International Journal of Mathematical Archive-8(11), 2017, 167-170
IJMA Available online through www.ijma.info ISSN 2229 – 5046

CONVERGENCE OF LAGRANGE-HERMITE INTERPOLATION ON UNIT CIRCLE

SWARNIMA BAHADUR*[1], VARUN[2]

Department Of Mathematics and Astronomy, University Of Lucknow - 226007, India.

(Received On: 05-10-17; Revised & Accepted On: 02-11-17)

ABSTRACT

The aim of this paper is to study a Lagrange-Hermite interpolation on the nodes, which are obtained by projecting vertically the zeroes of the $(1-x^2)P_n(x)$ on the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial. We prove the regularity of the problem, give explicit forms and establish a convergence theorem for the same.

Mathematics Subject Classification: *41A05, 30E10.*

Keywords: *Legendre polynomial, Explicit representation, Convergence.*

1. INTRODUCTION

In 1991 Zi Yu Wang and Shan Ji Tian [6] considered the zeroes of $(1-x^2)P_{n-1}'(x)$, where $P_{n-1}'(x)$ is the derivative of $(n-1)^{th}$ Legendre polynomial and obtained the estimate for the same.

Later in 1994, Siqing Xie [5] showed regularity of (0, 1,…r-2,r) interpolation on the set obtained by projecting vertically the zeroes of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ onto the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for the n^{th} Jacobi polynomial. In 2011, Author[1] [1] presented a method for computing the convergence of (0; 0, 1) interpolation on unit circle.

In 2012, Giuseppe Mastroianni, Gradimir V. Milovanovic and Incoronata Notarangelo [4] considered a Lagrange–Hermite polynomial, interpolating a function at the Jacobi zeroes with its first $(r-1)$ derivatives at the points at ±1 and gave necessary and sufficient conditions on the weights for the uniform boundedness of the related operator.

In 2014, Author[1] (with M.Shukla) [2] considered a Lagrange-Hermite Interpolation on the nodes, which are obtained by vertically projected zeroes of the $(1-x^2)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for the Jacobi polynomial and studied explicit forms and established a convergence theorem for the same.

These have motivated us to consider the Lagrange-Hermite interpolation on the set of nodes on unit circle different from above.

In this paper, we considered a Lagrange Hermite interpolation on the nodes, which are obtained by projecting vertically the zeroes of the $(1-x^2)P_n(x)$ on the unit circle where $P_n(x)$ stands for n^{th}Legendre polynomial. Here the functions are prescribed at all points, whereas the derivatives only at ±1 .

In section 2, we give some preliminaries and in section 3, we describe the problem and give the existence theorem of the interpolatory polynomials, whereas in section 4, we give the explicit formulae of the interpolatory polynomials. Lastly in section 5 and 6, we give estimates and convergence of interpolatory polynomials respectively.

2. PRELIMINARIES

In this section we shall give some well known results, which we shall use.

(2.1) $\{z_0 = 1, z_{2n+1} = -1, z_k = \cos\theta_k + i\sin\theta_k, z_{n+k} = -z_k, k = 1(1)n\}$

be the vertical projections on unit circle of the zeroes of $(1-x^2)P_n(x)$, where $P_n(x)$stands for n^{th}Legendre polynomial having zeroes $x_k = cos\theta_{k,}\ k = 1(1)n$ such that $1 > x_1 > x_2 > \cdots > x_n > -1$.

Corresponding Author: Swarnima Bahadur[1],*
Department of Mathematics and Astronomy, University of Lucknow- 226007, India.

(2.2) $W(z)=\prod_{k=1}^{2n}(z-z_k) = K_n P_n\left(\frac{1+z^2}{2z}\right)z^n$

(2.3) $K_n = \frac{(2^n n!)}{(2n-1)!!}$

(2.4) $R(z) = (z^2-1)W(z)$

The differential equation satisfied by $P_n(x)$ is

(2.5) $(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$

Fundamentals polynomials of Lagrange Interpolation based on the nodes as the zeroes of $W(z)$ and $R(z)$ respectively are given by

(2.6) $L_k(z) = \frac{R(z)}{(z-z_k)R'(z_k)}, k = 0(1)2n+1$

(2.7) $L_{1k}(z) = \frac{W(z)}{(z-z_k)W'(z_k)}, k = 1(1)2n$

For $-1 \leq x \leq 1$ we have,

(2.8) $|z^2-1| = 2\sqrt{1-x^2}$

(2.9) $(1-x^2)^{\frac{1}{4}}|P_n(x)| \leq \sqrt{\frac{2}{\pi n}}$

Let $x_k's$ be the zeroes of $P_n(x)$, then

(2.10) $(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$

(2.11) $\left|P_n'(x_k)\right| \geq ck^{-\frac{3}{2}}n^2$

3. THE PROBLEM AND THE REGULARITY

Let $Z_n = \{z_k; k = 0(1)2n+1\}$ satisfying (2.1)

Here we are interested in determining the interpolatory polynomial $L_n(z)$ of degree$\leq$ 2n+3 satisfying the following conditions.

