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FUZZY AND NORMAL FUZZY *WI*-IDEALS OF LATTICE WAJSBERG ALGEBRAS

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ABSTRACT

In this paper, we introduce the notion of fuzzy WI-ideal of lattice Wajsberg algebra and obtain some properties. Further, we introduce normal fuzzy WI-ideal. Moreover, we establish the concepts of maximal fuzzy and completely normal fuzzy WI-ideals in lattice Wajsberg algebra. We show that every maximal fuzzy WI-ideal is a completely normal.

Keywords: *Wajsberg algebra, Lattice Wajsberg algebra, WI-ideal, Fuzzy WI-ideal, Fuzzy lattice ideal, Normal fuzzy WI-ideal.*

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1. INTRODUCTION

Fuzzy logic is a form of many-valued logic in which the truth values of variables may be any real number between 0 and 1. The term fuzzy logic was introduced with the 1965 proposal of fuzzy set theory by Lotfi Zadeh[9]. Idea of Zadeh's have been applied in the field of algebraic structures, the study of fuzzy algebras has achieved great success. Mordewaj Wajsberg[7] proposed the concept of Wajsberg algebras in 1935. Rose *et al.* [6] published the proof of Wajsberg algebras in 1958. In 1984, Font *et al.* [3] extended Wajsberg algebras as an alternative model for the infinite valued Łukasiewicz logic and introduced lattice structure of Wajsberg algebra. Lattice Wajsberg algebras provide the foundation to establish the corresponding logic system in algebraic view point. Further, they [3] introduced the notions of implicative filters and family of implicative filters in a lattice Wajsberg algebras and investigated their properties. Fuzzy subset has been applied to the theories of filters and ideals in various non-classical logical algebras. Basheer Ahamed and Ibrahim [1, 2] introduced the definitions of fuzzy implicative and an anti fuzzy implicative filters of lattice Wajsberg algebras. Recently, we [4] introduced the notion of Wajsberg implicative ideal (*WI*-ideal) of lattice Wajsberg algebra and discussed some properties.

The aim of this paper is to introduce the definition of fuzzy Wajsberg implicative ideal (fuzzy *WI*-ideal) of lattice Wajsberg algebras and discuss some properties with examples. Further, we introduce normal fuzzy *WI*-ideal. Moreover, we obtained the definitions of maximal fuzzy and completely normal fuzzy *WI*-ideals in lattice Wajsberg algebra. Finally, we show that every maximal fuzzy *WI*-ideal of a lattice Wajsberg algebra is completely normal.

2. PRELIMINARIES

In this section, we recall some basic definitions and their properties which are helpful to develop our main results.

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Definition 2.1[3]: Let (*A*, $\rightarrow$, *****, 1) be an algebra with quasi complement "*" and a binary operation "$\rightarrow$" is called a Wajsberg algebra if and only if it satisfies the following axioms for all $x, y, z \in A$,

(i) $1 \rightarrow x = x$

(ii) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$

(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$

Proposition 2.2[3]: A Wajsberg algebra (*A*, $\rightarrow$, *****, 1) satisfies the following axioms for all $x, y, z \in A$,

(i) $x \rightarrow x = 1$

(ii) If $(x \rightarrow y) = (y \rightarrow x) = 1$ then $x = y$

(iii) $x \rightarrow 1 = 1$

(iv) $(x \rightarrow (y \rightarrow x)) = 1$

(v) If $(x \rightarrow y) = (y \rightarrow z) = 1$ then $x \rightarrow z = 1$

(vi) $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$

(ix) $(x^*)^* = x$

(x) $(x^* \rightarrow y^*) = y \rightarrow x$.

