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GLOBAL EXISTENCE OF SOLUTIONS FOR NONLINEAR NONLOCAL VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT

We prove the global existence of solutions of nonlinear second order nonlocal Volterra integrodifferential equations in a Banach space. The technique used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative and rely on a priori bounds of solutions.

1. INTRODUCTION

Let X be a Banach space with norm $\|\,.\,\|$. Let $B = C([t_0,\ t_0+\beta], X)$ be the Banach space of all continuous functions from $[t_0,\ t_0+\beta]$ into X endowed with supremum norm

$$\| x \|_B = \sup\{\| x(t) \| : t \in [t_0,\ t_0+\beta]\}.$$

In this paper, we discuss the global existence of solution for nonlinear Volterra integrodifferential equation with nonlocal condition of the type:

$$[r(t)x'(t)]' = f(t,\ x(t),\ \int_{t_0}^{t} k(t,\ s)h(s,\ x(s))ds,\ t \in [t_0,\ t_0+\beta], \tag{1.1}$$

$$x(t_0) + g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot)) = x_0,\ x_0(t_0) = 0, \tag{1.2}$$

where $0 \le t_0 < t_1 < t_2 < \cdots < t_p \le t_0+\beta$, $(p \in N)$, $r(t)$ is a real-valued positive and sufficiently smooth function defined on $[t_0,\ t_0+\beta]$, $f : [t_0,\ t_0+\beta] \times X \times X \to X$, $h : [t_0,\ t_0+\beta] \times X \to X$, $k : [t_0,\ t_0+\beta] \times [t_0,\ t_0+\beta] \to R$, $g(t_1,\ t_2,\ \cdots,\ t_p,\ \cdot) : X \to X$ are functions and x_0 is a given element of X.

The notion of 'nonlocal condition' has been introduced to extend the study of the classical initial value problems, see, for example [1, 2, 3, 5]. It is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial time. The study of abstract nonlocal initial value problem was initiated by Byszewski [4]. In [4], By szewski using the method of semigroups and the Banach fixed point theorem proved the existence and uniqueness of mild, strong and classical solution of first order IVP:

$$u'(t) + Au(t) = f(t,\ u(t)), \qquad t \in [t_0,\ t_0+a], \tag{1.3}$$

$$u(t_0) + g(t_1,\ t_2,\ \cdots,\ t_p,\ u(\cdot)) = u_0, \tag{1.4}$$

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where $0 \le t_0 < t_1 < t_2 < \cdots < t_p \le t_0 + a,\ (p \in N)$, $u_0 \in X$, $-A$ is the infinitesimal generator C_0 semigroup of $T(t),\ t \ge 0$ on a Banach space X and $f : [t_0,\ t_0 + a] \times X \to X$, $g(t_1,\ t_2,\ \cdots,\ t_p,\ \cdot) : X \to X$ are given functions. The symbol $g(t_1,\ t_2,\ \cdots,\ t_p,\ u(\cdot))$ is used in the sense that in the place of '$\cdot$' we can substitute only elements of the set $\{t_1,\ t_2,\ \cdots,\ t_p\}$. For example $g(t_1,\ t_2,\ \cdots,\ t_p,\ u(\cdot))$ can be defined by the formula

$$g(t_1,\ t_2,\ \cdots,\ t_p,\ u(\cdot)) = C_1 u(t_1) + C_2 u(t_2) + \ldots + C_p u(t_p),$$

where $C_i\ (i = 1, 2, \cdots, p)$ are given constants.

Several authors have studied the problems such as existence, uniqueness, boundedness and other properties of solutions of these equations (1.1)–(1.2) or their special forms by using various techniques, see [6, 7, 9, 10, 11, 12, 13] and the references cited therein. In an interesting paper [7], M. B. Dhakne and G. B. Lamb have studied the global existence of solutions of (1.1) with classical condition $x(0) = x_0$. We are motivated by the work of M. B. Dhakne and G. B. Lamb [7] and influenced by the work of Byszewski [4]. The results obtained in this paper generalize the some result of [7].

The aim of the present paper is to study the global existence of solutions of the equations (1.1)–(1.2). The tools used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence.

The paper is organized as follows. In section 2, we present the preliminaries and the statement of our result. Section 3 deals with proof of result.

2. PRELIMINARIES AND RESULT

Before proceeding to the statement of result, we shall set forth some preliminaries and hypotheses that can be used in our further discussion.

