be a minimum isolate dominating set of $G-v$ then $|S| = \gamma_0(G-v) = \gamma_0(G)$.

Case (1): Suppose v is adjacent to some vertex of S then S is a minimum isolate dominating set of G not containing v.

Which contradiction is condition (1).

Case (2): Suppose v is not adjacent any vertex of S then $N[v] \cap S = \emptyset$ which is equivalent to the fact that S is subset of $V(G)-N[v]$ also $|S| \leq \gamma_0(G)$ and S is an isolate dominating set of $G-v$.

This again contradiction condition (2).

From case (1) and case (2) it follows that $\gamma_0(G-v) = \gamma_0(G)$ is not possible.

Suppose $\gamma_0(G-v) < \gamma_0(G)$.

Let S be a minimum isolate dominating set of $G-v$ that S cannot be isolate dominating set of G because $|S| < \gamma_0(G)$ this means that v is not adjacent to any other vertex of S then S is subset of $V(G)-N[v]$, $|S| \leq \gamma_0(G)$ and S is an isolate dominating set of $G-v$.

This again contradiction condition (2).

Therefore $\gamma_0(G-v) < \gamma_0(G)$ is also not possible.

Hence $\gamma_0(G-v) > \gamma_0(G)$

Proposition 4.10: Let G be a graph and v be a non isolated vertex of G if $\gamma_0(G-v) < \gamma_0(G)$ then $\gamma_0(G-v) = \gamma_0(G)-1$.

Proof: Let S_1 be a minimum isolate dominating set of $G-v$ then S_1 cannot be an isolate dominating set of G because $|S_1| < \gamma_0(G)$.

Let $S = S_1 \cup \{v\}$ then S is an isolate dominating set of G.

Since $\gamma_0(G-v) < \gamma_0(G)$, S must be a minimum isolate dominating set of G.

Thus $\gamma_0(G) = |S| = |S_1| + 1 = \gamma_0(G-v) + 1$

Now we state and prove a necessary and sufficient condition under which the isolate domination number decreases when a non isolated vertex is removed from the graph.

Theorem 4.11: Let G be a graph and v be a non isolated vertex in G then $\gamma_0(G-v) < \gamma_0(G)$ if and only if there is a minimum isolate dominating set S containing v and some other isolate such that $P_n[v,\ S] = \{v\}$.

Proof: Suppose $\gamma_0(G-v) < \gamma_0(G)$.

Let S_1 be a minimum isolate dominating set of $G-v$. Then S_1 cannot be an isolate dominating set of G.

It follows that v cannot be adjacent to any vertex of S_1.

Let $S = S_1 \cup \{v\}$ then S is aminimum isolate dominating set of $G$and $v \in S$.

Since v is not adjacent to any other vertex of S, $v \in P_n[v,\ S]$.

Suppose $x \neq v$ & $x \in P_n[v,\ S]$ then $x \notin S_1$. Since x is a vertex of $G-v$, x is adjacent to some vertex y of S_1.

Thus x is adjacent to v also.

Thus x is adjacent to two distinct vertices of G.

Which contradict the fact that $x \in P_n[v,\ S]$.

Therefore $P_n[v,\ S] = \{v\}$.

Obviously, S contains atleast two isolates and one of them is v.

Conversely, Suppose that there is a minimum isolate dominating set S of G such that $v \in S$ and $P_n[v,\ S] = \{v\}$ and S contains atleast two isolates.

Let $S_1 = S - \{v\}$ then S_1 is an isolate dominating set of $G-v$.

Therefore $\gamma_0(G-v) \leq |S_1| < |S| = \gamma_0(G)$.

Thus $\gamma_0(G-v) < \gamma_0(G)$

Corollary 4.12: Let G be a graph and u , v be two vertices of G such that $\gamma_0(G-v) > \gamma_0(G)$ and $\gamma_0(G-u) < \gamma_0(G)$ then u & v cannot be adjacent vertices.

Proof: By the above theorem 4.11, there is a minimum isolate dominating set Scontaining u such that $P_n[u,\ S] = \{u\}$ this means that u is not adjacent to any other vertex of S.

Since $\gamma_0(G-v) > \gamma_0(G)$, $v \in S$.

Therefore u is not adjacent to v

Theorem 4.13: Let G be a graph and v be a non isolated vertex in G then $\gamma_0(G-v) > \gamma_0(G)$ and let S be a minimum isolate dominating set of G which contains an isolate different from v then $v \in S$ and $P_n[v,\ S]$ contains two non adjacent vertices.

Proof: Since $\gamma_0(G-v) > \ \gamma_0(G)$, $v \in S$ by the theorem 4.9.

Since S is a minimal isolate dominating sets $P_n[v,\ S] \neq \emptyset$.

If $P_n[v\ ,\ S] = \{v\}$ then $\gamma_0(G-v) < \gamma_0(G)$ by the theorem 4.11.

Therefore there is a vertex $x \neq v$ & $\in P_n[v,\ S]$.

Suppose $P_n[v,\ S] = \{x\}$ then $x \notin S$.

Let $S_1 = S - \{v\} \cup \{x\}$ then S_1 is a minimum isolate dominating set of G not containing v.

This is a contradiction.

Suppose $P_n[v,\ S] = \{v, y\}$for some vertex y$\neq v$.

Let $S_1 = S - \{v\} \cup \{y\}$ then S_1 is a minimum isolate dominating set of G not containing v.

Which is again a contradiction.

Therefore $P_n[v\ ,\ S]$ contains atleast two distinct vertices different from v .

Suppose any two vertices in the $P_n[v,\ S]$ which are different from v are adjacent. Then let $y_{1,}\ y_2$ be two distinct vertices in the $P_n[v,\ S]$ such that $y_1 \neq v,\ y_2 \neq v$.

Now y_1 & y_2 are adjacent.

Let $S_1 = S - \{v\} \cup \{y_1\}$ then S_1 is a minimum isolate dominating set of G not containing v.

Which is again a contradiction.

Therefore there must be exist two distinct vertices in $P_n[v,\ S]$ which are non adjacent.

Thus the theorem is prove

Remark: Let G be a graph, $v \in V(G)$ such that $\gamma_0(G - v) > \gamma_0(G)$ and suppose there is a minimum isolate dominating set S containing an isolate different from .

By the above Theorem 4.12, $v \in S$ and $P_n[v,\ S]$ contains atleast two non adjacent vertices say v_1 & v_2

Note that $v_1 \neq v$, $v_2 \neq v$ and both v_1 & v_2 are adjacent to.

Since $v_1 \notin S$, $\gamma_0(G - v_1) > \gamma_0(G)$ is not possible. Also since v and v_1 are adjacent vertices, by corollary 4.11, $\gamma_0(G - v_1) < \gamma_0(G)$ is also not possible.

Therefore it must be true that $\gamma_0(G - v_1) = \gamma_0(G)$.

Similarly, $\gamma_0(G - v_2) = \gamma_0(G)$.

CONCLUDING REMARK

If G is a graph, $v \in V(G)$ and suppose G has a minimum isolate dominating set S which has an isolate different from v then v gives rises to two distinct vertices v_1 & v_2 $\ni$ $\gamma_0(G - v_i) = \gamma_0(G)$ or $i = 1{,}2$.

Thus it follows that for any graph G which contains a minimum isolate dominating set of S (with isolate u) and if there is a vertex v in V_0^+ then $V_0^0 \geq 2|V_0^+|$.

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Source of support: Nil, Conflict of interest: None Declared.