Proposition 2.3[3]: A Wajsberg algebra (*A*, $\rightarrow$, *****, 1) satisfies the following axioms for all $x, y, z \in A$,

(i) If $x \le y$ then $x \rightarrow z \ge y \rightarrow z$ and $z \rightarrow x \le z \rightarrow y$

(ii) $x \le y \rightarrow z$ if and only if $y \le x \rightarrow z$

(iii) $(x \vee y)^* = (x^* \wedge y^*)$

(iv) $(x \wedge y)^* = (x^* \vee y^*)$

(v) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

(vi) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$

(vii) $(x \rightarrow y) \vee (y \rightarrow x) = 1$

(viii) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$

(ix) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$

(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(xi) $(x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Definition 2.4[3]: Wajsberg algebra *A* is called a lattice Wajsberg algebra, if it satisfies the following conditions for all $x, y \in A$,

The Partial ordering "$\le$" on a lattice Wajsberg algebra *A*, such that $x \le y$ if and only if

(i) $x \rightarrow y = 1$

(ii) $(x \vee y) = (x \rightarrow y) \rightarrow y$

(iii) $(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$. Thus, (*A*, $\vee$, $\wedge$, *****, 0, 1) is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Definition 2.5[4]: The lattice Wajsberg algebra *A* is called a lattice *H*-Wajsberg algebra, if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in A$.

In a lattice *H*-Wajsberg algebra *A*, the following hold

(i) $x \rightarrow (x \rightarrow y) = (x \rightarrow y)$

(ii) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Definition 2.6[3]: Let *A* be aWajsberg algebra, a subset *F* of *A* is called an implicative filter of *A*, if it satisfies the following axioms for all $x, y \in A$,

(i) $1 \in F$

(ii) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

Definition 2.7[3]: Let L be a lattice. An ideal I of L is a nonempty subset of L is called a lattice ideal, if it satisfies the following axioms for all $x, y \in I$,

(i) $x \in I, y \in L$ and $y \leq x$ imply $y \in I$

(ii) $x, y \in I$ implies $x \vee y \in I$.

Definition 2.8[4]: Let A be a lattice Wajsberg algebra. Let I be a nonempty subset of A, then I is called a *WI*-ideal of lattice Wajsberg algebra A satisfies,

(i) $0 \in I$

(ii) $(x \to y)^* \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in A$.

Definition 2.9[9]: Let A be a set. A function $\mu : A \to [0,1]$ is called a fuzzy subset on A, for each $x \in A$, the value of $\mu(x)$ describes a degree of membership of x in μ.

Definition 2.10[9]: Let μ be a fuzzy set in a set A. Then for $t \in [0,1]$, the set $\mu_t = \{x \in A / \mu(x) \geq t\}$ is called level subset of μ.

Definition 2.11[9]: Let μ be a fuzzy subset of a set A. Then for $t \in [0,1]$, the set $\mu^t = \{x \in A / \mu(x) \leq t\}$ is called the lower t-level cut of μ.

Definition 2.12[8]: The height of a fuzzy set A is the largest membership grade obtained by any element in that set $h(A) = \sup_{x \in X} A(x)$.

Definition 2.13[8]: A fuzzy set A is called normal when $h(A) = 1$.

3. MAIN RESULTS

3.1. Fuzzy Wajsberg implicative ideal (fuzzy *WI*-ideal)

In this section, we define fuzzy *WI*-ideal in lattice Wajsberg algebra and obtain some useful results with illustrations.

Definition 3.1.1: Let A be a lattice Wajsberg algebra. A fuzzy subset μ of A is called a fuzzy *WI*-ideal of A if for any $x, y \in A$,

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu((x \to y)^*), \mu(y)\}$.

Example 3.1.2: Let $A = \{0, a, b, c, d, 1\}$ be a set with Figure (1) as a partial ordering. Define a quasi complement "$*$" and a binary operation "$\to$" on A as in Table (1) and Table (2).

