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L^1-CONVERGENCE OF DERIVATIVE OF FOURIER SERIES USING MODIFIED SUMS

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ABSTRACT

In this paper we discuss the L^1-convergence of the r-th derivative of Fourier series using modified trigonometric sums introduced by Rees and Stanojevic [15] and by Kumari and Ram [12]. It is shown that results concerning L^1-convergence of r-th derivative of trigonometric series can be better established using modified trigonometric sums as compared to classical partial sums. Previously obtained results in this direction by Bhatia and Ram [3] and Kaur and Bhatia [9] have been generalised.

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1. INTRODUCTION

Consider the cosine series.

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k coskx \qquad (1.1)$$

and let its partial sum be denoted by $S_n(x)$ and also $f(x) = \lim_{n\to\infty} S_n(x)$

It is well known that the Fourier series of a function f may not converge to f in L^1-metric ([1], Vol. II, Ch. VIIIξ 22). Many authors introduced different classes of coefficients to study the integrability and L^1-convergence of trigonometric series. The work was initiated by Young [24] in 1913 and Kolmogorov [11] in 1923 by taking the classes of convex sequences and quasiconvex sequences respectively. Sidon [20] introduced a condition which is weaker than quasi convex and is described by Telyakovskii [21] as

Definition [20, 21]: A sequence $\{a_k\}$ is said to belong to class S if $a_k = o(1)$, $k \to \infty$ and there exists a sequence $\{A_k\}$ such that (a) $A_k \downarrow 0$, $k \to \infty$ (b) $\sum_{k=0}^{\infty} A_k < \infty$ (c) $|\Delta a_k| \le A_k$ for all k

These authors proved the following necessary and sufficient condition for L^1-convergence of Fourier series;

$$||f - S_n|| = o(1),\ n \to \infty, \text{ iff } a_n \log n = o(1),\ n \to \infty \quad (*)$$

Regarding the integrability and L^1-convergence of trigonometric series, some other classes of coefficients with modifications and generalisations of above mentioned classes have been introduced by authors like Fomin [4], Moricz [13], Tomovski [23] and Tikhonov [22]. All these authors confirmed that (*) is necessary and sufficient condition for L^1-convergence of Fourier series.

In order to find better results, many authors like Rees and Stanojevic [15], Kumari and Ram [12], K. Kaur, Bhatia and Ram [7] , J. Kaur [10] and Krasniqi [8] introduced modified cosine sums and modified sine sums and proved that these modified sums approximate their limits better than classical partial sums. But out of all the modified sums introduced so far, the sums introduced by Rees and C.V. Stanojevic [15], Kumari and Ram [12] have been widely studied along with their complex forms.

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In this paper we will consider only modified cosine sums as results on modified sine sums can be interpreted on similar lines.

Rees and Stanojevic [15] introduced following modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_j) \cos kx$$

and it is also studied by many other authors like Garett and Stanojevic [5, 6], Ram [14] and Singh and Sharma [16, 17].

Garett and Stanojevic by considering the class of bounded variation and class C [5] proved the following result;

Theorem A: [5] If $\{a_k\}$ belongs to class C and is of bounded variation, then $||f - g_n|| = o(1)$, $n \to \infty$

Ram proved the following theorem by considering the class S.

Theorem B: [14] If $\{a_k\}$ belongs to class S, then $||f - g_n|| = o(1)$, $n \to \infty$

Kaur and Bhatia proved the following result considering r-th derivative of (1.1)

Theorem C: [9] If $\{a_k\}$ belongs to class S_r, then $||f^r - g^r_n|| = o(1)$, $n \to \infty$,

From above results, it is clear that authors proved the results without considering the condition $a_n \log n = o(1)$; $n \to \infty$. Thus this is the improvement as compared to classical partial sums where the condition $a_n \log n = o(1)$; $n \to \infty$ is necessary and sufficient for the L^1-convergence of Fourier series

Kumari and Ram [12] introduced modified cosine sums as

$$h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j}\right) k \cos kx$$

and proved the following theorem

Theorem D: [12] If $\{a_k\}$ belongs to class S, then

$||f - h_n|| = o(1)$, $n \to \infty$ if and only if $a_n \log n = o(1)$, $n \to \infty$

Bor by considering the class S(δ)[see [2]] proved the following result.

