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ON HYPERSURFACE OF A SPECIAL FINSLER SPACE WITH A METRIC $\frac{(\alpha+\beta)^2}{\alpha}$

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ABSTRACT

The purpose of the present paper is to investigate the various kinds of hypersurfaces of Finsler space with special (α, β) metric $\frac{(\alpha+\beta)^2}{\alpha}$.

Key Words: *Special Finsler hypersurface, (α, β)-metic, hyperplane of 1st kind, hyperplane of 2nd kind, Hyperplane of 3rd kind.*

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1. INTRODUCTION

We consider an n-dimensional Finsler space $F^n = (M^n, L)$, i.e., a pair consisting of an n-dimensional differential manifold M^n equipped with a fundamental function L(x, y). The concept of the $L(\alpha, \beta)$-metric was introduced by M. Matsumoto [5] and has been studied by many authors ([1], [2], [11], [7]). As well known examples, there are Randers metric $\alpha + \beta$, Kropina metric $\frac{\alpha^2}{\beta}$ and generalized Kropina metric $\frac{\alpha^{m+1}}{\beta^m}$ (m≠ 0, -1) whose studies have greately contributed to the growth of Finsler geometry. A Finsler metric L(x, y) is called an (α, β)- metric $L(\alpha, \beta)$ if L is a positively homogeneous function of α *and* β of degree one, where $\alpha^2 = a_{ij}(x)\, y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M^n.

A hypersurface M^{n-1} of the M^n may be represented parametrically by the equation $x^i = x^i(u^\alpha), \alpha = 1, \ldots\ldots, n-1$, where u^α are Gaussian coordinates on M^{n-1}.

In this present paper, we consider an n-dimensional Finsler space $F^n = (M^n, L)$ *with* (α, β)-metric $L(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha}$ and the hypersurface of F^n with $b_i(x) = \partial_i b$ being the gradient of a scalar function b(x). We prove the condition for this hypersurface to be a hyperplane of 1st kind, 2nd kind and also prove that this hypersurface is not a hyperplane of 3rd kind.

2. PRELIMINARIES

We are devoted to a special Finsler space $F^n = (M^n, L)$ with the metric

$$L(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha}. \tag{2.1}$$

The derivatives of the (2.1) with respect to α and β are given by

$$\begin{aligned} L_\alpha &= \frac{\alpha^2-\beta^2}{\alpha^2}, \\ L_\beta &= \frac{2(\alpha+\beta)}{\alpha}, \\ L_{\alpha\alpha} &= \frac{2\beta^2}{\alpha^3}, \\ L_{\beta\beta} &= \frac{2}{\alpha}, \\ L_{\alpha\beta} &= -\frac{2\beta}{\alpha^2}. \end{aligned} \tag{2.2}$$

Where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$ and $L_{\alpha\beta} = \partial L_\alpha / \partial \beta$.

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If in the special Finsler space $F^n = (M^n, L)$, where $L = \frac{(\alpha+\beta)^2}{\alpha}$, we get $\alpha = \left(a_{ij}(x)y^i y^j\right)^{\frac{1}{2}}$, $\beta = b_i\,(x)y^i$

then normalized element of support l_i=$\partial_i L$ and the angular metric tensor h_{ij} are given by [8]:

$$l_i = \alpha^{-1} L_\alpha\, Y_i + L_\beta\, b_i\,, \tag{2.3}$$

$$h_{ij} = p a_{ij} + q_0\, b_i b_j + q_1\left(b_i Y_j + b_j Y_i\right) + q_2 Y_i Y_j\,, \tag{2.4}$$

Where

$$\begin{aligned} &Y_i = a_{ij} y^j,\\ &p = L L_\alpha \alpha^{-1} = \frac{(\alpha+\beta)^2(\alpha^2-\beta^2)}{\alpha^4},\\ &q_0 = L L_{\beta\beta} = \frac{2(\alpha+\beta)^2}{\alpha^2},\\ &q_1 = L L_{\alpha\beta}\alpha^{-1} = -\frac{2\beta(\alpha+\beta)^2}{\alpha^4},\\ &q_2 = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \frac{(\alpha+\beta)^2(3\beta^2-\alpha^2)}{\alpha^6}\,. \end{aligned} \tag{2.5}$$

The fundamental tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$ is given by [9]

$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_1\left(b_i Y_j + b_j Y_i\right) + p_2 Y_i Y_j\,, \tag{2.6}$$