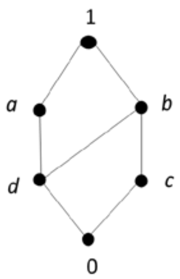

Figure-(1)

x	x^*
0	1
a	c
b	d
c	a
d	b
1	0

Table-(1)

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table-(2)

Define $\vee$ and $\wedge$ operations on A as follow,
$(x \vee y) = (x \to y) \to y$,
$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$.
Then, A is a lattice Wajsberg algebra.

Consider the fuzzy subset μ on A as, $\mu(x) = \begin{cases} 0.6 \text{ if } x = 0 & \text{for all } x \in A \\ 0.2 \text{ otherwise} & \text{for all } x \in A \end{cases}$

Then, we have μ is fuzzy *WI*-ideal of *A*.

Example 3.1.3: Let $A = \{0, a, b, c, 1\}$ be a set with Figure (2) as a partial ordering. Define a quasi complement "$*$" and a binary operation "$\to$" on A as in Table (3) and Table (4)

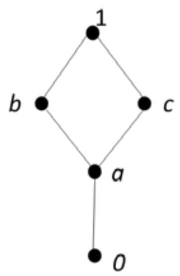

Figure-(2)

x	x^*
0	1
a	a
b	c
c	b
1	0

Table-(3)

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	a	a	c	1	1
b	c	1	1	1	1
c	b	1	1	1	1
1	0	a	b	c	1

Table-(4)

Define $\vee$ and $\wedge$ operations on A as follow,
$(x \vee y) = (x \to y) \to y$,
$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$.

Then, A is a lattice Wajsberg algebra.

Consider the fuzzy subset μ on A as, $\mu(x) = \begin{cases} 0.4 \text{ if } x = 0 & \text{for all } x \in A \\ 0.1 \text{otherwise} & \text{for all } x \in A \end{cases}$

Then, we have μ is fuzzy *WI*-ideal of *A*.

Example 3.1.4: Let $A = \{0, a, b, c, 1\}$ be a set with Figure (3) as a partial ordering. Define a quasi complement "$*$" and a binary operation "$\to$" on A as in Table (5) and Table (6).

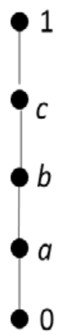

Figure-(3)

x	x^*
0	1
a	c
b	b
c	a
1	0

Table-(5)

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	1	1	1
1	0	a	b	c	1

Table-(6)

Define $\vee$ and $\wedge$ operations on A as follow,
$(x \vee y) = (x \to y) \to y$,
$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$.
Then, A is a lattice Wajsberg algebra.

Consider the fuzzy subset μ on *A* as,

$$\mu(x) = \begin{cases} 0.5 \text{ if } x \in \{0, b\} & \text{for all } x \in A \\ 0.3 \text{ if } x \in \{a, c, 1\} & \text{for all } x \in A \end{cases}$$

Then, we get μ is not a fuzzy *WI*-ideal of lattice Wajsberg algebra *A*.

Since, we have $\mu(a) = 0.3$. But, $\min\{\mu(a \to b)^*, \mu(b)\} = \min\{\mu(0), \mu(b)\} = 0.5$.

Thus, we have $\mu(a) \ngeq \min\{\mu(a \to b)^*, \mu(b)\}$.

Example 3.1.5: Let *A* be a lattice Wajsberg algebra defined in Example 3.1.3, fuzzy subset μ of *A* defined by $\mu(0) = \mu(c)$ and $\mu(0) \geq \mu(x)$ for any $x \in \{a, b, 1\}$, then μ is a fuzzy *WI*-ideal of *A*.

Proposition 3.1.6: Every fuzzy *WI*-ideal μ of a lattice Wajsberg algebra *A* is order reversing.

Proof: If $x, y \in A$ and $y \leq x$ then $(y \to x)^* = 1^* = 0$

and so, $\mu(y) \geq \min\{\mu((y \to x)^*), \mu(x)\}$

$= \min\{\mu(0), \mu(x)\}$

$= \mu(x)$

This shows that μ is order reversing.