Theorem E: [2] If $\{a_k\}$ belongs to class S(δ), then

$||f - h_n|| = o(1)$, $n \to \infty$ if and only if $a_n \log n = o(1)$, $n \to \infty$

The result of Theorem E is an improvement over result of Theorem D where as the result of Theorem E has been further modified by authors Singh and Kaur [18] in which the condition $a_n \log n = o(1)$; $n \to \infty$; has been removed while proving the result.

Bhatia and Ram [3] proved the following result considering r-th derivative of (1.1);

Theorem F: [3] If $\{a_k\}$ belongs to class R_r, then $||f^r - h_n^r|| = o(1)$, $n \to \infty$,

In this paper we will prove some results regarding L^1-convergence of r-th derivative of (1.1) using modified sums $g_n(x)$ and deduce the results for $h_n(x)$. We have attempted to consider both the sums $g_n(x)$ and $h_n(x)$ simultaneously while proving our results. Thus our results will generalise the results proved earlier by various authors like Bhatia and Ram [3] and Kaur and Bhatia [9] in a certain way though these authors have taken different classes of coefficients while proving the results.

Definition: A sequence $\{a_k\}$ is said to be generalised quasi-convex if

$$\sum_{k=1}^{\infty} k^{r+1} |\Delta^2 a_k| < \infty \,;\ r = 0,1,2,3 \ldots \qquad (1.2)$$

Clearly for r=0, (1.2) reduces to class of quasi-convex.

The first result of this paper is the following theorem:

Theorem 1.1: If a null sequence $\{a_k\}$ is generalised quasi-convex, then

(i) $||f^r(x) - g_n^r(x)|| = o(1)$, $n \to \infty$

(ii) $||f^r(x) - g_n^r(x)|| = o(1)$, $(n \to \infty)$ implies $||f^r(x) - h_n^r(x)|| = o(1)$ as $a_n n^r \to 0$, $n \to \infty$

(iii) $||f^r(x) - S_n^r(x)|| = o(1)$, $n \to \infty$, iff $a_{n+1} n^r \log n = o(1)$, $n \to \infty$

Sheng [19] generalised the notion of class S in the following way:

Definition [19]: A sequence $\{a_k\}$ is said to belong to class $S_{p\alpha}$ if $a_k = o(1)$, $k \to \infty$ and there exists a sequence $\{A_k\}$ such that

(a) $A_k \downarrow 0$, $k \to \infty$
(b) $\sum_{k=0}^{\infty} k^{\alpha} A_k < \infty$, for some $\alpha \ge 0$
(c) $\frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta a_k|}{A_k^P} = O(1)$, $1<p\le 2$, $n \to \infty$

The second result of this paper is the following theorem:

Theorem 1.2: Let the sequence $\{a_k\}$ belongs to the class $S_{p\alpha}$, $\alpha \ge 0$, $r \in \{0,1,2,\ldots[\alpha]\}$ and $|a_{n+1}| n^r \log n = o(1)$, $n \to \infty$. Then

$$||f^r(x) - g_n^r(x)|| = o(n^{r-\alpha}),\ n \to \infty \text{ and } ||f^r(x) - h_n^r(x)|| = o(1) \text{ as } a_n n^r \to 0,\ n \to \infty$$

2. LEMMAS

We require the following lemmas for the proof of our results.

2.1. Lemma [19]:

If $D_n(x)$ and $\tilde{D}_n(x)$ are Dirichlet and conjugate Dirichlet kernels respectively and are defined by

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}},\ \tilde{D}_n(x) = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

Then

(a) $\|D_n^r(x)\| = \frac{4}{\pi}(n^r \log n) + O(n^r)$, $r = 0,1,2,3\ldots$, where $D_n^r(x)$ represent r-th derivative of the Dirichlet kernel.

