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CONSTRUCTION OF RECTANGULAR DESIGNS FROM GENERALIZED ORTHOGONAL CONSTANT COLUMN MATRICES

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ABSTRACT

A series of Rectangular designs has been constructed from Generalized Row Orthogonal Constant Column Matrices (GROCM).It is shown that in general, a GROCM is an incidence matrix of a Rectangular Design.

MSC: *05B05*

Keywords: *Rectangular Designs, Generalized Row Orthogonal Constant Column Matrices (GROCM), Hadamard Matrix, Circulant Matrix.*

1. INTRODUCTION

1.1 Normalized Hadamard Matrix-A square matrix H of order n and entries 1,-1 is called a Hadamard matrix if $HH^T=nI_n$ where I_n is an nxn identity matrix. A Hadamard matrix is in normalized form if its first row and first column have all entries 1[3].

1.2 Generalized Hadamard Matrix:
A Generalized Hadamard matrix GH (nq, G) over the group G of order n is an $nq \times nq$ matrix

GH (nq,G)= $\left(h_{ij}\right)$ such that

(i) $h_{ij} \in G \ \forall i, j \in \{1, 2, ..., nq\}$

(ii) $\sum_{l=1}^{nq} h_{il} h_{jl}^{-1} = \sum_{g \in G} qg \ whenever \ i \neq j$ where the summation belongs to the group ring Z[G].

1.3 Circulant Matrix

An $n \times n$ matrix C= $\left[c_{ij}\right]_{0 \le i, j \le n-1}$ where $c_{ij} = c_{j-i(\text{mod}\, n)}$ is a circulant matrix of order n.

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & . & . & . & c_{n-1} \\ c_{n-1} & c_0 & c_1 & . & . & . & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & . & . & . & c_{n-3} \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ c_1 & c_2 & c_3 & . & . & . & c_0 \end{pmatrix} = circ(c_0, c_1, c_2 ..., c_{n-1})$$

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1.4 m-Class Association Scheme (AS):

Let X be a non-empty set of order v. A set $\Omega = \{R_O = I, R_1, ..., R_m\}$ of non-empty relations on X is an m -class AS if following properties are satisfied

(i) $R_0 = \{(x,x): x \in X\}$

(ii) Ω is a partition of $X \times X$ i.e.

$$\bigcup_{i=0}^{m} R_i = X \times X, R_i \bigcap R_j = \Phi \ if \ i \neq j.$$

(iii) $R_i^T = R_i \ where \, R_i^T = \{(x,y):(y,x) \in R_i\}, i = 0,1,...,m$.

(iv) Let (x, y) $\in$ R_i. For $i, j, k \in \{0,1,2,...,m\}$

$p_{jk}^i = \left| \{z:(x,z) \in R_j \bigcap (z,y) \in R_k\} \right| = p_{kj}^i$, which is independent of (x, y) $\in R_i$.

The non-negative integers p_{jk}^i are called parameters of an m-Class AS. If (x,y) $\in R_i$ then x and y are called ith associates.

1.5 Association Matrices-

These matrices were introduced by Bose and Mesner [1].

The i-th association matrix $B_i = \left[b_{\alpha\beta}^i \right]_{\substack{0 \le i \le m \\ \alpha,\beta \in X}}$ of an m-class AS is a symmetric matrix of order v where

$$b_{\alpha\beta}^i = \begin{cases} 1 if \ \alpha \ and \ \beta \ are \, mutually \, i-th \, associates \\ 0 \, otherwise \end{cases}$$

1.5.1 Properties Of Association Matrices-

(i) $B_0 = I_v$ (ii) $\sum_{i=0}^{m} B_i = J_v$ (iii) $B_i B_j = \sum_{k=0}^{m} p_{ij}^k B_k = B_j B_i (i, j = 0,1,2,...,m).$

1.6 Partially Balanced Incomplete Block (PBIB) Design

Let X be non-empty set with cardinality v. The elements of X are called treatments. A PBIB design based on an m-class association scheme is a family of b subsets of X, each of size k such that each treatment occurs in r blocks, any two treatments occur together in λ_i (i=0,1,…,m) blocks if they are mutually ith associates. v, b, r, k, λ_i are called parameters of a PBIB design.[2]

1.7 Rectangular AS

Rectangular AS, introduced by Vartak [14], is an arrangement of v=mn treatments in a rectangular array of m rows and n columns such that any two treatments belonging to the same row are first associates, any two treatments belonging to the same column are second associates and remaining pairs of treatments are third associates.