Definition 3.1.7: A fuzzy subset μ of a lattice Wajsberg algebra *A* is called a fuzzy lattice ideal if for any $x, y \in A$,

(i) If $y \leq x$ then $\mu(y) \geq \mu(x)$

(ii) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$.

Example 3.1.8: Let *A* be a lattice *W*-algebra defined in Example 3.1.2, {0, *d*} is a non-trivial lattice ideal, and a fuzzy subset μ of *A* defined by $\mu(0) = \mu(d) \geq \mu(x)$ for any $x \in \{a, b, c, 1\}$ is a non-trivial fuzzy ideal of *A*.

Proposition 3.1.9: Let *A* be a lattice Wajsberg algebra. Every fuzzy *WI*-ideal of *A* is a fuzzy lattice ideal.

Proof: Suppose μ is a fuzzy *WI*-ideal of *A*.

By Proposition 3.1.6 shows that $\mu(y) \geq \mu(x)$ if $y \leq x$.

By $((x \vee y) \to y)^* = ((x \to y) \wedge (y \to y))^*$

$= (x \to y)^*$

$\leq x$

We get, $\mu(x \vee y) \geq \min\{\mu((x \vee y) \to y)^*), \mu(y)\}$

$\geq \min\{\mu(x), \mu(y)\}$.

The following example shows that the converse of Proposition 3.1.9 is not true.

Example 3.1.10: Let *A* be a lattice Wajsberg algebra defined in Example 3.1.2 and μ is a fuzzy subset of *A* defined by,

$$\mu(x) = \begin{cases} 0.8 \text{ if } x \in \{0, d\} & \text{for all } x \in A \\ 0.3 \text{ if } x \in \{a, b, c, 1\} & \text{for all } x \in A \end{cases}$$

Then, μ is a fuzzy lattice ideal of *A*, but not a fuzzy *WI*-ideal for

$\mu(a) \ngeq \min\{\mu((a \to d)^*), \mu(d)\}$.

Proposition 3.1.11: In a lattice *H*-Wajsberg algebra *A*, every fuzzy lattice ideal is a fuzzy *WI*-ideal.

Proof: Since $0 \leq x$, it follows that $\mu(0) \geq \mu(x)$ for any $x \in A$.

Let $x, y \in A$, we have $\mu(x) \geq \mu(x \vee y)$

$= \mu(y \vee (x^* \vee y^*)^*)$

$= \mu(y \vee (x \to y)^*)$

$\geq \min\{\mu(y), \mu(x \to y)^*\}$

Hence, μ is a fuzzy *WI*-ideal.

Proposition 3.1.12: Let μ be a fuzzy subset of a lattice Wajsberg algebra *A*. Then, μ is a fuzzy *WI*-ideal if and only if μ_t is a *WI*-ideal when $\mu_t \neq \varphi,\ t \in [0,1]$

Proof: Assume that μ is a fuzzy *WI*-ideal of *A* and $t \in [0,1]$ such that $\mu_t \neq \phi$. Clearly, $0 \in \mu_t$. Suppose $x, y \in A$, $(x \to y)^* \in \mu_t$ and $y \in \mu_t$. Then, $\mu((x \to y)^*) \geq t$ and $\mu(y) \geq t$. It follows that, $\mu(x) \geq \min\{\mu(x \to y)^*,\ \mu(y)\} \geq t.$ So that, $x \in \mu_t$. Hence, μ_t is a *WI*-ideal of *A*.

Conversely, suppose μ_t $(t \in [0,1])$ is a *WI*-ideal of *A*. When $\mu_t \neq \phi$ for any $x \in A$, $x \in \mu_{\mu(x)}$, it follows that $\mu_{\mu(x)}$ is a *WI*-ideal of *A* and hence $0 \in \mu_{\mu(x)}$ that is $\mu(0) \geq \mu(x)$ for any $x, y \in A$. Let $t = \min\{\mu(x \to y)^*,\ \mu(y)\}$, it follows that μ_t is a *WI*-ideal and $(x \to y)^* \in \mu_t$, $y \in \mu_t$, this implies that $x \in \mu_t$ and $\mu(x) \geq t = \min\{\mu(x \to y)^*,\ \mu(y)\}$.