Where

$$\begin{aligned} &p_0 = q_0 + L_\beta^2 = \frac{6(\alpha+\beta)^2}{\alpha^2},\\ &p_1 = q_1 + L^{-1} p L_\beta = \frac{2(\alpha^3 - 3\alpha\beta^3 - 2\beta^3)}{\alpha^4},\\ &p_2 = q_2 + p^2 L^{-2} = \frac{4\beta^4 + 6\alpha\beta^3 - 2\alpha^3\beta}{\alpha^6}\,. \end{aligned} \tag{2.7}$$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1} a^{ij} - S_0 b^i b^j - S_1(b^i j^j + b^j y^i) - S_2 y^i y^j, \tag{2.8}$$

Where

$$\begin{aligned} &b^i = a^{ij} b_j, \qquad S_0 = (p p_0 + (p_0 p_2 - p_1^2)\alpha^2)/\zeta p,\\ &S_1 = (p p_1 - (p_0 p_2 - p_1^2)\beta/\zeta p,\\ &S_2 = \frac{p p_2 + (p_0 p_2 - p_1^2) b^2}{\zeta p}, \qquad b^2 = a_{ij} b^i b^j,\\ &\zeta = p(p + p_0 B^2 + p_1\beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2)\,. \end{aligned} \tag{2.9}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [11]

$$2p C_{ijk} = p_1\left(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j\right) + \gamma_1 m_i m_j m_k\,, \tag{2.10}$$

Where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3 p_1 q_0, \qquad m_1 = b_i - \alpha^{-2}\beta Y_i\,. \tag{2.11}$$

Here m_i is non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\left\{ {i \atop jk} \right\}$ be the components of Christoffel symbols of the associated Riemannian space R^n and ∇_k be covariant differentiation with respect to x^k relative to this Christoffel symbols.

We put

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji}, \tag{2.12}$$

Where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ be the Cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \left\{ {i \atop jk} \right\}$ of the special Finsler space F^n is given by

$$D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s \big(C^i_{jm} C^m_{sk} + CkmiCsjm - CjkmCmsi. \tag{2.13}$$

Where

$$B_k = p_0 b_k + p_1 Y_k, \qquad B^i = g^{ij} B_j, \qquad F^k_i = g^{kj} F_{ji}$$

$$B_{ij} = \left\{ p_1 \big(a_{ij} - \alpha^{-2} Y_i Y_j \big) + \frac{\partial p_0}{\partial \beta} m_i m_j \right\} / 2,$$

$$B^k_i = g^{kj} B_{ji}, \tag{2.14}$$

$$A^m_k = b^m_k E_{00} + B^m E_{ko} + B_k F^m_0 + B_0 F^m_k \;,$$

$$\lambda^m = B^m E_{00} + 2B_0 F^m_0, \qquad B_0 = B_i y^i.$$

Where '0' denote contraction with y^i expect for the quantities p_0, q_0 *and* S_0 .

3. INDUCED CARTAN CONNECTION

Let F^{n-1} be a hypersurface of F^n given by the equations $x^i = x^i(u^\alpha)$. The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is

$$y^i = B^i_\alpha(u) v^\alpha \;. \tag{3.1}$$

The metric tensor $g_{\alpha\beta}$ and HV-torsion tensor $C_{\alpha\beta\gamma}$ *of* F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta \,, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma \,. \tag{3.2}$$

At each point u^α *of* F^{n-1}, a unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}\big(x(u,v), y(u,v)\big) B^i_\alpha N^j = 0, \qquad g_{ij}\big(x(u,v), y(u,v)\big) N^i N^j = 1\,. \tag{3.3}$$

As for the angular metric tensor h_{ij} , we have

$$h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \qquad h_{ij} B^i_\alpha N^j = 0, \qquad h_{ij} N^i N^j = 1. \tag{3.4}$$

If (B^α_i, N_i) denote the inverse of (B^i_α, N^i), then we have

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \;\; B^i_\alpha B^\beta_i = \delta^\beta_\alpha,$$

$$B^\alpha_i N^i = 0, \qquad B^i_\alpha N_i = 0, \qquad N_i = g_{ij} N^j, \tag{3.5}$$

$$B^k_i = g^{kj} B_{ji}, \tag{3.6}$$

$$B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma},\ G^{\alpha}_{\beta},\ C^{\alpha}_{\beta\gamma})$ of F^{n-1} induced from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk},\ \Gamma^{*i}_{0k},\ C^{i}_{jk})$ is given by [6]

$$\Gamma^{*\alpha}_{\beta\gamma} = B^{\alpha}_{i}\left(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}\right) + M^{\alpha}_{\beta}H_{\gamma}\ , \tag{3.7}$$

$$G^{\alpha}_{\beta} = B^{\alpha}_{i}(B^{i}_{0\beta} + \Gamma^{*i}_{0j}B^{j}_{\beta}), \tag{3.8}$$

$$C^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}C^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}, \tag{3.9}$$

