International Journal of Mathematical Archive-8(11), 2017, 16-28
IJMA Available online through www.ijma.info ISSN 2229 – 5046

THE BETA TRANSMUTED WEIGHTED EXPONENTIAL DISTRIBUTION

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(Received On: 21-09-17; Revised & Accepted On: 29-10-17)

ABSTRACT

The beta transmuted weighted exponential distribution is introduced in this paper. It contains a number of distributions as special cases. The mathematical properties of the distribution are provided and explicit expressions are derived for the mean deviations and moment generating function. The distribution and moments of order statistics of the new distribution are discussed. Estimation of the model parameters by the methods of moments and maximum likelihood are studied. The observed information matrix is obtained. Finally, the usefulness of the new distribution in analyzing real data set is illustrated.

Key words: *Beta transmuted weighted exponential distribution, moment generating function, order statistics, moments estimation, maximum likelihood estimation.*

1. INTRODUCTION

Gupta and Kundu (2009) introduced the weighted exponential distribution as a generalization of the exponential distribution. It is a life time model that has been extensively used in engineering and medicine. Alqallaf *et al.* (2015) estimated the parameters of the weighted exponential distribution by different estimation methods. Oguntunde (2015) studied exponentiated weighted exponential distribution. Following Nasiru (2015), Oguntunde *et al.* (2016) presented another weighted exponential distribution based on a modified weighted version of Azzalini's (1985) with probability density function (pdf) as follows

$$g\left(x\right)=\left(\beta+1\right)\alpha\, e^{-(\beta+1)\alpha x},\ \alpha>0,\beta\succ 0,x\succ 0,$$

where α is a scale parameter, and β is a shape parameter. Many generalized univariate continuous distributions have been introduced in the statistical literature. The generalization of any distribution is important in order to make its shape more flexible to capture the diversity present in the observe data. One of the classes of generator that is used to generalize the well known distributions is the beta generator class. If G denotes the baseline cumulative distribution function (cdf) of a random variable, then the beta-generalized distribution is defined as

$$F\left(x\right)=I_{G(x)}(a,b)=\frac{1}{B(a,b)}\int_{0}^{G(x)} w^{a-1}(1-w)^{b-1}\,dw,\ a,b>0, \tag{1}$$

where $I_z(a,b)=B_z(a,b)/B(a,b)$ is the incomplete beta function ratio, $B_z(a,b)$ is the incomplete beta function, and $B(a,b)$ is the beta function. The pdf for the beta-generalized distribution in (1) is

$$f\left(x\right)=\frac{1}{B(a,b)}G(x)^{a-1}\left[1-G(x)\right]^{b-1}g(x),\ x>0, \tag{2}$$

where $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$, $\Gamma(.)$ is the gamma function. Based on the above generalization, Nadarajah and Kotz (2005) introduced beta exponential distribution, Famoye *et al.* (2005) studied beta weibull distribution, Akinsete *et al.* (2008) explained beta pareto distribution, Badmus and Bamiduro (2014) investigated some statistical properties of exponentiated weighted weibull distribution, and finally Badmus *et al.* (2015) made a detailed study of beta weighted exponential distribution.

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Recently, Abdelall (2016) introduced another generalization of the weighted exponential which they called the transmuted weighted exponential distribution. A random variable X is said to have transmuted weighted exponential distribution, with parameters $\alpha, \beta \succ 0$ and $|\lambda| \le 1$ if it has pdf given by

$$g(x) = \alpha(\beta+1)\, e^{-\alpha(\beta+1)x}\left[1-\lambda+2\lambda\, e^{-\alpha(\beta+1)x}\right], \quad x \succ 0, \tag{3}$$

where α is a scale parameter, β is a shape parameter and λ is the transmuted parameter. Hence, the cdf of transmuted weighted exponential distribution is

$$G(x) = \left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda\, e^{-\alpha(\beta+1)x}\right]. \tag{4}$$

The rest of this article is outlined as follows: In Section 2, the beta transmuted weighted exponential (BTWE) distribution is introduced. In Section 3, we obtain expansions of the cdf and pdf of the distribution using power series. Moments and moment generating function are studied in Section 4. Quantile function and mean deviation are derived in Section 5 and 6. Order statistics and their moments are derived in Section 7. Estimation of parameters by the moments and maximum likelihood methods are discussed in Section 8. In section 9, the distribution is applied for analyzing real life data. Finally, in Section 10, we make some concluding remarks on our study.