3.2. Normal Fuzzy *WI*-ideals

In this section, we introduce the notions of normal fuzzy, maximal fuzzy and completely normal fuzzy *WI*-ideal in lattice Wajsberg algebra. We also show that every maximal fuzzy *WI*-ideal of a lattice Wajsberg algebra is completely normal.

Definition 3.2.1: A fuzzy *WI*-ideal μ of a lattice Wajsberg algebra *A* is said to be a normal if there exists $x \in A$ such that $\mu(x) = 1$.

Example 3.2.2: Let *A* be a lattice Wajsberg algebra in Example 3.1.2 and define in a fuzzy subset μ of *A* by

$$\mu(x) = \begin{cases} 1 & if\ \ x \in \{0, c\} \quad for\ all\ x \in A \\ 0.3 & if\ \ x \in \{a, b,\ d, 1\} \quad for\ all\ x \in A \end{cases}$$

Then μ is a normal fuzzy *WI*-ideal of *A*.

Note: If μ is a normal fuzzy *WI*-ideal of *A*, then $\mu(0) = 1$. Hence, μ is normal if and only if $\mu(0) = 0$.

Definition 3.2.3: Let *A* be a lattice Wajsberg algebra, $\mathcal{F}(A)$ be the set of all fuzzy subset of *A* and $\mu \in \mathcal{F}(A)$, we define the set as $A_\mu = \{x \in A /\ \mu(x) = \mu(0)\}$.

Proposition 3.2.4: Given a fuzzy *WI*-ideal μ of *A*, let μ' be a fuzzy subset of *A* defined by $\mu'(x) = \mu(x) + 1 - \mu(0)$ for any $x \in A$. Then μ' is a normal fuzzy *WI*-ideal of *A* containing μ.

Proof: For any fuzzy *WI*-ideal *W* of *A*, the characteristic function χ_w of *W* is a normal fuzzy *WI*-ideal of *A*. It is clear that, μ is normal if and only if $\mu = \mu'$. If μ is a fuzzy *WI*-ideal of *A*. Then $(\mu') = (\mu')'$. Moreover, if μ is normal, then $\mu = (\mu')'$.

According to Definition 3.2.1, We know that if μ and σ are fuzzy *WI*-ideal of *A* such that $\mu \subseteq \sigma$ and $\mu(0) = \sigma(0)$, then $A_\mu \subseteq A_\sigma$. If μ is fuzzy *WI*-ideal of *A* and there exists a fuzzy *WI*-ideal σ of *A* such that $\sigma' \subseteq \mu$, then μ is normal.

Proposition 3.2.5: Let μ be a fuzzy *WI*-ideal of *A* and $f : [\ 0,\ \ \mu\ (0)] \to [0,\ 1]$ an increasing function. Then, the fuzzy subset $\mu_f(x) = f(\mu(x))$ for any $x \in A$ is a fuzzy *WI*-ideal of *A*. In particular, if $f(\ \mu\ (0)) = 1$ then μ_f is normal, and if $f(t) \geq t$ for any $t \in [0,\ \mu(0)]$, then μ is contained in μ_f.