1.8 Rectangular Design

Rectangular design is a 3- class PBIB design based on a rectangular AS of v=mn treatments arranged in a rectangular array of m rows and n columns in b blocks such that each block contains k distinct treatments, each treatment occurs in exactly r blocks and any two treatments which are first associates occur together in λ_1 blocks, whereas second treatments occur together in λ_2 blocks and the treatment which are third associates occur together in λ_3 blocks.v,b,r,k,λ_1,λ_2,λ_3 are called parameters of a rectangular design.

Rectangular designs have also been studied by Suen[13], Sinha [8], Sinha et al. [6,9,10,11,12], Kageyama and Miao[5]and so on. The rectangular designs are useful for factorial experiments, having factorial balance as well as orthogonality. [4]

For convenience, I_n denotes the identity matrix of order n, $J_{m \times n}$ denotes the $m \times n$ matrix with all its entries 1, in particular $J_n = J_{n \times n}$ and $K_n = J_n - I_n$. $A \otimes B$ denotes the Kronecker product of two matrices A and B. $\alpha^i = circ.(o,o,o,...,1,...,o)$ is a circulant matrix of order n with 1 at (i+1)th Position such that $\alpha^n = I_n$.

2. GROCM AND ITS REDUCTION TO AN INCIDENCE MATRIX OF A RECTANGULAR DESIGN

2.1 Definition of GROCM

Singh and Prasad [7] defined Generalized Orthogonal Combinatorial matrix (GOCM). Here we define GROCM.

Let $N=[N_{ij}]$, i, j$\epsilon\{1,2,\ldots,m\}$ where N_{ij} are {0,1} matrices of order $n\times s_j$. Let$R_i = (N_{i1}, N_{i2},\ldots, N_{im})$ be the ith row of blocks. We define inner product of two row of blocks R_i and R_j as

$R_i.R_j=R_iRj^T=\sum_{k=1}^{m} N_{ik}N_{jk}^T$. N is called a Generalized Row Orthogonal Matrices (GROM) if there exists fixed positive integer r and fixed non-negative integers $\lambda_1, \lambda_2, \lambda_3$ such that

$$R_i.R_j= R_iRj^T = \sum_{k=1}^{m} N_{ik}N_{jk}^T = \begin{cases} r\,\mathrm{I}_n+\lambda_1 K_n \ if\ i=j, \\ \lambda_2 I_n+\lambda_3 K_n \quad if\ i\neq j. \end{cases}$$

A GROM with constant column sum will be called GROCM. GROCM is an extension of Generalized Hadamard Matrices.

Theorem 2.2: A GROCM is in general an incidence matrix of a rectangular design.
Proof.

$$NN^T=\begin{pmatrix} rI_n+\lambda_1 K_n & \dots & \lambda_2 I_n+\lambda_3 K_n \\ \vdots & \ddots & \vdots \\ \lambda_2 I_n+\lambda_3 K_n & \cdots & rI_n+\lambda_1 K_n \end{pmatrix} = r(I_m\otimes I_n)+\lambda_1(I_m\otimes K_n)+\lambda_2(K_m\otimes I_n)+\lambda_3(K_m\otimes K_n)$$

$B_0=I_m\otimes I_n, B_1=I_m\otimes K_n, B_2=K_m\otimes I_n, B_3=K_m\otimes K_n$ are the association matrices of at most three classes association scheme. We have

$$\sum_{i=0}^{3} B_i = J_{m\times n}$$

$B_1B_2 = B_3$, $B_1B_3= (n-1) B_2+(n-2)B_3$, $B_2B_3=(m-1)B_1+(m-2)B_3$.
$B_1^2 = (n-1)B_0+(n-2)B_1$, $B_2^2=(m-1)B_0+(m-2)B_2$,
$B_3^2= (m-1)(n-1)B_0+(m-1)(n-2)B_1+(n-1)(m-2)B_2+(m-2)(n-2)B_3$.

$$P_0=\left(p_{ij}^0\right)=\begin{pmatrix} n-1 & 0 & 0 \\ 0 & m-1 & 0 \\ 0 & 0 & (m-1)(n-1) \end{pmatrix}, P_1=\left(p_{ij}^1\right)=\begin{pmatrix} n-2 & 0 & 0 \\ 0 & 0 & m-1 \\ 0 & m-1 & (m-1)(n-2) \end{pmatrix},$$

$$P_2=(p_{ij}^2)=\begin{pmatrix} 0 & 0 & n-1 \\ 0 & m-2 & 0 \\ n-1 & 0 & (m-2)(n-1) \end{pmatrix}, P_3=\left(p_{ij}^3\right)=\begin{pmatrix} 0 & 1 & n-2 \\ 1 & 0 & m-2 \\ n-2 & m-2 & (m-2)(n-2) \end{pmatrix}$$

The above matrices give the values of $p_{ij}^k(0\le i,j,k\le 3)$ which are the parameters of a rectangular association scheme. Hence a GROCM is the incidence matrix of a rectangular design in general, which is defined by an $m\times n$ array.