2. THE BETA TRANSMUTED WEIGHTED EXPONENTIAL DISTRIBUTION

The five-parameter BTWE distribution is obtained by taking $G(x)$ in (1) to be the cdf of a transmuted weighted exponential distribution given by (4). The BTWE cdf then becomes

$$\begin{aligned} F(x) &= I_{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda e^{-\alpha(\beta+1)x}\right]}(a,b) \\ &= \frac{1}{B(a,b)} B_{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda e^{-\alpha(\beta+1)x}\right]}(a,b), \quad x \succ 0,\ \alpha,\beta,a,b>0, \text{ and } |\lambda| \le 1. \end{aligned} \tag{5}$$

where $B_z(a,b) = \int_0^z w^{a-1}(1-w)^{b-1}\,dw,\ a,b>0$, is the incomplete beta function. The cdf can be expressed in a closed form using the hypergeometric function (see Gradshteyn and Ryzhik (2000)) as follows:

$$F(x) = \frac{\left\{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda e^{-\alpha(\beta+1)x}\right]\right\}^a}{a\ B(a,b)} \times {}_2F_1\left(a, 1-b; 1+b; \left\{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda\, e^{-\alpha(\beta+1)x}\right]\right\}\right),$$

where $B_z(a,b) = \dfrac{z^a}{a}\ {}_2F_1(a, 1-b; 1+b; z)$, ${}_2F_1(a_1, a_2; b; x) = \sum_{k=0}^{\infty} \dfrac{a_{1(k)} a_{2(k)}}{b_{(k)}} \dfrac{x^k}{k!}$ is the Gauss hypergeometric function with Pochhammer symbol $a_{(k)}$ defined as:

$$a_{(k)} = a(a+1)\ldots(a+k-1), \quad k = 1, 2, \ldots$$
$$a_{(0)} = 1.$$

The pdf $f(x)$ and hazard rate function $h(x)$ are obtained as:

$$\begin{aligned} f(x) &= \frac{1}{B(a,b)} \alpha(\beta+1) e^{-\alpha(\beta+1)x}\left[1-\lambda+2\lambda\, e^{-\alpha(\beta+1)x}\right] \\ &\left\{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda\, e^{-\alpha(\beta+1)x}\right]\right\}^{a-1}\left[1-\left\{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda\, e^{-\alpha(\beta+1)x}\right]\right\}\right]^{b-1}, x>0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} h(x) &= \frac{1}{\left[B(a,b) - B_{G(x)}(a,b)\right]} \alpha(\beta+1) e^{-\alpha(\beta+1)x}\left[1-\lambda+2\lambda\, e^{-\alpha(\beta+1)x}\right] \\ &\left\{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda\, e^{-\alpha(\beta+1)x}\right]\right\}^{a-1}\left[1-\left\{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda\, e^{-\alpha(\beta+1)x}\right]\right\}\right]^{b-1}, x>0, \end{aligned}$$

where $G(x) = \left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda\, e^{-\alpha(\beta+1)x}\right]$. Figures 1, 2 and 3 illustrates some of the possible shapes of the pdf, cdf and hazard rate function of beta transmuted weighted exponential distribution for selected values of parameters $\alpha, \beta, \lambda, a$ and b.