Proof: Now $\mu(x) \leq \mu(0)$ for any $x \in A$
Since, *f* is an increasing function it follows that,

$$\begin{aligned} \mu_f\ (0) &= f(\mu(0)) \\ &\geq f(\mu(x)) \\ &> \mu_f(x) \ \text{ for any } x \in A. \end{aligned}$$

Let $x, y \in A$ then,

$$\begin{aligned}\min\{\mu_f(x \to y)^*,\ \mu_f(y)\} &= \min\{f(\mu(x \to y)^*),\ f(\mu(y))\}\\ &= f(\min\{\mu(x \to y)^*,\ \mu(y)\})\\ &\le f(\mu(x))\\ &= \mu_f(x)\end{aligned}$$

Hence, μ_f is a fuzzy *WI*-ideal of *A*. If $f(\mu(0)) = 1$ then μ_f is normal. If $f(t) \ge t$ for any $t \in [0, \mu(0)]$, then $\mu_f(x) = f(\mu(x)) \ge \mu(x)$ for any $x \in A$, so $\mu \subseteq \mu_f$.

Note: (i) We denote *N*(*A*) as the set of all normal fuzzy *WI*-ideal.
(ii) *N*(*A*) is a poset under the set inclusion.

Proposition 3.2.6: If μ and σ are non-empty fuzzy *WI*-ideals of *A* such that for any $x, y \in A$, $\mu(x) \le \mu(y)$ if and only if $\sigma(x) \le \sigma(y)$ then $\mu \circ \sigma$ is also a fuzzy *WI*-ideal of *A* and $A_{\mu\circ\sigma} = A_\mu \cap A_\sigma$ where $(\mu \circ \sigma)(x) = \mu(x)\sigma(x)$ for any $x \in A$.

Proof: It is easy to prove that, $\mu \circ \sigma$ is a *WI*-ideal of *A* and $A_{\mu\circ\sigma} \supseteq A_\mu \cap A_\sigma$

Let $x \in A_{\mu\circ\sigma}$. Then $(\mu \circ \sigma)(x) = \mu \circ \sigma(0)$

$$\mu(x)\sigma(x) = \mu(0)\sigma(0)$$

Hence, $\mu(x) \ne 0$ and $\sigma(x) \ne 0$. If $\mu(x) < \mu(0)$, then $\mu(x)\sigma(x) < \mu(0)\sigma(0)$.

This is a contradiction. Similarly, it is also a contradiction when $\sigma(x) < \sigma(0)$.

Hence, $\mu(x) = \mu(0)$ and $\sigma(x) = \sigma(0)$. That is, $x \in A_\mu \cap A_\sigma$.

Remark 3.2.7: A fuzzy *WI*-ideal is called a maximal *WI*-ideal if it is not *A*, and it is maximal element of the set of all fuzzy *WI*-ideal with respect to fuzzy set inclusion.

Proposition 3.2.8: If μ is a maximal fuzzy *WI*-ideal of *A* then, the following hold

(i) μ is normal
(ii) μ takes only the values 0 and 1
(iii) A_μ is a maximal *WI*-ideal of *A*.

Proof: (i) Suppose μ is not normal, then for any $x \in A$

$$\mu(x) \le \mu(x) + 1 - \mu(0) = \mu'(x)$$

$$\mu(0) < 1 = \mu(0) + 1 - \mu(0) = \mu'(0)$$

It follows that μ is not maximal fuzzy *WI*-ideal of *A*. This is a contradiction. Thus, μ is normal.

(ii) $\mu(0) = 1$, since μ is normal

Let $a \in A$ and $\mu(a) \ne 1$. We claim that, $\mu(a) = 0$. If not then $0 < \mu(a) < 1$

Let σ be a fuzzy subset of *A* defined by

$$\sigma(x) = \frac{1}{2}(\mu(x) + \mu(a)) \text{ for all } x \in A$$

$$\begin{aligned}\text{Then, } \sigma(0) &= \frac{1}{2}(\mu(0) + \mu(a))\\ &= \frac{1}{2}(1 + \mu(a))\\ &\ge \frac{1}{2}(\mu(x) + \mu(a)) = \sigma(x)\end{aligned}$$