Definition 2.1: Let $f \in L^1(t_0,\ t_0 + \beta; X)$. The function $x \in B$ given by

$$x(t) = [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} f(\tau,\ x(\tau), \int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma,\ x(\sigma)) d\sigma) d\tau ds \qquad (2.1)$$

is called the solution of the initial value problem (1.1)–(1.2).

Theorem 2.2: ([8], p-61) Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let $\in (F) = \{x \in S : x = \lambda F x \quad \text{for some } 0 < \lambda < 1\}$. Then either $\in (F)$ is unbounded or F has a fixed point.

We list the following hypotheses for our convenience.

(H_1) There exists a constant G such that

$$G = \max_{x \in B} \left\| g(t_1,\ t_2, \cdots,\ t_p,\ x(\cdot)) \right\|.$$

(H_2) There exists a continuous function $p : [t_0,\ t_0 + \beta] \to R^+$ and a continuous nondecreasing function $\Omega_1 : R^+ \to (0, \infty)$ such that

$$\left\| f(t,\ x,\ y) \right\| \le p(t) \Omega_1(\|x\| + \|y\|), \text{ for every } t \in [t_0,\ t_0 + \beta] \text{ and } x,\ y \in X\ .$$

(H_3) There exists a continuous function $q:[t_0,\ t_0+\beta]\to R^+$ and a continuous nondecreasing function $\Omega_2 : R^+ \to (0,\infty)$ such that

$$\|h(t,\ x(t))\| \le q(t)\Omega_2(\|x\|),\ \text{for every}\ t\in[t_0,\ t_0+\beta]\ \text{and}\ x\in X\ .$$

(H_4) There exists a constant M such that

$$|k(t,\ s)| \le M, \quad \text{for}\ t \ge s \ge t_0.$$

(H_5) For each $t\in[t_0,\ t_0+\beta]$ the function $f(t,\cdot,\cdot):[t_0,\ t_0+\beta]\times X\times X\to X$ is continuous and for each $x,\ y\in X$ the function $f(\cdot,x,y):[t_0,\ t_0+\beta]\times X\times X\to X$ is strongly measurable.

(H_6) For each $t\in[t_0,\ t_0+\beta]$ the function $h(t,\cdot):[t_0,\ t_0+\beta]\times X\to X$ is continuous and for each $x\in X$ the function $h(\cdot,x):[t_0,\ t_0+\beta]\times X\to X$ is strongly measurable.

(H_7) For every positive integer m there exists $\alpha_m\in L^1(t_0,t_0+\beta)$ such that

$$\sup_{\|x\|\le m,\ \|y\|\le m} \|f(t,x,y)\| \le \alpha_m(t), \quad t\in[t_0,t_0+\beta] \quad \text{a. e.}$$

With these preparations we are now in a position to state our result to be proved in this paper.

Theorem 2.3: Suppose that the hypotheses (H_1) – (H_7) are satisfied. Then initial value problem (1.1)–(1.2) has a solution x defined on $[t_0,t_0+\beta]$ provided β satisfies

$$\int_{t_0}^{t_0+\beta} N(s)ds < \int_{c}^{\infty} \frac{ds}{[\beta\Omega_1(s)+\Omega_2(s)]}, \tag{2.2}$$

where $(\|x_0\|+G)\ =c$ and $P=\max\limits_{t\in[t_0,\ t_0+\beta]}\{p(t)\}$ and $N(t)\ =\ \max\limits_{t\in[t_0,\ t_0+\beta]}\{\frac{1}{r(t)}P, Mq(t)\}$.

3. PROOF OF THEOREM

Proof: First we establish the priori bounds for the initial value problem $(1.1)_\lambda-(1.2)$, $\lambda\in(0,1)$ where

$$[r(t)x'(t)]' = \lambda f(t,\ x(t),\ \int_{t_0}^{t} k(t,\ s)h(s,\ x(s))ds),\ t\in[t_0,\ t_0+\beta]. \tag{1.1$_\lambda$}$$

Let $x(t)$ be a mild solution of the initial value problem $(1.1)_\lambda-(1.2)$. Then it satisfies the equivalent integral equation

$$x(t)\ =\ [\ x_0-g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))]\ +\lambda\int_{t_0}^{t}\frac{1}{r(s)}\int_{t_0}^{s} f(\tau,\ x(\tau),\int_{t_0}^{\tau} k(\tau,\sigma)h(\sigma,\ x(\sigma))d\sigma)d\tau ds \tag{3.1}$$