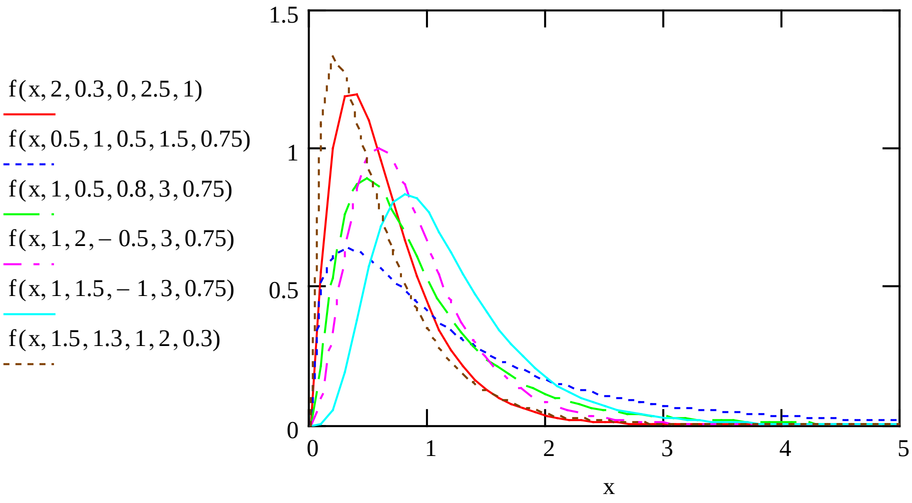

Figure-1: The pdf of various BTWE distributions.

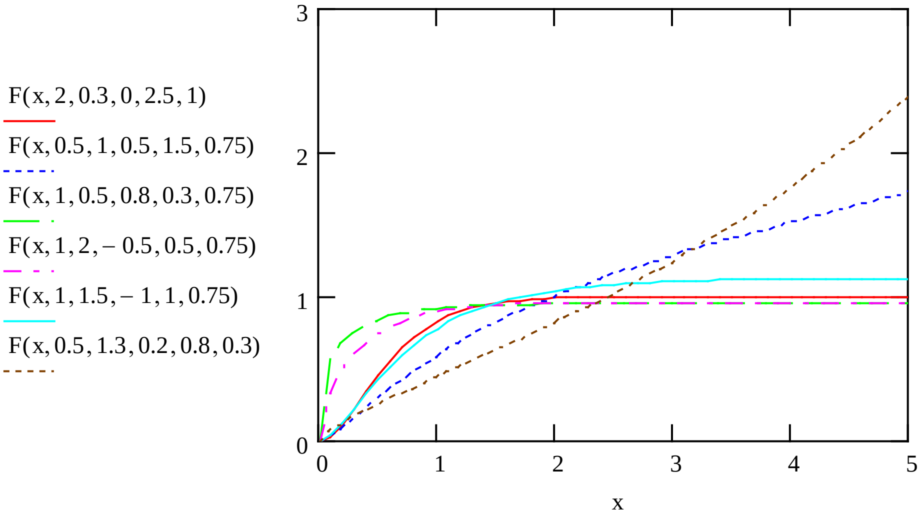

Figure-2: The cdf of various BTWE distributions.

