



ON HYPERSURFACE OF A SPECIAL FINSLER SPACE WITH A METRIC $\frac{(\alpha+\beta)^2}{\alpha}$

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ABSTRACT

The purpose of the present paper is to investigate the various kinds of hypersurfaces of Finsler space with special (α, β) metric $\frac{(\alpha+\beta)^2}{\alpha}$.

Key Words: Special Finsler hypersurface, (α, β) -metric, hyperplane of 1st kind, hyperplane of 2nd kind, Hyperplane of 3rd kind.

1. INTRODUCTION

We consider an n-dimensional Finsler space $F^n = (M^n, L)$, i.e., a pair consisting of an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. The concept of the $L(\alpha, \beta)$ -metric was introduced by M. Matsumoto [5] and has been studied by many authors ([1], [2], [11], [7]). As well known examples, there are Randers metric $\alpha + \beta$, Kropina metric $\frac{\alpha^2}{\beta}$ and generalized Kropina metric $\frac{\alpha^{m+1}}{\beta^m}$ ($m \neq 0, -1$) whose studies have greatly contributed to the growth of Finsler geometry. A Finsler metric $L(x, y)$ is called an (α, β) -metric $L(\alpha, \beta)$ if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x) y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M^n .

A hypersurface M^{n-1} of the M^n may be represented parametrically by the equation $x^i = x^i(u^\alpha), \alpha = 1, \dots, n-1$, where u^α are Gaussian coordinates on M^{n-1} .

In this present paper, we consider an n-dimensional Finsler space $F^n = (M^n, L)$ with (α, β) -metric $L(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha}$ and the hypersurface of F^n with $b_i(x) = \partial_i b$ being the gradient of a scalar function $b(x)$. We prove the condition for this hypersurface to be a hyperplane of 1st kind, 2nd kind and also prove that this hypersurface is not a hyperplane of 3rd kind.

2. PRELIMINARIES

We are devoted to a special Finsler space $F^n = (M^n, L)$ with the metric

$$L(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha}. \tag{2.1}$$

The derivatives of the (2.1) with respect to α and β are given by

$$\begin{aligned} L_\alpha &= \frac{\alpha^2 - \beta^2}{\alpha^2}, \\ L_\beta &= \frac{2(\alpha+\beta)}{\alpha}, \\ L_{\alpha\alpha} &= \frac{2\beta^2}{\alpha^3}, \\ L_{\beta\beta} &= \frac{2}{\alpha}, \\ L_{\alpha\beta} &= -\frac{2\beta}{\alpha^2}. \end{aligned} \tag{2.2}$$

Where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$ and $L_{\alpha\beta} = \partial L_\alpha / \partial \beta$.

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If in the special Finsler space $F^n = (M^n, L)$, where $L = \frac{(\alpha+\beta)^2}{\alpha}$, we get $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$, $\beta = b_i(x)y^i$

then normalized element of support $l_i = \partial_i L$ and the angular metric tensor h_{ij} are given by [8]:

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i, \quad (2.3)$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j, \quad (2.4)$$

Where

$$\begin{aligned} Y_i &= a_{ij} y^j, \\ p &= L L_\alpha \alpha^{-1} = \frac{(\alpha+\beta)^2 (\alpha^2 - \beta^2)}{\alpha^4}, \\ q_0 &= L L_\beta \beta = \frac{2(\alpha+\beta)^2}{\alpha^2}, \\ q_1 &= L L_{\alpha\beta} \alpha^{-1} = -\frac{2\beta(\alpha+\beta)^2}{\alpha^4}, \\ q_2 &= L \alpha^{-2} (L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \frac{(\alpha+\beta)^2 (3\beta^2 - \alpha^2)}{\alpha^6}. \end{aligned} \quad (2.5)$$

The fundamental tensor $g_{ij} = \frac{1}{2} \partial_i \partial_j L^2$ is given by [9]