Where

$$M_{\beta\gamma} = N_{i}C^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma},\ M^{\alpha}_{\beta} = g^{\alpha\gamma}M_{\beta\gamma}, \tag{3.10}$$

$$H_{\beta} = N_{i}(B^{i}_{0\beta} + \Gamma^{*i}_{0j}B^{j}_{\beta}), \tag{3.11}$$

and $B^{i}_{\beta\gamma} = {\partial B^{i}_{\beta}}/{\partial u^{\gamma}}$, $B^{i}_{0\beta} = B^{i}_{\alpha\beta}v^{\alpha}$. The quantities $M_{\beta\gamma}\ and\ H_{\beta}$ are called the second fundamental v-tensor and normal curvature vector respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]

$$H_{\beta\gamma} = N_{i}\left(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk}B^{j}_{\beta}B^{k}_{\gamma}\right) + M_{\beta}H_{\gamma}\ , \tag{3.12}$$

Where

$$M_{\beta} = N_{i}C^{i}_{jk}B^{j}_{\beta}N^{k}. \tag{3.13}$$

The relative h and v-covariant derivatives of projection factor B^{i}_{α} with respect to I$C\Gamma$ are given by

$$B^{i}_{\alpha|\beta} = H_{\alpha\beta}N^{i}, \qquad B^{i}_{\alpha}|_{\beta} = M_{\alpha\beta}N^{i}. \tag{3.14}$$

The equation (3.12) shows that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta}H_{\gamma} - M_{\gamma}H_{\beta}. \tag{3.15}$$

The above equation yield

$$H_{0\gamma} = H_{\gamma}\ ,\quad H_{\gamma 0} = H_{\gamma} + M_{\gamma}H_{0}. \tag{3.16}$$

We use following lemmas which are due to Matsumoto [6] :

Lemma: 3.1 The normal curvature $H_0 = H_{\beta}v^{\beta}$ vanishes if and only if the normal curvature vector H_{β} vanishes.

Lemma: 3.2 A hypersurface F^{n-1} is a hyperplane of the 1st kind if and only if $H_{\alpha} = 0$.

Lemma: 3.3 A hypersurface F^{n-1} is a hyperplane of the 2nd kind with respect to the connection $C\Gamma$ if and only if $H_{\alpha} = 0\ and\ H_{\alpha\beta} = 0$.

Lemma: 3.4 A hypersurface F^{n-1} is a hyperplane of the 3rd kind with respect to the connection $C\Gamma$ if and only if $H_{\alpha} = 0\ \ and\ \ H_{\alpha\beta} = M_{\alpha\beta} = 0$.

4. HYPERSURFACE $F^{n-1}(c)$ OF THE SPECIAL FINSLER SPACE

Let us consider special Finsler metric $L = \frac{(\alpha+\beta)^2}{\alpha}$ with a gradient $b_i(x) = \partial_i b$ for a scalar function b(x) and a hypersurface $F^{n-1}(c)$ given by the equation b(x) = c (constant) [10]. From parametric equation

$x^i = x^i(u^{\alpha})\ of\ F^{n-1}(c)$, we get $\partial_{\alpha}b\big(x(u)\big) = 0 = b_iB^{i}_{\alpha}$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$ we have

$$b_iB^{i}_{\alpha} = 0 \qquad and \qquad b_iy^i = 0. \tag{4.1}$$

The induced metric L(u, v) of $F^{n-1}(c)$ is given by

$$L(u,v) = a_{\alpha\beta}v^{\alpha}v^{\beta}, \qquad a_{\alpha\beta} = a_{ij}B^i_{\alpha}B^j_{\beta} \tag{4.2}$$

Which is the Riemannian metric.

At a point of $F^{n-1}(c)$, from (2.5), (2.7) and (2.9), we have

$$p = 1,\ \ q_0 = 2, \quad q_1 = 0, \quad q_2 = -\alpha^{-2},\ \ p_0 = 6, \ \ p_1 = \frac{2}{\alpha} \tag{4.3}$$

$$p_2 = 0,\ \ \zeta = 1 + 2b^2, \quad S_0 = {}^{2}/_{(1+2b^2)}, \quad S_1 = {}^{2}/_{\alpha(1+2b^2)}, \quad S_2 = {}^{-4b^2}/_{\alpha^2(1+2b^2)}$$

Therefore, from (2.8) we get

$$g^{ij} = a^{ij} - \frac{2}{1+2b^2}b^ib^j - \frac{1}{\alpha(1+2b^2)}(b^iy^j + b^jy^i) + \frac{4b^2}{\alpha^2(1+2b^2)}y^iy^j. \tag{4.4}$$

Thus along $F^{n-1}(c)$, (4.4) and (4.1) lead to $g^{ij}b_ib_j = \frac{b^2}{1+2b^2}$.