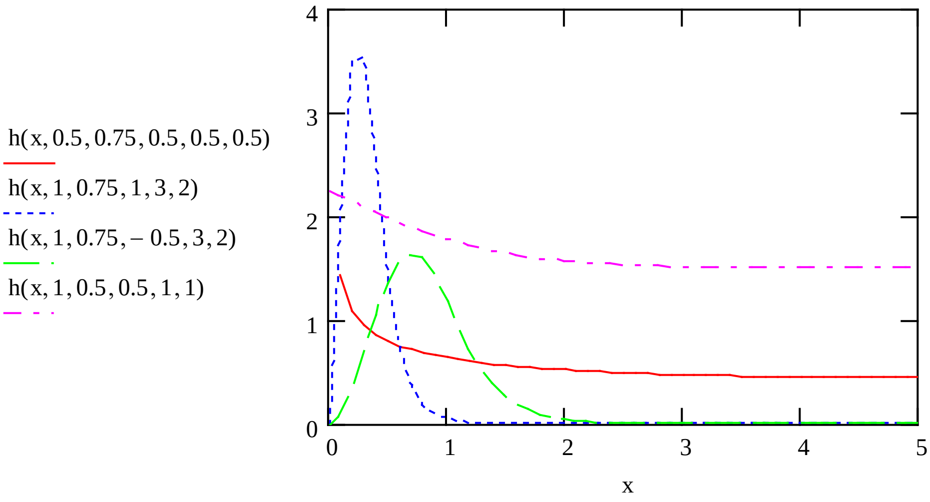

Figure-3: The hazard rate function of various BTWE distributions.

The following distributions are obtained from the BTWE distribution by proper choice of its parameters:

Parameters	Distribution
$b=1$	Exponentiated Transmuted Weighted Exponential (ETWE)
$a=1$	LehmannType II Transmuted Exponential Weighted(LTWE)
$\lambda=0$	Beta Weighted Exponential (BWE)
$\beta=0$	Beta Transmuted Exponential (BTE)
$a=b=1$	Transmuted Weighted Exponential (TWE)
$b=1,\lambda=0$	Exponentiated Weighted Exponential (EWE)
$b=1,\beta=0$	Exponentiated Transmuted Exponential (ETE)
$a=1,\lambda=0$	LehmannType II Exponential Weighted(LWE)
$a=b=1,\lambda=0$	Weighted Exponential (WE)
$a=b=1,\beta=0$	Transmuted Exponential (TE)
$b=1,\beta=0,\lambda=0$	Exponentiated Exponential (EE)
$a=b=1,\beta=0,\lambda=0$	Exponential (E)

3. EXPANSIONS FOR THE CDF AND PDF

Here we express $F(x)$ and $f(x)$ of the BTWE distribution in terms of infinite (finite) weighted sums of cdf's and pdf's of weighted exponential distributions, respectively. We note that for $b \succ 0$ real non-integer, we can replace $(1-w)^{b-1}$ under the integral in (1) by the power series expansion of binomials and integrate to obtain

$$\frac{1}{B(a,b)}\int_0^{G(x)} w^{a-1}(1-w)^{b-1}\,dw=\frac{1}{B(a,b)}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\frac{G(x)^{a+j}}{(a+j)},$$

where the binomial term is defined for any real b as follows $\binom{b-1}{j}=\dfrac{\Gamma(b)}{j!\,\Gamma(b-j)}$.

Then from (5) we get

$$F(x)=\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\frac{\left\{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda e^{-\alpha(\beta+1)x}\right]\right\}^{a+j}}{B(a,b)(a+j)},\quad x\succ 0$$

Using the binomial expansion another two times we have:

$$F(x)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+j}\binom{b-1}{j}\binom{a+j}{k}\binom{a+j}{l}\frac{\lambda^l e^{-\alpha(k+l)(\beta+1)x}}{B(a,b)(a+j)}$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+j}\binom{b-1}{j}\binom{a+j}{k}\binom{a+j}{l}\frac{\lambda^l\left[1-G_1(x;\alpha(k+l),\beta)\right]}{B(a,b)(a+j)},\quad x\succ 0, \tag{7}$$

where $G_1(x;\alpha(k+l),\beta)$ is the weighted exponential cdf with scale $\alpha(k+l)$ and shape β.