$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j, \quad (2.6)$$

Where

$$\begin{aligned} p_0 &= q_0 + L_\beta^2 = \frac{6(\alpha+\beta)^2}{\alpha^2}, \\ p_1 &= q_1 + L^{-1} p L_\beta = \frac{2(\alpha^3 - 3\alpha\beta^3 - 2\beta^3)}{\alpha^4}, \\ p_2 &= q_2 + p^2 L^{-2} = \frac{4\beta^4 + 6\alpha\beta^3 - 2\alpha^3\beta}{\alpha^6}. \end{aligned} \quad (2.7)$$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1} \alpha^{ij} - S_0 b^i b^j - S_1 (b^i Y^j + b^j Y^i) - S_2 Y^i Y^j, \quad (2.8)$$

Where

$$\begin{aligned} b^i &= \alpha^{ij} b_j, \quad S_0 = (p p_0 + (p_0 p_2 - p_1^2) \alpha^2) / \zeta p, \\ S_1 &= (p p_1 - (p_0 p_2 - p_1^2) \beta) / \zeta p, \\ S_2 &= \frac{p p_2 + (p_0 p_2 - p_1^2) b^2}{\zeta p}, \quad b^2 = a_{ij} b^i b^j, \end{aligned} \quad (2.9)$$

$$\zeta = p(p + p_0 B^2 + p_1 \beta) + (p_0 p_2 - p_1^2) (\alpha^2 b^2 - \beta^2).$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2} \partial_k g_{ij}$ is given by [11]

$$2p C_{ijk} = p_1 (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k, \quad (2.10)$$

Where

$$\gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3p_1 q_0, \quad m_1 = b_i - \alpha^{-2} \beta Y_i. \quad (2.11)$$

Here m_i is non-vanishing covariant vector orthogonal to the element of support y^i .

Let $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ be the components of Christoffel symbols of the associated Riemannian space R^n and ∇_k be covariant differentiation with respect to x^k relative to this Christoffel symbols.

We put

$$2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji}, \quad (2.12)$$

Where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma_{jk}^*{}^i, \Gamma_{0k}^*{}^i, C_{jk}^i)$ be the Cartan connection of F^n . The difference tensor $D_{jk}^i = \Gamma_{jk}^*{}^i - \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ of the special Finsler space F^n is given by

$$D_{jk}^i = B^i E_{jk} + F_k^i B_j + F_j^i B_k + B_j^i b_{0k} + B_k^i b_{0j} - b_{0m} g^{im} B_{jk} - C_{jm}^i A_k^m - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m + C_{kmi} C_{sjm} - C_{jkm} C_{msi}). \quad (2.13)$$

Where

$$B_k = p_0 b_k + p_1 Y_k, \quad B^i = g^{ij} B_j, \quad F_i^k = g^{kj} F_{ji}$$

$$B_{ij} = \left\{ p_1 (a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j \right\} / 2,$$

$$B_i^k = g^{kj} B_{ji}, \quad (2.14)$$

$$A_k^m = b_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m,$$

$$\lambda^m = B^m E_{00} + 2B_0 F_0^m, \quad B_0 = B_i y^i.$$

Where '0' denote contraction with y^i expect for the quantities p_0, q_0 and S_0 .

3. INDUCED CARTAN CONNECTION

Let F^{n-1} be a hypersurface of F^n given by the equations $x^i = x^i(u^\alpha)$. The element of support y^i of F^n is to be taken tangential to F^{n-1} , that is

$$y^i = B_\alpha^i(u) v^\alpha. \quad (3.1)$$

The metric tensor $g_{\alpha\beta}$ and HV-torsion tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, \quad C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k. \quad (3.2)$$

At each point u^α of F^{n-1} , a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}(x(u, v), y(u, v)) B_\alpha^i N^j = 0, \quad g_{ij}(x(u, v), y(u, v)) N^i N^j = 1. \quad (3.3)$$