(3.1) $\begin{cases} L_n(f, z_k) = f(z_k) & , k = 0(1)2n+1 \\ L_n'(f, \pm 1) = \alpha_{\pm 1} \end{cases};$

where $f(z_k)$ and $\alpha_{\pm 1}$ are arbitrary complex constants.

We establish convergence theorem for the same.

Theorem 3.1: $L_n(z)$ is regular on Z_n.

Proof: It is sufficient if we show the unique solution of (3.1) is $L_n(z) \equiv 0$

In this case, consider $L_n(z) = R(z)q(z)$, where $q(z)$ is a polynomial of degree ≤ 1

Obviously, $L_n(z_k) = 0$ for $k = 0(1)2n+1$.

By $L_n'(\pm 1) = 0$, we get $q(\pm 1) = 0$.

Therefore, we have

(3.2) $q(z) = az + b$

Now for $z = 1$ and -1, we get $a = b = 0$

Hence the theorem follows.

4. EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS

We shall write,

(4.1) $L_n(z) = \sum_{k=0}^{2n+1} f(z_k)A_k(z) + \sum_{0,2n+1} \alpha_{\pm 1}B_k(z)$

where $A_k(z)$ and $B_k(z)$ are unique polynomial , each of degree at most $2n+3$ satisfying the conditions

(4.2) $\begin{cases} A_k(z_j) = \delta_{kj} & j,k = 0(1)2n+1 \\ A_k'(z_j) = 0, \quad k = 0(1)2n+1, & j = 0, 2n+1 \end{cases}$

(4.3) $$\begin{cases} B_k(z_j) = 0, \ j = 0(1)2n+1, k = 0,2n+1 \\ B_k'(z_j) = \delta_{kj} \qquad j,k = 0,2n+1 \end{cases}$$

Theorem 4.1: For $k = 0,2n+1$ we have

(4.4) $$B_k(z) = \frac{R(z)(z+z_k)}{4K_n}$$

Proof: Let $B_k(z) = (z^2-1)W(z)t(z)$, where $t(z)$ is a polynomial of degree one and $B_k(z)$ satisfying conditions given in (4.3)

Using (2.2) and (2.4), we have the theorem.

Theorem 4.2: For $k = 1(1)2n$

(4.5) $$A_k(z) = L_k(z) + \frac{(z+z_k)R(z)}{(z_k^2-1)R'(z_k)}$$

For $k = 0,2n+1$

(4.6) $$A_k(z) = (z+z_k)L_k(z)[\frac{1}{2z_k} - (\frac{1}{4z_k^2} + \frac{L_k'(z_k)}{2z_k})(z-z_k)]$$

Proof-: For $k = 1(1)2n$

Let $$A_k(z) = L_k(z) + c_k(z+z_k)R(z)$$
where c_k is a constant.Using conditions in (4.2), we have $c_k = \frac{1}{(z_k^2-1)R'(z_k)}$

Therefore, we have (4.5)

Now for $k = 0,2n+1$

Let $A_k(z) = (z+z_k)L_k(z)t_k(z)$, where $t_k(z)$ is a polynomial of degree one .Using conditions in (4.2), we have
$$t_k(z) = \frac{1}{2z_k} - \left(\frac{1}{4z_k^2} + \frac{L_k'(z_k)}{2z_k}\right)(z-z_k),$$

We have the desired result (4.6). Hence the theorem follows.

5. ESTIMATION OF FUNDAMENTAL POLYNOMIALS

Lemma 5.1 [5]: Let $L_k(z)$ be given by (2.6)
$$\max_{|Z|=1} \sum_{k=0}^{2n+1} |L_k(z)| \leq c\log n$$

Lemma 5.2: Let $B_k(z)$ be given by (4.4), then for $k = 0, 2n+1$
$$|B_k(z)| \leq \sqrt{\frac{2}{\pi n}}$$

Proof: From (4.4) we have
$$|B_k(z)| \leq \frac{1}{2\sqrt{1-x^2}} |P_n(x)||z+z_k|.$$

Using (2.2), (2.6) and (2.9),

We have lemma.