Therefore, we get

$$b_i\big(x(u)\big) = \sqrt{\frac{b^2}{1+2b^2}}\,N_i, \qquad b^2 = a^{ij}b_ib_j. \tag{4.5}$$

Where b is the length of the vector b^i.

Again from (4.4) and (4.5) we get

$$b_i = a^{ij}b_j = \sqrt{b^2(1+2b^2)}\,N^i + b^2\alpha^{-1}y^i. \tag{4.6}$$

Thus we have

Theorem: 4.1 Let F^n be a special Finsler space with $L = \frac{(\alpha+\beta)^2}{\alpha}$ and a gradient $b_i(x) = \partial_i b(x)\ and\ F^{n-1}(c)$ be a undersurface of F^n which is given by b(x) = C (constant). Suppose the Riemannian metric $a_{ij}(x)dx^idx^j$ be positive definite and b_i be non-zero field. They the induced metric on F^{n-1} is a Riemannian metric given by (4.2) and relations (4.5) and (4.6) hold.

The angular metric tensor and metric tensor of F^n are given by

$$h_{ij} = a_{ij} + 2b_ib_j - \frac{Y_iY_j}{\alpha^2}, \tag{4.7}$$

$$g_{ij} = a_{ij} + 6b_ib_j + \frac{2}{\alpha}\big(b_iY_j + b_jY_i\big). \tag{4.8}$$

From (4.1), (4.7) and (3.4) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$, then along

$F^{n-1}(c), h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$.

From (2.7), we get

$$\frac{\partial p_0}{\partial\beta} = \frac{12\beta}{\alpha^2}.$$

Thus along $F^{n-1}(c)$, $\frac{\partial p_0}{\partial\beta} = 0$ and therefore (2.11) gives $\gamma_1 = \frac{12}{\alpha},\ m_i = b_i$.

Therefore the hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{\alpha}\big(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j\big) + \frac{12}{\alpha}b_ib_jb_k \tag{4.9}$$

In a special Finsler hypersurface $F^{n-1}(c)$

Therefore, (3.4), (3.10), (3.13), (4.1) and (4.9) give

$$M_{\alpha\beta} = \frac{1}{\alpha}\sqrt{\frac{b^2}{(1+2b^2)}}\, h_{\alpha\beta} \quad and\ M_\alpha = 0. \tag{4.10}$$

From (3.15) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

Theorem: 4.2 The second fundamental v-tensor of special Finsler hypersurface F^{n-1}(c) is given by (4.10) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

Next from (4.1), we get $b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0$. Therefore, from (3.14) and using$b_{i|\beta} = b_{i|j}B^j_\beta + b_{i|j}\, N^j H_\beta$, we get

$$b_{i|j}\, B^i_\alpha B^j_\beta + \ b_i\,|_j\, B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0. \tag{4.11}$$

Since $b_{i|j} \ = -b_h C^h_{ij}$, we get

$b_i\,|_j\, B^i_\alpha N^j = 0.$

Thus (4.11) gives

$$\sqrt{\frac{b^2}{1+2b^2}}\, H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0\ . \tag{4.12}$$

It is noted that $b_{i|j}$ is symmetric. Furthermore, contracting (4.12) with v^β and then with v^α and using (3.1), (3.16) and (4.10) we get

$$\sqrt{\frac{b^2}{1+2b^2}}\, H_\alpha + b_{i|j}B^i_\alpha y^j = 0, \tag{4.13}$$

$$\sqrt{\frac{b^2}{1+2b^2}}\, H_0 + b_{i|j}y^i y^j = 0. \tag{4.14}$$

In view of Lemma (3.1) and (3.2), the hypersurface $F^{n-1}(c)$ is hyperplane of the first kind if and only if $H_0 = 0$. Thus from (4.14) it follows that $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $b_{i|j}y^i y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n depends on y^i.