By using (3.1), hypotheses (H_1) – (H_7) and the fact that $\lambda\in(0,1)$, we have

$$\begin{aligned}\| x(t)\| &\le (\| x_0\| + G) + \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} \| f(\tau, x(\tau), \int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma, x(\sigma)) d\sigma) \| d\tau ds \\ &\le (\| x_0\| + G) + \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} p(\tau)\Omega_1[\| x(\tau)\| + \int_{t_0}^{\tau} | k(\tau,\sigma)| \ \| h(\sigma, x(\sigma))\| d\sigma] d\tau ds \\ &\le (\| x_0\| + G) + \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} p(\tau)\Omega_1[\| x(\tau)\| + \int_{t_0}^{\tau} Mq(\tau)\Omega_2(\| x(\sigma)\|) d\sigma] d\tau ds. \end{aligned} \tag{3.2}$$

Denote the right side of the above inequality by $u(t)$, then

$$\| x(t)\| \le u(t), \quad u(t_0) = (\| x_0\| + G) = c$$

and

$$\| u(t)\| \le (\| x_0\| + G) + \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} p(\tau)\Omega_1[u(\tau) + \int_{t_0}^{\tau} Mq(\tau)\Omega_2(u(\sigma)) d\sigma] d\tau ds. \tag{3.3}$$

Clearly $u(t)$ is an increasing function. Differentiating $u(t)$, we obtain

$$u'(t) \le \frac{1}{r(s)} \int_{t_0}^{t} p(s)\Omega_1[u(s) \int_{t_0}^{s} Mq(\tau)\Omega_2(u(\tau)) d\tau] ds. \tag{3.4}$$

Setting $v(t) = u(t) + \int_{t_0}^{t} Mq(s)\Omega_2(u(s))ds$, then $u(t) \le v(t)$, $v(t_0) = u(t_0) = c$ and the equation (3.4) becomes

$$v'(t) \le \frac{1}{r(t)} \int_{t_0}^{t} p(s)\Omega_1(v(s)) ds. \tag{3.5}$$

Differentiating $v(t)$ and using the equation (3.5), we get

$$\begin{aligned} v'(t) &= u'(t) + Mq(t)\Omega_2(u(t)) \\ &\le \frac{1}{r(t)} \int_{t_0}^{t} p(s)\Omega_1(v(s)) ds + Mq(t)\Omega_2(v(t)) \\ &\le \frac{1}{r(t)} P\beta\, p(t)\Omega_1(v(t)) + Mq(t)\Omega_2(v(t)) \end{aligned}$$

$$\frac{v'(t)}{\beta\Omega_1(v(t)) + \Omega_2(v(t))} \le N(t), \tag{3.6}$$

where

$$P = \max_{t\in[t_0, t_o+\beta]} \{p(t)\} \quad \text{and} \quad N(t) = \max_{t\in[t_0, t_o+\beta]} \{\frac{1}{r(t)} P, Mq(t)\}.$$

Integrating the equation (3.6) from t_0 to t and use of the change of variable $t \to s = v(t)$, and the condition (2.2), we have

$$\int_{c}^{v(t)} \frac{ds}{[\beta\Omega_1(s)+\Omega_2(s)]} \le \int_{t_0}^{t} N(s)ds \le \int_{t_0}^{t_0+\beta} N(s)ds < \int_{c}^{\infty} \frac{ds}{[\beta\Omega_1(s)+\Omega_2(s)]}. \qquad (3.7)$$

From (3.7) we conclude by the mean value theorem that there exists a constant γ independent of $\lambda \in (0,1)$ such that $v(t) \le \gamma$, for $t \in [t_0, t_0+\beta]]$. Then we have successively $u(t) \le \gamma$, $\|x(t)\| \le \gamma$ for $t \in [t_0, t_0+\beta]$ and consequently $\|x\|_B = \sup\{\|x(t)\| : t \in [t_0, t_0+\beta]\} \le \gamma.$

Now, we rewrite the problem (1.1)–(1.2) as follows: If $y \in B$ and

$x(t) = [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y(t)$, $t \in [t_0, t_0+\beta]$ where $y(t)$ satisfies $y(t_0)=0$;

$$y(t) = \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} f(\tau, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y(\tau),$$

$$\int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y(\sigma)) d\sigma) d\tau ds,$$

if and only if $x(t)$ satisfies

$$x(t) = [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} f(\tau, x(\tau), \int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma, x(\sigma)) d\sigma) d\tau ds.$$

Define $F : B_0 \to B_0$, $B_0 = \{y \in B : y(t_0) = 0\}$ by

$$(Fy)(t) = \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} f(\tau, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y(\tau),$$

$$\int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y(\sigma)) d\sigma) d\tau ds, \quad t \in [t_0, t_0+\beta]. \qquad (3.8)$$