Differentiate (7) with respect to x gives a useful expansion of $f(x)$ as

$$f(x)=\sum_{k,l=0}^{\infty}u_{kl}\;g(x;\alpha(k+l),\beta)\;,x\succ 0, \tag{8}$$

where $u_{k,l}=\sum_{j=0}^{\infty}(-1)^{k+j+1}\binom{b-1}{j}\binom{a+j}{k}\binom{a+j}{l}\dfrac{\lambda^l}{B(a,b)(a+j)}$, and $g(x;\alpha(k+l),\beta)$ is the weighted exponential pdf with scale $\alpha(k+l)$ and shape β. If $b\succ 0$ is an integer, the index j in the sum stops at $b-1$, and if a is an integer, the indices k and l in the sum stop at $(a+j)$.

4. MOMENTS AND MOMENT GENERATING FUNCTION

4.1 Moments

The r^th non-central moment of X, denoted by $\mu_r^{'}$, of the BTWE distribution can be easily expressed as functions of weighted exponential by using expression (8) of its pdf. If X has a weighted exponential distribution with scale α and shape β ., then the r^th non-central moment of X is given by

$$\mu_r^{'} = E\left[X^r\right] = \frac{\Gamma(r+1)}{\left[\alpha\,(\beta+1)\right]^r} \;.$$

Hence, for X distributed BTWE with density given by (8) we get

$$\mu_r^{'} = \Gamma(r+1)\sum_{k,l=0}^{\infty} u_{k\,l}\,\frac{1}{\left[\alpha\,(k+l)\,(\beta+1)\right]^r} \;.$$

Therefore, the expected value μ and variance σ^2 of a BTWE distribution are, respectively, given by

$$\mu = E(X) = \sum_{k,l=0}^{\infty} u_{k\,l}\,\frac{1}{\left[\alpha\,(k+l)\,(\beta+1)\right]} \;, \qquad (9)$$

$$\sigma^2 = \sum_{k,l=0}^{\infty} u_{k\,l}\,\frac{2}{\left[\alpha\,(k+l)\,(\beta+1)\right]^2} - \mu^2 \;.$$

Also, the r^th central moment of X μ_r , $r = 1, 2, 3, \ldots$ is related to the r^th non-central moment $\mu_r^{'}$ as

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j\, \mu_{r-j}^{'}\, \mu^j \,,\; \mu_0^{'} = 1, \mu_1^{'} = \mu.$$

The coefficient of skewness and coefficient of kurtosis are respectively given as:

$$CS = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}}{\left(\mu_2^{'} - \mu_1^{'2}\right)^{3/2}} \;,$$

$$CK = \frac{\mu_4}{(\mu_2)^2} = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3\mu_1^{'4}}{\left(\mu_2^{'} - \mu_1^{'2}\right)^2} \;.$$

4.2 Moment Generating Function

The moment generating function of X, denoted by $M_X(t)$, of the BTWE distribution can be easily expressed as functions of weighted exponential by using expression (8) of its pdf. If X has a weighted exponential distribution with scale α and shape β ., then the moment generating function of X is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r\,\Gamma(r+1)}{r!\left[\alpha\,(\beta+1)\right]^r}.$$

Therefore, The moment generating function of BTWE is given by

$$M_X(t) = \sum_{k,l=0}^{\infty} u_{k\,l} \sum_{r=0}^{\infty} \frac{t^r\;\Gamma(r+1)}{r!\,\left[\alpha\,(k+l)\,(\beta+1)\right]^r} \;.$$

5. QUANTILE FUNCTION AND SIMULATION

The quantile function x_q corresponding to the BTWE distribution with cdf (5) is

$$x_q = \left[\frac{-1}{\alpha\,(\beta+1)}\log q\right], \;\; q \in (0,1), \qquad (10)$$

where q is a solution to the quadratic equation

$$\lambda\, q^2 + (1-\lambda)q - \left(1 - I_x^{-1}(a,b)\right) = 0 \,,$$

where $I_x^{-1}(a,b)$ denotes the inverse of incomplete beta function with parameters a and b. Clearly,

$$q=\frac{1}{2\lambda}\left[\sqrt{(1-\lambda)^2+4\lambda\left(1-I_x^{-1}(a,b)\right)}-(1-\lambda)\right].$$

The simulation of X is easily obtained from (10) as follows:

$$x_q=\left[\frac{-1}{\alpha(\beta+1)}\log\left(\frac{\sqrt{(1-\lambda)^2+4\lambda(1-B)}-(1-\lambda)}{2\lambda}\right)\right],$$

where B is a random number following beta distribution with parameters a and b.