Since b_i is a gradient vector, from (2.12) we have $E_{ij} = b_{ij},\ F_{ij} = 0\ \ and\ \ F^i_j = 0$. Thus (2.13) reduces to

$$D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms})\ . \tag{4.15}$$

In view of (4.3) and (4.4), the relations in (2.14) become to

$$B_i = 6b_i + \alpha^{-1} Y_i\ , \qquad B^i = \frac{2b^i}{1+2b^2} + \frac{y^i}{\alpha(1+2b^2)}\,, \tag{4.16}$$

$$B_{ij} = \frac{1}{\alpha}(a_{ij} - \alpha^{-2} Y_i Y_j)\ , \qquad B^i_j = \frac{1}{\alpha}((\delta^i_j - \alpha^{-2} Y_j\, y^i),$$

$$A^m_k = B^m_k b_{00} + B^m b_{k0}, \qquad \lambda^m = B^m b_{00}.$$

By virtue of (4.16) we have $B^i_0 = 0,\ \ B_{i0} = 0$ which leads $\ \ A^m_0 = B^m b_{00}$.

Therefore we have

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}\ \ , \tag{4.17}$$

$$D^i_{00} = B^i b_{00} = \left[\frac{2b^i}{1+2b^2} + \frac{y^i}{\alpha(1+2b^2)}\right] b_{00}.$$

Thus from the relation (4.1), we get

$$b_i D^i_{j0} = \frac{2b^2}{1+2b^2} b_{j0} - 2b^m b_i C^i_{jm} b_{00}, \tag{4.19}$$

$$b_i D^i_{00} = \frac{2b^2}{1+2b^2} b_{00}. \tag{4.20}$$

From (4.9) it follows that

$$b^m b_i C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Therefore, the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and equations (4.19), (4.20) give

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{1}{1+2b^2} b_{00}.$$

Consequently, (4.13) and (4.14) may be written as

$$\sqrt{b^2} H_\alpha + \frac{1}{\sqrt{1+2b^2}} b_{i0} B^i_\alpha = 0, \tag{4.21}$$

$$\sqrt{b^2} H_0 + \frac{1}{\sqrt{1+2b^2}} b_{00} = 0, \tag{4.22}$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend on y^i. Since y^i is to satisfy (4.1), the condition is written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i.$$

From (4.1) and (4.23) it follows that $b_{00} = 0,\ b_{ij} B^i_\alpha B^j_\beta = 0,\quad b_{ij} B^i_\alpha y^j = 0$. Hence (4.21) gives$H_\alpha = 0$. Again from (4.23) and (4.16) we get $b_{i0} b^i = \frac{c_0 b^2}{2},\quad \lambda^m = 0,\ A^i_j\, B^j_\beta = 0$ and $B_{ij} B^j_\beta = \frac{1}{\alpha} h_{\alpha\beta}$. Thus (3.10), (4.4), (4.5), (4.6), (4.10) and (4.15) give

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^2}{2\alpha(1+2b^2)^2} h_{\alpha\beta}. \tag{4.23}$$

Therefore, eqn. (4.12) reduces to

$$\sqrt{\frac{b^2}{1+2b^2}} H_{\alpha\beta} + \frac{c_0 b^2}{4\alpha(1+2b^2)^2} h_{\alpha\beta} = 0. \tag{4.24}$$

Hence the hypersurface $F^{n-1}(c)$ is umbilic.

Theorem: 4.3 The necessary and sufficient condition for $F^{n-1}(c)$ to be hyperplane of 1st kind is (4.23) and in this case the second fundamental tensor of $F^{n-1}(c)$ is proportional to its angular metric tensor.

In view of Lemma (3.3), $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (4.24), we get $c_0 = c_i(x) y^i = 0$. Therefore, there exist a function $e(x)$ *such that* $c_i(x) = e(x) b_i(x)$. Thus (4.23) gives

$$b_{ij} = e b_i b_j. \tag{4.25}$$

Theorem: 4.4 The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of 2nd kind is (4.25).

Finally from (4.10) and Lemma (3.4) show that $F^{n-1}(c)$ doest not become a hyperplane of third kind.

Theorem: 4.5 The hypersurface $F^{n-1}(c)$ is not a hyperplane of the 3rd kind.

CONCLUSION

The present paper has investigated the various kinds of hypersurface of Finsler space with special (α, β) metric$\frac{(\alpha+\beta)^2}{\alpha}$. By using the angular metric tensor, fundamental tensors and induced cartan connections we proved the hypersurface F^{n-1} is a hyperplane of first kind, second kind as well as third kind. Subsequently we proved that the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric with respect to the Riemannian metric and got the result hypersurface F^{n-1}(c) as umblic. In the special Finsler space, eventually we found that the necessary and sufficient condition for hypersurface F^{n-1}(c) to be a hyperplane only in first kind and second kind but F^{n-1}(c) is not a hyperplane of third kind.

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