For any $x, y \in A$, we obtain $\sigma(x) = \frac{1}{2}(\mu(x) + \mu(a))$

$$\geq \frac{1}{2}(\min\{\mu(x \to y)^*,\ \mu(y)\} + \mu\,(\mathrm{a}))$$
$$= \min\{\frac{1}{2}(\mu(x \to y)^* + \mu(a)),\ \frac{1}{2}(\mu(y) + \mu(a))\}$$
$$= \min\{\sigma(x \to y)^* + \sigma(y)\}$$

Hence, σ is a fuzzy *WI*-ideal of *A*. It follows from the Proposition 3.2.4 that for any $x \in A$

$$\sigma^*(x) = \frac{1}{2}(\mu(x) + \mu(a)) + 1 - \frac{1}{2}(\mu(0) + \mu(a))$$
$$= \frac{1}{2}(\mu(x) + 1)$$
$$\geq \mu(x)$$
$$\sigma^*(a) = \sigma(a) + 1 - \sigma(0)$$
$$= \frac{1}{2}(\mu(a) + 1)$$
$$< 1 = \sigma^*(0)$$

Hence, σ' is a non-constant and μ is not a maximal element of *N*(*A*). This is a contradiction.
Therefore, μ takes only the value of 0 and 1.

(iii) A_μ is a *WI*-ideal of *A*. If $A_\mu \subseteq I \neq A$ and *I* is a *WI*-ideal, then the characteristic functions χ_A and χ_{A_μ} are fuzzy *WI*-ideal and $\mu = \chi_{A_\mu} \subseteq \chi_I$.

It follows that, $\mu = \chi_I$. Because μ is a maximal fuzzy *WI*-ideal and hence $A_\mu = I$.

This proves that, A_μ is a maximal *WI*-ideal of *A*.

Definition 3.2.9: A normal fuzzy *WI*-ideal μ of *A* is said to be completely normal if there exist $x \in A$ such that $\mu(x) = 0$ denoted by $\mathcal{C}(A)$ the set of all completely normal fuzzy *WI*-ideal of *A*.

Note: If μ is a completely normal fuzzy *WI*-ideal of *A*, then clearly $\mu(0) = 1$, if μ is completely normal if and only if $\mu(1) = 0$.

Proposition 3.2.10: Let μ be a non-constant fuzzy *WI*-ideal of *A* and define a fuzzy subset $\overline{\mu}$ of *A* by $\overline{\mu}(x) = \frac{\mu(x) - \mu(1)}{\mu(0) - \mu(1)}$ for any $x \in A$. Then $\overline{\mu} \in \mathcal{C}(A)$.

Proof: By Proposition 3.1.6, μ is order reversing, and then $\overline{\mu}$ is well defined.
Clearly, $\overline{\mu}(1) = 0$ and $\overline{\mu}(0) = 1 \geq \overline{\mu}(x)$ for any $x \in A$.

Let $x, y \in A$, Then, $\min\{\overline{\mu}(x \to y)^*,\ \overline{\mu}(y)\} = \min\{\frac{\mu((x \to y)^* - \mu(1))}{\mu(0) - \mu(1)},\ \frac{\mu(y) - \mu(1)}{\mu(0) - \mu(1)}\}$

$$= \frac{1}{\mu(0) - \mu(1)} \min\{\mu(x \to y)^* - \mu(1),\ \mu(y) - \mu(1)\}$$
$$= \frac{1}{\mu(0) - \mu(1)} (\min\{\mu(x \to y)^*,\ \mu(y)\} - \mu(1))$$
$$= \frac{1}{\mu(0) - \mu(1)} ((\mu(x) - \mu(1))$$
$$= \overline{\mu}(x)$$

Therefore, we have $\overline{\mu} \in \mathcal{C}(A)$.

Proposition 3.2.11: If $\mu \in \mathcal{C}(A)$, then $\overline{\mu} = \mu$.