As for the angular metric tensor h_{ij} , we have

$$h_{\alpha\beta} = h_{ij} B_\alpha^i B_\beta^j, \quad h_{ij} B_\alpha^i N^j = 0, \quad h_{ij} N^i N^j = 1. \quad (3.4)$$

If (B_i^α, N_i) denote the inverse of (B_α^i, N^i) , then we have

$$B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j, \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta,$$

$$B_i^\alpha N^i = 0, \quad B_\alpha^i N_i = 0, \quad N_i = g_{ij} N^j, \quad (3.5)$$

$$B_i^k = g^{kj} B_{ji}, \quad (3.6)$$

$$B_\alpha^i B_j^\alpha + N^i N_j = \delta_j^i.$$

The induced connection $IC\Gamma = (\Gamma_{\beta\gamma}^{*\alpha}, G_{\beta}^{\alpha}, C_{\beta\gamma}^{\alpha})$ of F^{n-1} induced from the Cartan's connection $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^i)$ is given by [6]

$$\Gamma_{\beta\gamma}^{*\alpha} = B_i^{\alpha} (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_{\beta}^j B_{\gamma}^k) + M_{\beta}^{\alpha} H_{\gamma}, \quad (3.7)$$

$$G_{\beta}^{\alpha} = B_i^{\alpha} (B_{0\beta}^i + \Gamma_{0j}^{*i} B_{\beta}^j), \quad (3.8)$$

$$C_{\beta\gamma}^{\alpha} = B_i^{\alpha} C_{jk}^i B_{\beta}^j B_{\gamma}^k, \quad (3.9)$$

Where

$$M_{\beta\gamma} = N_i C_{jk}^i B_{\beta}^j B_{\gamma}^k, \quad M_{\beta}^{\alpha} = g^{\alpha\gamma} M_{\beta\gamma}, \quad (3.10)$$

$$H_{\beta} = N_i (B_{0\beta}^i + \Gamma_{0j}^{*i} B_{\beta}^j), \quad (3.11)$$

and $B_{\beta\gamma}^i = \frac{\partial B_{\beta}^i}{\partial u^{\gamma}}$, $B_{0\beta}^i = B_{\alpha\beta}^i v^{\alpha}$. The quantities $M_{\beta\gamma}$ and H_{β} are called the second fundamental v-tensor and normal curvature vector respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]

$$H_{\beta\gamma} = N_i (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_{\beta}^j B_{\gamma}^k) + M_{\beta} H_{\gamma}, \quad (3.12)$$

Where

$$M_{\beta} = N_i C_{jk}^i B_{\beta}^j N^k. \quad (3.13)$$

The relative h and v-covariant derivatives of projection factor B_{α}^i with respect to $IC\Gamma$ are given by

$$B_{\alpha|\beta}^i = H_{\alpha\beta} N^i, \quad B_{\alpha}^i|_{\beta} = M_{\alpha\beta} N^i. \quad (3.14)$$

The equation (3.12) shows that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta} H_{\gamma} - M_{\gamma} H_{\beta}. \quad (3.15)$$

The above equation yield

$$H_{0\gamma} = H_{\gamma}, \quad H_{\gamma 0} = H_{\gamma} + M_{\gamma} H_0. \quad (3.16)$$

We use following lemmas which are due to Matsumoto [6] :

Lemma: 3.1 The normal curvature $H_0 = H_{\beta} v^{\beta}$ vanishes if and only if the normal curvature vector H_{β} vanishes.

Lemma: 3.2 A hypersurface F^{n-1} is a hyperplane of the 1st kind if and only if $H_{\alpha} = 0$.

Lemma: 3.3 A hypersurface F^{n-1} is a hyperplane of the 2nd kind with respect to the connection $C\Gamma$ if and only if $H_{\alpha} = 0$ and $H_{\alpha\beta} = 0$.