6. MEAN DEVIATION

The mean deviation about the mean and the median are useful measures of variation for population. If X has a BTWE distribution, then we can derive the deviations about the mean μ and about the median M as

$$\eta_1(\mu)=\int_0^\infty |x-\mu|\, f(x)\,dx\ , \quad \eta_1(M)=\int_0^\infty |x-M|\, f(x)\,dx.$$

The mean of the BTWE distribution is obtained from (9), and the median is obtained by solving the equation

$$I_{\left[1-e^{-\alpha(\beta+1)x}\right]\left[1+\lambda e^{-\alpha(\beta+1)x}\right]}(a,b)=\frac{1}{2}.$$

Thus, the above measures can be derived from the following relations:

$$\eta_1(\mu)=2\mu F(\mu)-2J(\mu), \quad \eta_1(M)=\mu-2J(M)\ , \tag{11}$$

where $J(r)=\int_0^r x\, f(x)\,dx$. From (8) we have

$$\begin{aligned} J(r) &= \sum_{k,l=0}^{\infty} u_{kl}\int_0^r \alpha(k+l)(\beta+1)\,x\,e^{-\alpha(k+l)(\beta+1)x}\,dx \\ &= \sum_{k,l=0}^{\infty}\frac{u_{kl}}{\alpha(k+l)(\beta+1)}\int_0^{\alpha(k+l)(\beta+1)r} z\,e^{-z}\,dz \\ &= \sum_{k,l=0}^{\infty}\frac{u_{kl}}{\alpha(k+l)(\beta+1)}\,\gamma\left(\alpha(k+l)(\beta+1)r\ ,\ 2\right), \end{aligned}$$

where $\gamma(x,\sigma)=\int_0^x w^{\sigma-1}e^{-w}\,dw,\ \sigma>0$ is an incomplete gamma function. Using (7), one can easily find $\eta_1(\mu),\ \eta_2(M)$ from (11).

7. ORDER STATISTICS

Let $X_{(1)}\le X_{(2)}\le\ldots\le X_{(n)}$ denote the ordered statistics observations in a data set from the BTWE distribution given by (5) and (6), then the pdf of $X_{(j)},\ j=1,2,\ldots,n$ is given by

$$f_{X_{(j)}}(x)=\frac{1}{B(j,n-j+1)}\,f(x)\left[F(x)\right]^{j-1}\left[1-F(x)\right]^{n-j}.$$

Using expressions (7) and (8) for $F(x)$ and $f(x)$ respectively, and applying the binomial expansion yields:

$$f_{X_{(j)}}(x)=\frac{1}{B(j,n-j+1)}\,f(x)\sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\left[F(x)\right]^{s+j-1}$$

$$f_{X_{(j)}}(x)=\frac{\alpha(\beta+1)}{B(j,n-j+1)}\left(\sum_{k,l=0}^{\infty}u_{kl}(k+l)e^{-\alpha(k+l)(\beta+1)x}\right)\sum_{s=0}^{n-j}(-1)^{s+1}\binom{n-j}{s}\left(\sum_{k,l=0}^{\infty}u_{kl}\,e^{-\alpha(k+l)(\beta+1)x}\right)^{s+j-1}.$$