Next, we shall prove that $F : B_0 \to B_0$ is continuous. Let $\{y_n\}$ be a sequence of elements of B_0 converging to y in B_0. Then from (3.8), we get

$$(Fy_n)(t) = \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} f(\tau, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y_n(\tau),$$

$$\int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y_n(\sigma)) d\sigma) d\tau ds, \quad t \in [t_0, t_0+\beta].$$

Since $\{y_n\}$ be the sequence of elements of B_0 converging to y in B_0 and by hypotheses (H_5) – (H_7), we have

$$f(\tau, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y_n(\tau),$$

$$\int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y_n(\sigma)) d\sigma d\tau ds$$

$$\to f(\tau, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y(\tau),$$

$$\int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma, [x_0 - g(t_1, t_2, \cdots, t_p, x(\cdot))] + y(\sigma)) d\sigma d\tau ds$$

for each $t \in [t_0,\ t_0+\beta]$.

Now, $\| Fy_n - Fy \|_B = \sup_{t\in[t_0,t_0+\beta]} \| (Fy_n)(t) - (Fy)(t) \|$. By using hypotheses ($H_6$) – ($H_7$) and the dominated convergence theorem, we have

$$\| Fy_n - Fy \|_B \le \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} \| [f(\tau,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y_n(\tau),$$
$$\int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y_n(\sigma)) d\sigma) d\tau ds$$
$$- f(\tau,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\tau),$$
$$\int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\sigma)) d\sigma) d\tau ds] \| \to 0$$

and consequently $\| Fy_n - Fy \|_B \to 0$ as $n \to \infty$ i.e. $Fy_n \to Fy$ in B_0 as $y_n \to y \in B_0$. Thus F is continuous.

Now, we prove that F maps a bounded set of B_0 into a precompact set of B_0. Let

$B_m = \{ y \in B_0 : \| y \|_B \le m \}$ for $m \ge 1$. We first show that F maps B_m into an equicontinuous family of functions with values in X. Let $y \in B_m$ and $t_0 \le \theta_1 < \theta_2 < t_0 + \beta$. Using hypotheses ($H_1$) – ($H_4$), we obtain

$$\| (Fy)(\theta_1) - (Fy)(\theta_2) \| \le \int_{\theta_1}^{\theta_2} \frac{1}{r(s)} \int_{t_0}^{s} \| f(\tau,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\tau),$$
$$\int_{t_0}^{\tau} k(\tau,\sigma) h(\sigma,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\sigma)) d\sigma) \| d\tau ds$$
$$\le \int_{\theta_1}^{\theta_2} \frac{1}{r(s)} \int_{t_0}^{s} p(\tau) \Omega_1 [\, \| x_0 \| + \| g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot)) \| + \| y(\tau) \|$$
$$+ \int_{t_0}^{\tau} | k(\tau,\sigma) | \, \| h(\sigma,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\sigma)) \| d\sigma] d\tau ds$$
$$\le \int_{\theta_1}^{\theta_2} \frac{1}{r(s)} \int_{t_0}^{s} p(\tau) \Omega_1 [\| x_0 \| + G + m + \int_{t_0}^{\tau} M q(\sigma) \Omega_2 [\| x_0 \| + G + m] d\sigma] d\tau ds$$
$$\le \int_{\theta_1}^{\theta_2} N(s) \int_{t_0}^{s} \Omega_1 [k^* + \mathrm{N}(\tau) \beta \Omega_2 (k^*)] d\tau ds$$
$$\le \int_{\theta_1}^{\theta_2} N^* \beta \Omega_1 [k^* + N^* \beta \Omega_2 (k^*)] ds$$
$$\le N^* \beta \Omega_1 [k^* + N^* \beta \Omega_2 (k^*)] (\theta_2 - \theta_1), \tag{3.9}$$

where $k^* = (\|x_0\| + G + m)$ and $N^* = \sup_{t\in[t_0, t_0+\beta]}\{N(t)\}$. Hence the right hand side of (3.9) tends to zero as $(s-t) \to 0$. Thus FB_m is an equicontinuous family of functions with values in X.