Lemma: 3.4 A hypersurface F^{n-1} is a hyperplane of the 3rd kind with respect to the connection $C\Gamma$ if and only if $H_{\alpha} = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

4. HYPERSURFACE $F^{n-1}(c)$ OF THE SPECIAL FINSLER SPACE

Let us consider special Finsler metric $L = \frac{(\alpha+\beta)^2}{\alpha}$ with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and a hypersurface $F^{n-1}(c)$ given by the equation $b(x) = c$ (constant) [10]. From parametric equation

$x^i = x^i(u^{\alpha})$ of $F^{n-1}(c)$, we get $\partial_{\alpha} b(x(u)) = 0 = b_i B_{\alpha}^i$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$ we have

$$b_i B_{\alpha}^i = 0 \quad \text{and} \quad b_i \gamma^i = 0. \quad (4.1)$$

The induced metric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, \quad a_{\alpha\beta} = a_{ij} B_\alpha^i B_\beta^j \quad (4.2)$$

Which is the Riemannian metric.

At a point of $F^{n-1}(c)$, from (2.5), (2.7) and (2.9), we have

$$p = 1, \quad q_0 = 2, \quad q_1 = 0, \quad q_2 = -\alpha^{-2}, \quad p_0 = 6, \quad p_1 = \frac{2}{\alpha} \quad (4.3)$$

$$p_2 = 0, \quad \zeta = 1 + 2b^2, \quad S_0 = 2/(1 + 2b^2), \quad S_1 = 2/\alpha(1 + 2b^2), \quad S_2 = -4b^2/\alpha^2(1 + 2b^2)$$

Therefore, from (2.8) we get

$$g^{ij} = a^{ij} - \frac{2}{1+2b^2} b^i b^j - \frac{1}{\alpha(1+2b^2)} (b^i y^j + b^j y^i) + \frac{4b^2}{\alpha^2(1+2b^2)} y^i y^j. \quad (4.4)$$

Thus along $F^{n-1}(c)$, (4.4) and (4.1) lead to $g^{ij} b_i b_j = \frac{b^2}{1+2b^2}$.

Therefore, we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{1+2b^2}} N_i, \quad b^2 = a^{ij} b_i b_j. \quad (4.5)$$

Where b is the length of the vector b^i .

Again from (4.4) and (4.5) we get

$$b_i = a^{ij} b_j = \sqrt{b^2(1 + 2b^2)} N^i + b^2 \alpha^{-1} y^i. \quad (4.6)$$

Thus we have

Theorem: 4.1 Let F^n be a special Finsler space with $L = \frac{(\alpha+\beta)^2}{\alpha}$ and a gradient $b_i(x) = \partial_i b(x)$ and $F^{n-1}(c)$ be a undersurface of F^n which is given by $b(x) = C$ (constant). Suppose the Riemannian metric $a_{ij}(x) dx^i dx^j$ be positive definite and b_i be non-zero field. They the induced metric on F^{n-1} is a Riemannian metric given by (4.2) and relations (4.5) and (4.6) hold.

The angular metric tensor and metric tensor of F^n are given by

$$h_{ij} = a_{ij} + 2b_i b_j - \frac{Y_i Y_j}{\alpha^2}, \quad (4.7)$$

$$g_{ij} = a_{ij} + 6b_i b_j + \frac{2}{\alpha} (b_i Y_j + b_j Y_i). \quad (4.8)$$

From (4.1), (4.7) and (3.4) it follows that if $h_{\alpha\beta}^{(a)}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$, then along

$$F^{n-1}(c), h_{\alpha\beta} = h_{\alpha\beta}^{(a)}.$$

From (2.7), we get

$$\frac{\partial p_0}{\partial \beta} = \frac{12\beta}{\alpha^2}.$$

Thus along $F^{n-1}(c)$, $\frac{\partial p_0}{\partial \beta} = 0$ and therefore (2.11) gives $\gamma_1 = \frac{12}{\alpha}$, $m_i = b_i$.

