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COMMON FIXED POINT FOR COMPATIBLE MAPPINGS OF TYPE (A-1) IN METRIC SPACES

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ABSTRACT

The aim of this paper is to obtain a common fixed point theorem for compatible mappings of type (A-1) in a metric space which generalizes the result of A.K.Sharma, V.H.Badshah and V.K.Gupta [6].

Keywords: *Fixed point, self maps, compatible mappings of type (A-1), associated sequence.*

AMS (2010) Mathematics Classification: *54H25, 47H10.*

1. INTRODUCTION

In 1986, G.Jungck[1] introduced the concept of compatible maps which is more general than that of weakly commuting maps. In 1993, Jungck and Cho [7] introduced the concept of compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Further Pathak and Khan [10] introduced the concepts of A-compatibility and S-compatibility by splitting the definition of compatible mapping of type (A). In 2007, Pathak *et.al* [8] renamed A-compatibility and S-compatibility as compatible mappings of type (A-1) and compatible mappings of type (A-2) respectively.

The purpose of this paper is to prove a common fixed point theorem for four self maps in metric space using weaker conditions such as compatible mappings of type (A-1) and associated sequence related to four self maps.

2. DEFINITIONS AND PRELIMINARIES

2.1 Definition [1]: Two self maps S and T of a metric space (X, d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

2.2 Definition [7]: Two self maps S and T of a metric space (X, d) are said to be compatible mappings of type (A) if $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

2.3 Definition [8]: Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type(A-1) if $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

2.4 Definition [9]: Suppose P, Q, S and T are self maps of a metric space (X, d) such that $S(X) \subset Q(X)$ *and* $T(X) \subset P(X)$. Now for any arbitrary $x_0 \in X$, we have $Sx_0 \in S(X) \subset Q(X)$ so that there is a $x_1 \in X$ such that $Sx_0 = Qx_1$ and for this x_1, there is a point $x_2 \in$ X such that $Tx_1 = Px_2$ and so on. Repeating this process to obtain a sequence $\{y_n\}$ in X such that $y_{2n} = Px_{2n} = Tx_{2n-1}$ *and* $y_{2n+1} = Qx_{2n+1} = Sx_{2n}$ for $n \geq 0$. We shall call this sequence an associated sequence of x_0 relative to the four self maps P, Q, S and T.

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2.5 Proposition: Let S and T be self mappings of a metric space (X, d).

If the pair (S,T) is compatible mappings of type (A-1) and Sz = Tz for some in X , then TSz = SSz.

2.6 Lemma: Let P, Q, S and T be self mappings of a metric space (X, d) satisfying

$$S(X) \subset Q(X) \text{ and } T(X) \subset P(X) \tag{2.6.1}$$

and $$d(Sx, Ty) \le \left[\alpha + \beta \frac{d(\mathrm{S}x, \mathrm{P}x)}{1 + d(Px, Qy)} \right] d(\mathrm{Ty}, Qy) \tag{2.6.2}$$

for all x, y in X, where $\alpha, \beta \ge 0, \alpha + \beta < 1$.

Further if X is complete,then for any $x_0 \in X$ and for any of its associated sequence $\{y_n\} = \{Sx_0, Tx_1, Sx_2, Tx_3, ...Sx_{2n}, Tx_{2n+1}, ...\}$ relative to four self maps, converges to some point in X.

Proof: From (2.4) and (2.6.2), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Tx_{2n-1}, Sx_{2n}) \\ &= d(Sx_{2n}, Tx_{2n-1}) \\ &\le \left[\alpha + \beta \frac{d(\mathrm{S}x_{2n}, \mathrm{P}x_{2n})}{1 + d(Px_{2n}, Qx_{2n-1})} \right] d(\mathrm{T}x_{2n-1}, Qx_{2n-1}) \\ &= \left[\alpha + \beta \frac{d(y_{2n+1}, y_{2n})}{1 + d(y_{2n}, y_{2n-1})} \right] d(y_{2n}, y_{2n-1}) \\ &\le \alpha\, d(y_{2n}, y_{2n-1}) + \beta\, d(y_{2n+1}, y_{2n}) \text{ implies} \end{aligned}$$