Corollary 2.2.1: A GROCM is an incidence matrix of a Group Divisible design if either $\lambda_1=\lambda_3\ or\ \lambda_2=\lambda_3$

3. CONSTRUCTION METHODS

Theorem 3.1: Let H be a Hadamard matrix of order 4n ($n\ge 1$).Then there exists a rectangular designs with parameters $v=4ns, b=4ns(s-1), r=4n(s-1), k=4n, \lambda_1=0, \lambda_2=2n(s-1), \lambda_3=2n, m=4n, n=s$, where $s\ge 2$ is a positive integer.

Proof: Let H be a Hadamard matrix of order 4n (n≥1) in its normalized form. We replace 1 by I_s and -1 by α in H to obtain a matrix H^1 where α is a (0, 1) circulant matrix of order s such that $\alpha^s = I_s$. We obtain H^i $(2 \le i \le s-1)$ replacing α by α^i in H^1.We adjoin $H^1, H^2, H^3, ..., H^{s-1}$ and obtain $N_1=[H^1\, H^2 ... H^{s-1}]$, which is an incidence matrix of a rectangular design with the required parameters.

Inner product of any two same rows of N_1 contributes $4n(s-1)$ I_s's and 0's K_s's.

In an inner product of two distinct rows R_i and R_j ($i \neq j; i, j \neq 1$) of N_1,

Block matrices $\begin{pmatrix} I_s & I_s & . & . & . & I_s \\ \alpha & \alpha^2 & . & . & . & \alpha^{s-1} \end{pmatrix}$, $\begin{pmatrix} I_s \\ I_s \end{pmatrix}$, $\begin{pmatrix} \alpha & \alpha^2 & . & . & . & \alpha^{s-1} \\ \alpha & \alpha^2 & . & . & . & \alpha^{s-1} \end{pmatrix}$ and $\begin{pmatrix} \alpha & \alpha^2 & . & . & . & \alpha^{s-1} \\ I_s & I_s & . & . & . & I_s \end{pmatrix}$ occurs n, $n(s-1)$, $n(s-1)$ and n times respectively. Hence they contribute $2n(s-1)$ I_s's and $2n$ K_s's.

In an inner product of R_1 and other rows different from R_1, block matrices $\begin{pmatrix} I_s \\ I_s \end{pmatrix}$ and $\begin{pmatrix} I_s & I_s & . & . & . & I_s \\ \alpha & \alpha^2 & . & . & . & \alpha^{s-1} \end{pmatrix}$ occur $2n(s-1)$ and $2n$ times respectively. Hence they contribute $2n(s-1)$ I_s's and $2n$ K_s's.

$$R_i \circ R_j = R_i R_j^T = \begin{cases} 4n(s-1)I_s + 0K_s if i = j \\ 2n(s-1)I_s + 2nK_s if i \neq j \end{cases}$$

Hence N_1 represents an incidence matrix of a RD design with the required parameters. [vide theorem 2.2]

Example For n=1, s=3 we obtain a rectangular design with parameters v=12, b=24, r=8, k=4, $\lambda_1 = 0, \lambda_2 = 4, \lambda_3 = 2, m = 4, n = 3$
whose blocks are

$$\begin{aligned}&(1,4,7,10),(2,5,8,11),(3,6,9,12),(1,6,7,12),(2,4,8,10),(3,5,9,11),\\&(1,4,9,12),(2,5,7,10),(3,6,8,10),(1,6,9,10),(2,4,7,11),(3,5,8,12),\\&(1,4,7,10),(2,5,8,11),(3,6,9,12),(1,5,7,11),(2,6,8,12),(3,4,9,10),\\&(1,4,8,11),(2,5,9,12),(3,6,7,10),(1,5,8,10),(2,6,9,11),(3,4,7,12).\end{aligned}$$

which is a quasimultiple of the rectangular design with parameters
v=b=12, r=k=4, $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 1$ in Sinha *et al.*[12]