Therefore the hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{\alpha} (h_{ij} b_k + h_{jk} b_i + h_{ki} b_j) + \frac{12}{\alpha} b_i b_j b_k \quad (4.9)$$

In a special Finsler hypersurface $F^{n-1}(c)$

Therefore, (3.4), (3.10), (3.13), (4.1) and (4.9) give

$$M_{\alpha\beta} = \frac{1}{\alpha} \sqrt{\frac{b^2}{(1+2b^2)}} h_{\alpha\beta} \quad \text{and} \quad M_{\alpha} = 0. \quad (4.10)$$

From (3.15) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

Theorem: 4.2 The second fundamental v-tensor of special Finsler hypersurface $F^{n-1}(c)$ is given by (4.10) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

Next from (4.1), we get $b_{i|\beta} B_{\alpha}^i + b_i B_{\alpha|\beta}^i = 0$. Therefore, from (3.14) and using $b_{i|\beta} = b_{i|j} B_{\beta}^j + b_{i|j} N^j H_{\beta}$, we get

$$b_{i|j} B_{\alpha}^i B_{\beta}^j + b_i |_{j} B_{\alpha}^i N^j H_{\beta} + b_i H_{\alpha\beta} N^i = 0. \quad (4.11)$$

Since $b_{i|j} = -b_n C_{ij}^h$, we get

$$b_i |_{j} B_{\alpha}^i N^j = 0.$$

Thus (4.11) gives

$$\sqrt{\frac{b^2}{1+2b^2}} H_{\alpha\beta} + b_{i|j} B_{\alpha}^i B_{\beta}^j = 0. \quad (4.12)$$

It is noted that $b_{i|j}$ is symmetric. Furthermore, contracting (4.12) with v^{β} and then with v^{α} and using (3.1), (3.16) and (4.10) we get

$$\sqrt{\frac{b^2}{1+2b^2}} H_{\alpha} + b_{i|j} B_{\alpha}^i y^j = 0, \quad (4.13)$$

$$\sqrt{\frac{b^2}{1+2b^2}} H_0 + b_{i|j} y^i y^j = 0. \quad (4.14)$$

In view of Lemma (3.1) and (3.2), the hypersurface $F^{n-1}(c)$ is hyperplane of the first kind if and only if $H_0 = 0$. Thus from (4.14) it follows that $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $b_{i|j} y^i y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to Γ of F^n depends on y^i .

Since b_i is a gradient vector, from (2.12) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$ and $F_j^i = 0$. Thus (2.13) reduces to

$$D_{jk}^i = B^i b_{jk} + B_j^i b_{0k} + B_k^i b_{0j} - b_{0m} g^{im} B_{jk} - C_{jm}^i A_k^m - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m + C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i). \quad (4.15)$$

In view of (4.3) and (4.4), the relations in (2.14) become to

$$B_i = 6b_i + \alpha^{-1} Y_i, \quad B^i = \frac{2b^i}{1+2b^2} + \frac{y^i}{\alpha(1+2b^2)}, \quad (4.16)$$

$$B_{ij} = \frac{1}{\alpha} (a_{ij} - \alpha^{-2} Y_i Y_j), \quad B_j^i = \frac{1}{\alpha} (\delta_j^i - \alpha^{-2} Y_j y^i),$$

$$A_k^m = B_k^m b_{00} + B^m b_{k0}, \quad \lambda^m = B^m b_{00}.$$

By virtue of (4.16) we have $B_0^i = 0$, $B_{i0} = 0$ which leads $A_0^m = B^m b_{00}$.