$(1-\beta) d(y_{2n}, y_{2n+1}) \le \alpha d(y_{2n-1}, y_{2n})$ so that

$$d(y_{2n}, y_{2n+1}) \le \frac{\alpha}{(1-\beta)} d(y_{2n-1}, y_{2n}) = \mathrm{h}\, d(y_{2n-1}, y_{2n}), \text{where } h = \frac{\alpha}{1-\beta}.$$

That is, $d(y_{2n}, y_{2n+1}) \le h\, d(y_{2n-1}, y_{2n})$. (2.6.3)

Similarly, we can prove that $d(y_{2n+1}, y_{2n+2}) \le h\, d(y_{2n}, y_{2n+1})$. (2.6.4)

Hence, from $(2.6.3)$ and $(2.6.4)$, we get

$$d(y_n, y_{n+1}) \le h\, d(y_{n-1}, y_n) \le h^2 d(y_{n-2}, y_{n-1}) \le .... \le h^n d(y_0, y_1) \; . \tag{2.6.5}$$

Now for any positive integer p, we have

$$\begin{aligned} d(y_n, y_{n+p}) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n+p-1}, y_{n+p}) \\ &\le h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + ... + h^{n+p-1} d(y_0, y_1) \\ &= (h^n + h^{n+1} + ... + h^{n+p-1}) d(y_0, y_1) \\ &= h^n (1 + h + h^2 + ... + h^{p-1}) d(y_0, y_1) \\ &< \frac{h^n}{1-h} d(y_0, y_1) \to 0 \quad \text{as } \mathrm{n} \to \infty, \text{since h<1}. \end{aligned}$$

Thus the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete , the sequence $\{y_n\}$ converges to some point z in X.

2.7 Remark: The converse of the above Lemma is not true. That is, if P, Q, S and T are self maps of a metric space (X, d) satisfying (2.6.1), (2.6.2) and even if for any x_0 in X and for any of its associated sequence converges, then the metric space (X, d) need not be complete.

2.8 Example: Let $X = (0,1]$ with $d(x,y) = |x-y|$ for $x, y \in X$. Define the self maps S, T, P and Q on X by

$$Sx = Tx = \begin{cases} 1-x & if\ 0 < x \le \frac{1}{2} \\ \frac{1}{3} & if\ \frac{1}{2} < x \le 1 \end{cases}, \quad Px = \begin{cases} \frac{1}{2} & if\ 0 < x \le \frac{1}{2} \\ x & if\ \frac{1}{2} < x \le 1 \end{cases} \text{ and } Qx = \begin{cases} \frac{1}{2} & if\ 0 < x \le \frac{1}{2} \\ 2x-1 & if\ \frac{1}{2} < x \le 1 \end{cases}.$$

Then $S(X) = T(X) = \left[\frac{1}{2}, 1\right)$ while $P(X) = \left[\frac{1}{2}, 1\right]$ and Q(X) = (0,1] .

Clearly $S(X) \subset Q(X)$ *and* $T(X) \subset P(X)$. It is also easy to see that the sequence

$Sx_0, Tx_1, Sx_2, Tx_3, ...Sx_{2n}, Tx_{2n+1}, ...$ converges to $\frac{1}{2}$. Also the inequality (2.6.2) holds for $\alpha, \beta \ge 0, \alpha + \beta < 1$. Note that (X, d) is not complete.

Now we generalize the result of A.K.Sharma, V.H.Badshah and V.K.Gupta as follows.

3. MAIN RESULT

3.1 Theorem: Let P,Q,S and T be self maps of a metric space (X, d) satisfying

$$S(X) \subset Q(X) \text{ and } T(X) \subset P(X) \tag{3.1.1}$$

$$d(Sx, Ty) \le \left[\alpha + \beta \frac{d(Sx, Px)}{1 + d(Px, Qy)}\right] d(Ty, Qy) \tag{3.1.2}$$

for all x,y in X where α,β ≥ 0, α+β<1.

one of P and Q is continuous and (3.1.3)

the pairs (P,S) and (Q,T) are compatible mappings of type (A-1) . (3.1.4)

Further if there is point $x_0 \in X$ and an associated sequence $Sx_0, Tx_1, Sx_2, Tx_3......Sx_{2n}, Tx_{2n+1}...$ of x_0 relative to four self maps P, Q, S and T converges to some point $z \in X$, (3.1.5)

then z is a unique common fixed point of P,Q,S and T.