(b) $\|\tilde{D}_n^r(x)\| = O(n^r \log n)$, $r = 0,1,2,3\ldots.$

2.2. Lemma

If $K_n(x)$ denotes Fezer kernel defined by

$$Kn(x) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(x), \text{then}$$

$\|K_n(x)\| = o(1)$ and by Zygmund Theorem [[1]; Vol: II; p:458]; we have

$\|K_n^r(x)\| = O(n^r)$, $r = 0,1,2,3\ldots.$

Also we have well known results

(i) $\tilde{D}'_n(x) = (n+1)D_n(x) - (n+1)K_n(x)$
(ii) $\tilde{D}_n^{r+1}(x) = (n+1)D_n^r(x) - (n+1)K_n^r(x)$

3. PROOF OF THEOREMS

Proof of Theorem 1.1:

We have

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx$$
$$= S_n(x) - a_{n+1}D_n(x)$$

Then

$$g_n^r(x) = S_n^r(x) - a_{n+1}D_n^r(x)$$

Clearly $\lim_{n\to\infty} g_n^r(x) = \lim_{n\to\infty} S_n^r(x) = f^r(x)$ for $x \in (0, \pi]$.

Therefore

$$f^r(x) - g_n^r(x) = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1}D_n^r(x)$$
$$= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m} a_k k^r \cos\left(kx + \frac{k\pi}{2}\right) +\right] + a_{n+1}D_n^r(x)$$

Applying Abel's Transformation twice, we have

$$f^r(x) - g_n^r(x) = \lim_{m\to\infty}\left[\sum_{k=n+1}^{m-1} \Delta a_k D_k^r(x) + a_m D_m^r(x) - a_{n+1}D_n^r(x)\right] + a_{n+1}D_n^r(x)$$
$$= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m-1} \Delta a_k D_k^r(x) + a_m D_m^r(x)\right]$$
$$= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m-2} (k+1)\Delta^2 a_k K_k^r(x) + m\Delta a_{m-1} K_{m-1}^r(x) - (n+1)\Delta a_{n+1} K_n^r(x) + a_m D_m^r(x)\right]$$
$$= \sum_{k=n+1}^{\infty} (k+1)\Delta^2 a_k K_k^r(x) - (n+1)\Delta a_{n+1} K_n^r(x)$$

Thus

$$\| f^r(x) - g_n^r(x)\| \le \sum_{k=n+1}^{\infty}(k+1)|\Delta^2 a_k| \int_0^{\pi}|K_k^r(x)|dx - (n+1)|\Delta a_{n+1}| \int_0^{\pi}|K_k^r(x)|dx \tag{3.1}$$

Now since

$$|\Delta a_{n+1}| = |\sum_{k=n+1}^{\infty}\Delta^2 a_k| \le \sum_{k=n+1}^{\infty}\frac{(k+1)}{(k+1)}|\Delta^2 a_k|$$
$$\le \frac{1}{n+1}\sum_{k=n+1}^{\infty}(k+1)|\Delta^2 a_k|$$

And by Lemma 2.2, we have from (3.1)

$$\|f^r(x) - g_n^r(x)\| = O(\sum_{k=1}^{\infty} k^{r+1}|\Delta^2 a_k|)$$

Thus by hypothesis, we get

$$\|f^r(x) - g_n^r(x)\| = o(1), n \to \infty$$

For **Part (ii)** we have

$$g_n^r(x) = S_n^r(x) - a_{n+1}D_n^r(x)$$

And also

$$h_n^r(x) = S_n^r(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)$$

Clearly by using Lemma 2.2

$$g_n^r(x) - h_n^r(x) = -a_{n+1}K_n^r(x) \text{ which implies}$$
$$||g_n^r(x) - h_n^r(x)|| = |a_{n+1}|||K_n^r(x)||$$

Thus

$$\|f^r - h_n^r\| = \|f^r - g_n^r + g_n^r - h_n^r\|$$
$$\le \|f^r - g_n^r|| + ||g_n^r - h_n^r\|$$
$$= o(1) + \|a_{n+1}K_n^r\|$$
$$= o(1) \text{ as } a_n n^r \to 0,\ n \to \infty$$