Therefore we have

$$D_{j0}^i = B^i b_{j0} + B_j^i b_{00} - B^m C_{jm}^i b_{00}, \quad (4.17)$$

$$D_{00}^i = B^i b_{00} = \left[\frac{2b^i}{1+2b^2} + \frac{y^i}{\alpha(1+2b^2)} \right] b_{00}.$$

Thus from the relation (4.1), we get

$$b_i D_{j0}^i = \frac{2b^2}{1+2b^2} b_{j0} - 2b^m b_i C_{jm}^i b_{00}. \quad (4.19)$$

$$b_i D_{00}^i = \frac{2b^2}{1+2b^2} b_{00}. \quad (4.20)$$

From (4.9) it follows that

$$b^m b_i C_{jm}^i B_\alpha^j = b^2 M_\alpha = 0.$$

Therefore, the relation $b_{lij} = b_{ij} - b_r D_{ij}^r$ and equations (4.19), (4.20) give

$$b_{lij} y^i y^j = b_{00} - b_r D_{00}^r = \frac{1}{1+2b^2} b_{00}.$$

Consequently, (4.13) and (4.14) may be written as

$$\sqrt{b^2} H_\alpha + \frac{1}{\sqrt{1+2b^2}} b_{i0} B_\alpha^i = 0, \quad (4.21)$$

$$\sqrt{b^2} H_0 + \frac{1}{\sqrt{1+2b^2}} b_{00} = 0, \quad (4.22)$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend on y^i . Since y^i is to satisfy (4.1), the condition is written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i.$$

From (4.1) and (4.23) it follows that $b_{00} = 0$, $b_{ij} B_\alpha^i B_\beta^j = 0$, $b_{ij} B_\alpha^i y^j = 0$. Hence (4.21) gives $H_\alpha = 0$. Again from (4.23) and (4.16) we get $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A_j^i B_\beta^j = 0$ and $B_{ij} B_\beta^j = \frac{1}{\alpha} h_{\alpha\beta}$. Thus (3.10), (4.4), (4.5), (4.6), (4.10) and (4.15) give

$$b_r D_{ij}^r B_\alpha^i B_\beta^j = -\frac{c_0 b^2}{2\alpha(1+2b^2)^2} h_{\alpha\beta}. \quad (4.23)$$

Therefore, eqn. (4.12) reduces to

$$\sqrt{\frac{b^2}{1+2b^2}} H_{\alpha\beta} + \frac{c_0 b^2}{4\alpha(1+2b^2)^2} h_{\alpha\beta} = 0. \quad (4.24)$$

Hence the hypersurface $F^{n-1}(c)$ is umbilic.

Theorem: 4.3 The necessary and sufficient condition for $F^{n-1}(c)$ to be hyperplane of 1st kind is (4.23) and in this case the second fundamental tensor of $F^{n-1}(c)$ is proportional to its angular metric tensor.

In view of Lemma (3.3), $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (4.24), we get $c_0 = c_i(x) y^i = 0$. Therefore, there exist a function $e(x)$ such that $c_i(x) = e(x) b_i(x)$. Thus (4.23) gives

$$b_{ij} = e b_i b_j. \quad (4.25)$$

Theorem: 4.4 The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of 2nd kind is (4.25).

Finally from (4.10) and Lemma (3.4) show that $F^{n-1}(c)$ does not become a hyperplane of third kind.

Theorem: 4.5 The hypersurface $F^{n-1}(c)$ is not a hyperplane of the 3rd kind.

CONCLUSION

The present paper has investigated the various kinds of hypersurface of Finsler space with special (α, β) metric $\frac{(\alpha+\beta)^2}{\alpha}$. By using the angular metric tensor, fundamental tensors and induced cartan connections we proved the hypersurface F^{n-1} is a hyperplane of first kind, second kind as well as third kind. Subsequently we proved that the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric with respect to the Riemannian metric and got the result hypersurface $F^{n-1}(c)$ as umblic. In the special Finsler space, eventually we found that the necessary and sufficient condition for hypersurface $F^{n-1}(c)$ to be a hyperplane only in first kind and second kind but $F^{n-1}(c)$ is not a hyperplane of third kind.

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