Let $w = e^{-\alpha(\beta+1)x}$, the pdf of the j[th] order statistic for BTWE distribution can be expressed as

$$f_{X_{(j)}}(x) = \frac{\alpha(\beta+1)}{B(j, n-j+1)} \left(\sum_{k,l=0}^{\infty} u_{kl}(k+l) w^{(k+l)} \right) \sum_{s=0}^{n-j} (-1)^{s+1} \binom{n-j}{s} \left(\sum_{k,l=0}^{\infty} u_{kl} w^{(k+l)} \right)^{s+j-1} \tag{12}$$

We note that in (12) we can write

$$\sum_{k,l=0}^{\infty} u_{kl} w^{(k+l)} = \sum_{m=0}^{\infty} u_m^* w^m,$$

and $$\sum_{k,l=0}^{\infty} u_{kl}(k+l) w^{(k+l)} = \sum_{m=0}^{\infty} m u_m^* w^m,$$

where $u_m^* = \sum_{k,l:k+l=m} u_{kl}$. Further, from (Gradshteyn and Ryzhik (2000), Section 0.314), for any positive integer r,

$$\left(\sum_{k=0}^{\infty} a_k u^k \right)^r = \sum_{k=0}^{\infty} d_{r,k} u^k, \tag{13}$$

where the coefficients $d_{r,k}$, for $k = 1, 2, \ldots,$ can be determined from the recurrence relation

$$d_{r,k} = (k a_0)^{-1} \sum_{m=1}^{k} \left[m(r+1) - k \right] a_m d_{r,k-m}, d_{r,0} = a_0^r. \tag{14}$$

Hence, $d_{r,k}$ comes directly from $d_{r,0}, \ldots, d_{r,k-1}$ and, therefore, from $a_0, \ldots, a_k$.

Using (13) and (14) it follows that:

$$f_{X_{(j)}}(x) = \frac{\alpha(\beta+1)}{B(j, n-j+1)} \left(\sum_{m=0}^{\infty} m u_m^* w^m \right) \sum_{s=0}^{n-j} (-1)^{s+1} \binom{n-j}{s} \left(\sum_{m=0}^{\infty} d_{s+j-1,m} w^m \right),$$

where

$$d_{s+j-1,m} = (m u_0^*)^{-1} \sum_{q=1}^{m} [q(s+j) - m] u_m^* d_{s+j-1,m-q} ,$$

$$d_{s+j-1,0} = (u_0^*)^{s+j-1} = \left(\sum_{j=0}^{\infty} (-1)^{j+1} \binom{b-1}{j} \frac{1}{B(a,b)(a+j)} \right)^{s+j-1}.$$

Combing terms, we obtain

$$\begin{aligned} f_{X_{(j)}}(x) &= \frac{\alpha(\beta+1)}{B(j, n-j+1)} \sum_{s=0}^{n-j} (-1)^{s+1} \binom{n-j}{s} \left(\sum_{m=1}^{\infty} \sum_{t=0}^{\infty} m d_{s+j-1,t} u_m^* w^{m+t} \right) \\ &= \frac{1}{B(j, n-j+1)} \sum_{s=0}^{\infty} (-1)^{s+1} \binom{n-j}{s} \sum_{m=1}^{\infty} \sum_{t=0}^{\infty} \frac{m d_{s+j-1,t} u_m^*}{(m+t)} \left\{ (m+t)\alpha(\beta+1) e^{-\alpha(\beta+1)(m+t)x} \right\} \\ &= \sum_{m=1}^{\infty} \sum_{t=0}^{\infty} C_i(m,t) g(x; \alpha(m+t), \beta), \end{aligned}$$

where

$$C_i(m,t) = \frac{1}{B(j, n-j+1)} \frac{m u_m^*}{(m+t)} \sum_{s=0}^{\infty} (-1)^{s+1} \binom{n-j}{s} d_{s+j-1,t}, \tag{16}$$

and

$g(x; \alpha(m+t), \beta)$ denotes the pdf of the weighted exponential with scale parameter $\alpha(m+t)$ and β shape parameter.