Lemma 5.3: Let $A_k(z)$ be given in theorem (4.2), then
$$\sum_{k=0}^{2n+1} |A_k(z)| \leq cn^{\frac{1}{2}}\log n$$, where c is a constant independent of z and .

Proof-: Using (4.5) we have

(5.1) $$\sum_{k=0}^{2n} |A_k(z)| \leq cn^{\frac{1}{2}}\log n ,$$

Using (2.8), (2.10) and lemma 5.1

For $k = 0,2n+1$,

$$|A_k(z)| \leq \frac{3}{2}|L_k(z)| + |L_k(z)||L_k{}'(z_k)|$$

(5.2) $$|A_k(z)| \leq cn^{\frac{1}{2}}$$

Combining (5.1) and (5.2) we have the theorem.

6. CONVERGENCE

Theorem 6.1-: Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Let the arbitrary numbers $\alpha_{\pm 1}$'s be such that

(6.1) $$|\alpha_{\pm 1}| = O\left(n\omega_2\left(f, \frac{1}{n}\right)\right)$$

Then $\{L_n(z)\}$ defined by

(6.2) $$L_n(z) = \sum_{k=0}^{2n+1} f(z_k)A_k(z) + \sum_{0,2n+1} \alpha_{\pm 1}B_k(z)$$

satisfies the relation,

(6.3) $|L_n(z) - f(z)| = O(\omega_2(f, n^{-1})n^{\frac{1}{2}}logn)$, where $\omega_2(f, n^{-1})$ be the second modulus of continuity of $f(z)$.

Remark 6.1: Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$ and $f' \in Lip\alpha,\ \alpha > 0$, then the sequence $\{L_n(z)\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, which follows from (6.3) as

(6.4) $$\omega_2(f, n^{-1}) \leq n^{-1}\omega_1(f', n^{-1}) = O(n^{-1-\alpha}),$$

To prove the theorem (6.1), we shall need following

Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, Then there exists a polynomial $F_n(z))$ degree $\leq 2n+3$ satisfying Jackson's inequality.

(6.5) $$|f(z) - F_n(z)| \leq c\omega_2(f, n^{-1}), \qquad z = e^{i\theta}(0 \leq \theta < 2\pi)$$

Also an inequality due to O.Kiŝ [3]

(6.6) $\left|F_n^{(m)}(z)\right| \leq c\, n^m\omega_2(f, n^{-1}),\ m\,\mathcal{E}\, I^+$, where c is a constant.

Proof: Since $L_n(z)$ be the uniquely determined of degree $\leq 2n+3$ and the polynomial $F_n(z)$ satisfying (6.5) and (6.6) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k)A_k(z) + \sum_{k=0,2n+1} F_n'(z_k)B_k(z)$$

$$\begin{aligned}|L_n(z) - f(z)| &\leq |L_n(z) - F_n(z)| + |F_n(z) - f(z)| \\ &\leq \sum_{k=0}^{2n+1}|f(z_k) - F_n(z_k)||A_k(z)| + \sum_{k=0,2n+1}\left(|\alpha_{\pm 1}| + |F_n'(z_k)|\right)|B_k(z)| \\ &\quad + |F_n(z) - f(z)|\end{aligned}$$

Using (6.1), (6.4), (6.5) lemma 5.2 and lemma 5.3, we have the theorem 6.1.

REFERENCES

1. Bahadur.S; (0; 0, 1) Interpolation on the Unit Circle, International J. Math .Anal. Vol 5,2011, No. 29, 1429-1434
2. Bahadur.S and Shukla.M; Convergence Of Lagrange-Hermite Interpolation, Italian J. Pure and Appl. Math, 33,2014,255-262
3. Kiŝ.O; Remarks On Interpolation (Russian); Acta Math. Acad.Sci.Hunger., 11, 1960,49-64.
4. Mastroianni Giuseppe, Milovanovic Gradimir V. and Notarangelo Incoronata; On An Interpolation Process Of Lagrange-Hermite Type, Publications De L'Institut Mathematique, Nouvelle Serie Tome 91(105) (2012), 163-175.
5. Xie.S; Regularity Of $(0,1, \ldots \ldots r-2, r)$ and $(0,1, \ldots \ldots r-2, r)^*$ Interpolation On Some Sets Of Unit Circle, J.Approx.Theory, 1995, 82(1), 54-59.
6. Wang Zi Yu and Tian Ji Shan; On The Pointwise Estimations of Approximations of Functions And Their Derivatives By Hermite Interpolation Polynomial, Chinese J. Math 19(1991), no.3, 243-256.

Source of support: Nil, Conflict of interest: None Declared.