Proof: From (3.1.5), we have

$$Sx_{2n} \to z, Qx_{2n+1} \to z, Tx_{2n+1} \to z \text{ and } Px_{2n+2} \to z \text{ as } n \to \infty \tag{3.1.6}$$

Let the pair (Q,T) be compatible mappings of type(A-1) and Q be continuous.

Then we have $\lim_{n\to\infty} TQx_{2n+1} = \lim_{n\to\infty} QQx_{2n+1} = Qz$. (3.1.7)

Now by (3.1.2), we have

$$d(Sx_{2n}, TQx_{2n+1}) \le \left[\alpha + \beta \frac{d(Sx_{2n}, Px_{2n})}{1 + d(Px_{2n}, QQ_{2n+1})}\right] d(TQ_{2n+1}, QQ_{2n+1})$$

Letting n → ∞ and using (3.1.6) and (3.1.7), we obtain

$$d(z, Qz) \le [\alpha + 0] d(Qz, Qz)$$
$$\le 0, \text{ a contradiction.}$$

Thus we have Qz = z.

Again from (3.1.2) we get

$$d(Sx_{2n}, Tz) \le \left[\alpha + \beta \frac{d(Sx_{2n}, Px_{2n})}{1 + d(Px_{2n}, Qz)}\right] d(Tz, Qz)$$

Letting n → ∞ and using Qz = z , we obtain

$$d(z, Tz) \le [\alpha + 0] d(Tz, z)$$
$$= \alpha\, d(Tz, z)$$
$$\le d(Tz, z), \text{ a contradiction since } \alpha < 1.$$

Thus we have Tz = z.

Now , since Tz = z and $T(X) \subset P(X)$, there exists a u $\in X$ such that

Tz = Pu.

Hence from (3.1.2) ,we get

$$d(Su,Tz) \le \left[\alpha + \beta \frac{d(\text{Su},\text{Pu})}{1+d(Pu,Qz)} \right] d(\text{Tz},Qz)$$

Using Qz = Tz , we obtain

$d(\text{Su},\text{T}\,z) \le 0$, a contradiction.

Thus we have Su = Tz.

Hence Su = Pu = z. Since the pair (P,S) is compatible mappings of type (A-1)
and Su = Pu , so by the Proposition (2.5)

we have SPu = PPu implies Sz = Pz.

Now from (3.1.2), we get

$$d(Sz,Tz) \le \left[\alpha + \beta \frac{d(\text{Sz},\text{Pz})}{1+d(Pz,Qz)} \right] d(\text{Tz},Qz)$$

≤ 0, a contradiction.

Thus we have Sz = Tz.

Therefore $Sz = Pz = Qz = Tz = z$, showing that z is a common fixed point of P, Q, S and T.

Uniqueness: Let z and w be two common fixed points of P,Q,S and T. Then we have $z = Sz = Pz = Qz = Tz$ and $w = Sw = Pw = Qw = Tw$.

Using (3.1.2), we get

$$d(Sz,Tw) \le \left[\alpha + \beta \frac{d(\text{Sz},\text{Pz})}{1+d(Pz,Qw)} \right] d(\text{Tw},Qw) \text{ implies}$$

d(z,w) ≤ 0 , a contradiction.

Thus we have $d(z,w) = 0$ which implies $z = w$.

Hence z is a unique common fixed point of P, Q, S and T.

3.2 Remark: It is easy to verify that the self mappings P,Q,S and T defined in the example (2.8) satisfy all the conditions of the Theorem (3.1). It may be noted that ‘ $\frac{1}{2}$ ’ is the unique common fixed point of P, Q, S and T.

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