Remark: N_1 is the incidence matrix of a GD design if s=2. [vide corollary 2.2.1]

Corollary 3.1.1: There exists a rectangular design with parameters
$v = 4n^2, \mathrm{b} = 4n^2(\mathrm{n}-1), \mathrm{r} = 4\mathrm{n}(\mathrm{n}-1), \mathrm{k} = 4\mathrm{n}, \lambda_1 = 0, \lambda_2 = 2\mathrm{n}(\mathrm{n}-1), \lambda_3 = 2\mathrm{n}, \mathrm{m} = 4\mathrm{n}, \mathrm{n} = \mathrm{n}.$

Proof: On putting s=n in the previous theorem, we obtain a rectangular design with the required parameters.

Theorem 3.2: There exists a rectangular design with parameters
$v = \mathrm{s}(4\mathrm{n}-1), \mathrm{b} = \mathrm{s}(\mathrm{s}-1)(4\mathrm{n}-1), \mathrm{r} = (\mathrm{s}-1)(4\mathrm{n}-1), \mathrm{k} = 4\mathrm{n}-1, \lambda_1 = 0,$
$\lambda_2 = (\mathrm{s}-1)(2\mathrm{n}-1), \lambda_3 = 2\mathrm{n}, \mathrm{m} = 4n-1, \mathrm{n} = s.$

Proof: Let H be a Hadamard matrix of order 4n (n≥1). In the core C^1 Of Normalized Hadamard matrix, we replace -1 by α and 1 by I_s, where α is a circulant matrix of order s such that $\alpha^s=I_s$. We obtain C^i replacing α by $\alpha^i\,(2 \le i \le s-1)$ in C^1 .We adjoin $C^1, C^2, ..., C^{s-1}$ and obtain $N_2 = [C^1 C^2 ... C^{s-1}]$, which is an incidence matrix of a rectangular design with the required parameters. [Vide theorem 2.2]

Example: For s=4, n=1 we obtain a rectangular design with parameters

$$v = 12, b = 36, r = 9, k = 3, \lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 2, m = 3, n = 4.$$

Remark: N is the incidence matrix of a GD design if $s = \frac{2n}{2n-1} + 1$. [vide corollary 2.2.1]

4. CONCLUSION

A new combinatorial structure GROCM has been used to construct some series of rectangular designs. GROCM can also be used to construct some more PBIB designs.

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REFERENCES

1. Bose, R.C. and Mesner, D.M., On linear associative algebras corresponding to association schemes of partially balanced designs, Ann. Math. Statist., 23(1959), 21-38.
2. Dey, A., Theory of block designs, Wiley Eastern, New York,(1986),158-162.
3. Dey, A. and Mukherjee,R., Fractional Factorial Plans, John Wiley & sons, Inc., New York,(1999), 32-34.
4. John, J.A. and Dean,A.M., Single Replicate Factorial Experiments in Generalized Cyclic Designs:I. Symmetrical Arrangements, Wiley for the Royal Society, 37(1975), 63-71.
5. Kagyeyama,S. and Miao, Y., Some methods of constructions of rectangular PBIB designs, Combinatorics Theory,1(1995),186-198.
6. Kagyeyama, S. and Sinha, K., Some patterned construction of rectangular designs, J. Japan Statist. Soc., 33(2003),137-144.
7. Singh, M.K. and Prasad, Dinesh, Some new balanced block intersection designs from Williamson matrices, Acta Ciencia,3(2014),371-376.
8. Sinha, K., A construction of rectangular designs, J. Comb. Math. Comb. Computing, 9(1991), 199-200.
9. Sinha, K., Kagyeyama, S. and Singh, M.K., Construction of rectangular designs, Statistics, 25(1993), 63-70.
10. Sinha,K., Mitra, R.K. and Saha,G.M., Nested BIB designs, balanced bipartite weighing designs and rectangular designs, Utilitas Math., 49(1996), 216-22.
11. Sinha,K., and Mitra,R.K., Constructions of nested balanced incomplete block designs, rectangular designs and q-ary codes, Ann. Combin., 3(1999),71-80.
12. Sinha, K., Singh, M.K., Kagyeyama, S. and Singh, R.S., Some series of rectangular designs, J. Statist. Plann. Inference, 106(2002b), 39-46.
13. Suen,C., Some rectangular designs constructed by method of differences, J. Statist. Plann. Inference, 21 (1989), 273-276.
14. Vartak, M.N., On an application of Kronecker product of matrices to statistical designs, Ann. Math. Statist., 26(1955), 420-438.

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