Proof: The proof is straight forward**.**

Note: $\mathcal{C}(A) \subseteq N(A)$ and the restriction of the partial ordering "$\subseteq$" of $N(A)$ gives a partial ordering $\mathcal{C}(A)$.

Proposition 3.2.12: Every non-constant maximal element of $(N(A), \subseteq)$ is also a maximal element of $(\mathcal{C}(A), \subseteq)$.

Proof: Let μ be a non-constant maximal element of $(N(A), \subseteq)$. By Proposition 3.2.8, μ takes only the values 0 and 1, and $\mu(0) = 1$, $\mu(1) = 0$. Hence, $\mu \in \mathcal{C}(A)$. Assume that, there exists $\sigma \in \mathcal{C}(A)$ such that $\mu \subseteq \sigma$. It follows that, $\mu \subset \sigma \in N(A)$. If σ is constant, then $\sigma = 1$. So, μ is a maximal element of $(\mathcal{C}(A), \subseteq)$. If σ is not a constant, since μ is a non-constant maximal in $(N(A), \subseteq)$, therefore, $\mu = \sigma$. Thus, we get μ is a maximal element of $(\mathcal{C}(A), \subseteq)$.

Proposition 3.2.1: Every maximal fuzzy *WI*-ideal of *A* is completely normal.

Proof: Let μ be a maximal fuzzy *WI*-ideal of *A*. By Proposition 3.2.8, μ is normal and $\mu = \mu^*$ takes only the values 0 and 1, now we claim that $\mu(1) = 0$. If not, then $\mu(1) = 1$. Hence, $\mu(x) \geq \mu(1) = 1$ and so $\mu(x) = 1$ for any $x \in A$. This means that $\mu = 1$, which is a contradiction. Thus, $\mu(1) = 0$ and μ is completely normal.

4. CONCLUSION

In this paper, we have introduced the definition of fuzzy Wajsberg implicative ideal (fuzzy *WI*-ideal) of lattice Wajsberg algebra and discussed some properties. We have introduced normal fuzzy *WI*-ideal. We have obtained the concepts of maximal fuzzy and completely normal fuzzy *WI*-ideals in lattice Wajsberg algebra. Finally, we have shown that every maximal fuzzy *WI*-ideal of a lattice Wajsberg algebra is completely normal.

REFERENCES

1. Basheer Ahamed, M., and Ibrahim, A., *Fuzzy implicative filters of lattice Wajsberg Algebras*, Advances in Fuzzy Mathematics, Volume 6, Number 2 (2011), 235-243.
2. Basheer Ahamed, M., and Ibrahim, A., *Anti fuzzy implicative filters in lattice W-algebras*, International Journal of Computational Science and Mathematics, Volume 4, Number 1 (2012), 49-56.
3. Font, J. M., Rodriguez, A. J., and Torrens, A., *Wajsberg algebras*, STOCHASTICA Volume 8, Number 1, (1984), 5-31.
4. Ibrahim, A., and Shajitha Begum, C., *On WI-Ideals Of lattice Wajsberg algebras*, Global Journal of Pure and Applied Mathematics, Volume 13, Number 10(2017), 7237-7254.
5. Leroy B. Beasley, GI-Sang Cheon, Young Bae Jun and Seok Zun Song, *Fuzzy Implicative LI-ideals in lattice implication algebras*, ScieniaeMathematicaeJaponicae Online Volume 9, (2003), 37-48.
6. Rose, A., and Rosser, J. B., *Fragments of many valued statement calculi*, Transaction of American Mathematical Society 87, (1958), 1-53.
7. Wajsberg, M., *Beitragezum Metaaussagenkalkul* l, Monat. Mat. phys. 42 (1935), 221-242.
8. Werro, N., *Fuzzy Classification of Online Customer, Fuzzy Management Methods*, Springer internationalpubli -shing Switzerland (2015), 7-26.
9. Zadeh, L. A., *Fuzzy Sets*, Information and Control 8 (1965), 338-353.

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