For the proof **of (iii) part,** we have

$$||S_n^r(x) - f^r(x)|| = ||S_n^r(x) - g_n^r(x) + g_n^r(x) - f^r(x)||$$
$$\le ||S_n^r(x) - g_n^r(x)|| + ||g_n^r(x) - f^r(x)||$$
$$\le \|a_{n+1}D^r(x)\| + \|g_n^r(x) - f^r(x)\|$$

Conversely, we have

$$\|a_{n+1}D^r(x)\| = \|S_n^r(x) - g_n^r(x)\|$$
$$\le \|S_n^r(x) - f^r(x)\| + \|f^r(x) - g_n^r(x)\|.$$

Thus the result follows by Lemma 2.1 and (i) part of the theorem

Remark: The result of (ii) and (iii) part of above Theorem will hold true if we prove the result of (i) part by taking any other class of coefficients.

Proof of Theorem 1.2:

As in Theorem 1.1, we have

$$g_n^r(x) = S_n^r(x) - a_{n+1}D^r(x)$$

and

$$f^r(x) - g_n^r(x) = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1}D_n^r(x)$$

By making use of Abel's Transformation, we have

$$\|f^r(x) - g_n^r(x)\| \le \int_0^{\pi}|\sum_{k=n+1}^{\infty}\Delta a_k D_k^r(x)|dx + \int_0^{\pi}|a_{n+1}D_n^r(x)|\,dx$$
$$= \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}A_k\frac{\Delta a_k}{A_k}D_k^r(x)\right|dx + \int_0^{\pi}|a_{n+1}D_n^r(x)|\,dx$$
$$\le \int_0^{\pi}\left|\sum_{k=n}^{\infty}\Delta A_k\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}D_j^r(x)\right|dx + A_n\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}D_j^r(x)\right|dx + \int_0^{\pi}|a_{n+1}D_n^r(x)|\,dx$$
$$\le \sum_{k=n}^{\infty}|\Delta A_k|\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}D_j^r(x)\right|dx + A_n\int_0^{\pi}\left|\sum_{j=1}^{n}\frac{\Delta a_j}{A_j}D_j^r(x)\right|dx + \int_0^{\pi}|a_{n+1}D_n^r(x)|\,dx$$
$$= I_1 + I_2 + I_3$$

Now

$$I_1 = \sum_{k=n}^{\infty}|\Delta A_k| \int_0^{\pi/k} \left|\sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^r(x)\right| + \sum_{k=n}^{\infty}|\Delta A_k| \int_{\pi/k}^{\pi} \left|\sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^r(x)\right| dx$$

$$= I_{11} + I_{12}$$

But as proved in [19], $I_{11} = o(n^{r-\alpha})$ and $I_{12} = o(n^{r-\alpha})$

Similarly $I_2 = o(n^{r-\alpha})$ and

$$I_3 = \int_0^{\pi}|a_{n+1}D_n^r(x)|\, dx$$

$$\leq |a_{n+1}| \int_0^{\pi}|D_n^r(x)|\, dx$$

$$= o(1),\ n \to \infty, \text{ by hypothesis}$$

Hence $\|g_n^r(x) - f^r(x)\| = o(n^{r-\alpha}),\ n \to \infty$

Now we shall prove that the result

$$\|f^r(x) - g_n^r(x)\| = o(n^{r-\alpha}),\ (n \to \infty)$$

$$\Rightarrow \|f^r - h_n^r\| = o(1) \text{ as } a_n n^r \to 0,\ n \to \infty$$

As discussed in Theorem 1 (part(ii)), we have

$$g_n^r(x) - h_n^r(x) = -a_{n+1}K_n^r(x) \text{ which implies}$$

$$\|g_n^r(x) - h_n^r(x)\| = |a_{n+1}|\|K_n^r(x)\|$$

and

$$\|f^r - h_n^r\| = \|f^r - g_n^r + g_n^r - h_n^r\|$$

$$\leq \|f^r - g_n^r\| + \|g_n^r - h_n^r\|$$

$$= o(n^{r-\alpha}) + \|a_{n+1}K_n^r\|$$

$$= o(a_{n+1} n^r)$$

$$= o(1) \text{ as } a_n n^r \to 0,\ n \to \infty$$

Which prove the results

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