We next show that FB_m is uniformly bounded. From the definition of F in (3.8) and using hypotheses (H_1) – (H_4), (H_7) and the fact that $\|y\|_B \le m$, we obtain

$$\begin{aligned}
\|(Fy)(t)\| &\le \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} p(\tau)\Omega_1[\|x_0\| + G + m + \int_{t_0}^{\tau} Mq(\sigma)\Omega_2[\|x_0\| + G + m]d\sigma]d\tau ds \\
&\le \int_{t_0}^{t} N(s) \int_{t_0}^{s} \Omega_1[k^* + \mathrm{N}(\tau)\beta\Omega_2(k^*)]d\tau ds \\
&\le \int_{t_0}^{t} N^*\beta\Omega_1[k^* + N^*\beta\Omega_2(k^*)]ds \\
&\le N^*\beta^2\Omega_1[k^* + N^*\beta\Omega_2(k^*)].
\end{aligned}$$

This implies that the set $\{(Fy)(t) : \|y\|_B \le m,\ t_0 \le t \le t_0+\beta\}$ is uniformly bounded in X and hence $\{F_{B_m}\}$ is uniformly bounded.

We have already shown that F_{B_m} is an equicontinuous and uniformly bounded collection. To prove the set F_{B_m} is precompact in B, it is sufficient, by Arzela-Ascoli's argument, to show that the set $\{(Fy)(t) : \|y\|_B \le m\}$ is precompact in X for each $t\in[t_0,\ t_0+\beta]$. Let t_0<t< $t_0+\beta$ be fixed and $\in$ a real number satisfying $t_0 < \in <$t . For $y \in B_m$, we define

$$\begin{aligned}
(F_\in y)(t) = \int_{t_0}^{t-\in} \frac{1}{r(s)} \int_{t_0}^{s} & f(\tau,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\tau), \\
& \int_{t_0}^{\tau} k(\tau,\sigma)h(\sigma,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\sigma))d\sigma)d\tau ds. \qquad (3.10)
\end{aligned}$$

The set F_{B_m} is bounded in B, the set $Y_\in(t) = \{(F_\in y)(t) : y \in B_m\}$ is precompact in X for every $\in$, $t_0 < \in < t$. Moreover for every $y \in B_m$, we get

$$\begin{aligned}
(Fy)(t) - (F_\in y)(t) = \int_{t-\in}^{t} \frac{1}{r(s)} \int_{t_0}^{s} & f(\tau,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\tau), \\
& \int_{t_0}^{\tau} k(\tau,\sigma)h(\sigma,\ [\, x_0 - g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))] + y(\sigma))d\sigma)d\tau ds. \qquad (3.11)
\end{aligned}$$

By making use of hypotheses (H_1) – (H_7) and the fact that $\|y(s)\| \le m$, we have

$$
\begin{aligned}
\|(Fy)(t)-(F_{\in}y)(t)\| &\le \int_{t-\in}^{t}\frac{1}{r(s)}\int_{t_0}^{s} p(\tau)\Omega_1[\|x_0\|+\|g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))\|+\|y(\tau)\| \\
&\quad +\int_{t_0}^{\tau}|k(\tau,\sigma)|\ \|h(\sigma,\ [\ x_0-g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))]+y(\sigma))\|\,d\sigma]d\tau ds \\
&\le \int_{t-\in}^{t}\frac{1}{r(s)}\int_{t_0}^{s} p(\tau)\Omega_1[\|x_0\|+G+m+\int_{t_0}^{\tau}Mq(\sigma)\Omega_2[\|x_0\|+G+m]d\sigma]d\tau ds \\
&\le \int_{t-\in}^{t}N(s)\int_{t_0}^{s}\Omega_1[k^*+\mathrm{N}(\tau)\beta\Omega_2(k^*)]d\tau ds \\
&\le \int_{t-\in}^{t}N^*\beta\Omega_1[k^*+N^*\beta\Omega_2(k^*)]ds \\
&\le N^*\beta\Omega_1[k^*+N^*\beta\Omega_2(k^*)]\in .
\end{aligned}
$$

This shows that there exists precompact sets arbitrarily close to the set $\{(Fy)(t)\ :y\in B_m\}$. Hence the set $\{(Fy)(t)\ :y\in B_m\}$ is precompact in X . Thus we have shown that F is completely continuous operator.

Moreover, the set

$$\in(F)\ =\{y\in B_0 : y=\lambda Fy \quad \text{for some } 0<\lambda<1\},$$

is bounded in B, since for every y in $\in(F)$, the function $x(t)\ =\ [\ x_0-g(t_1,\ t_2,\ \cdots,\ t_p,\ x(\cdot))]+y(t)$ is a solution of $(1.1)_\lambda-(1.2)$ for which we have proved $\|x\|_B\le\gamma$ and hence $\|y\|_B\le\gamma+(\|x_0\|+G)$. Now, by virtue of Theorem 2.2, the operator F has a fixed point in B_0. Therefore, the initial value problem (1.1)–(1.2) has a solution on $[t_0,\ t_0+\beta]$. This completes the proof of the Theorem 2.3.

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