The moments of the order statistics of the BTWE distribution can be easily written in terms of those of the weighted exponential by using the expression (15) of the pdf of the order statistic distribution. we get

$$\mu_{X_{(j)}}{}^r = \Gamma(r+1) \sum_{m=1}^{\infty} \sum_{t=0}^{\infty} C_i(m,t) \left[\alpha(m+t)(\beta+1) \right]^{-r},$$ where $C_i(m,t)$ is given in equation (16).

8. PARAMETER ESTIMATION

In this section, the five parameters $(\alpha, \beta, \lambda, a, b)$ of BTWE distribution will be estimated using the method of moments and the method of maximum likelihood.

8.1 Moments Estimation

In order to estimate of the parameters $(\alpha, \beta, \lambda, a, b)$ of BTWE distribution based on the method of moments, where the first five non central population moments $(\mu_r^{'}, r = 1, 2, ..., 5)$ are equated to the five non central sample moments $(m_r^{'}, r = 1, 2, ..., 5)$. That is $\mu_r^{'} = m_r^{'}, r = 1, 2, ..., 5$. The resulting equations are:

$$\sum_{k,l=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k+j+1}\binom{\hat{b}-1}{j}\binom{\hat{a}+j}{k}\binom{\hat{a}+j}{l}\frac{\hat{\lambda}^{l}\,\Gamma(r+1)}{B(\hat{a},\hat{b})(\hat{a}+j)\left[\hat{\alpha}(k+l)(\hat{\beta}+1)\right]^{r}}=\frac{1}{n}\sum_{i=0}^{n}t_i^{r}, \qquad r=1,2,...,5. \tag{17}$$

Because of the complexity of existing equations, it is difficult to obtain a closed form analytic solution of the system in equation (17). Therefore a suitable numerical technique can be used to obtain the required moment estimates of the parameters $\alpha, \beta, \lambda, a$, and b.

8.2 Maximum likelihood Estimation

Let $\theta = (\alpha, \beta, \lambda, a, b)$ denote the parameter vector of the BTWE distribution with pdf given by (6). The likelihood function $l(\theta)$ can be written as

$$l(\theta)=\frac{1}{B^{n}(a,b)}\alpha^{n}(\beta+1)^{n}e^{-\alpha(\beta+1)\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left[1-\lambda+2\lambda e^{-\alpha(\beta+1)x_i}\right]$$
$$\prod_{i=1}^{n}\left\{\left[1-e^{-\alpha(\beta+1)x_i}\right]\left[1+\lambda e^{-\alpha(\beta+1)x_i}\right]\right\}^{a-1}\prod_{i=1}^{n}\left[1-\left\{\left[1-e^{-\alpha(\beta+1)x_i}\right]\left[1+\lambda e^{-\alpha(\beta+1)x_i}\right]\right\}\right]^{b-1}, \tag{18}$$

Then, the log-likelihood function, $\ell(\theta)$, becomes:

$$\ell(\theta)=n\log\alpha-n\log B(a,b)+n\log(\beta+1)-\alpha(\beta+1)\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left[1-\lambda+2\lambda e^{-\alpha(\beta+1)x_i}\right]+$$
$$(a-1)\sum_{i=1}^{n}\left\{\log\left[1-e^{-\alpha(\beta+1)x_i}\right]+\log\left[1+\lambda e^{-\alpha(\beta+1)x_i}\right]\right\}+$$
$$(b-1)\sum_{i=1}^{n}\log\left[1-\left\{\left[1-e^{-\alpha(\beta+1)x_i}\right]\left[1+\lambda e^{-\alpha(\beta+1)x_i}\right]\right\}\right].$$

The maximum likelihood estimates (MLE's), $\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}, \tilde{a}, \tilde{b})$, are obtained by taking the first derivative of log likelihood function with respect to $\alpha, \beta, \lambda, a, b$ and equating to zero using the notation $P_i = e^{-\alpha(\beta+1)x_i}$ as follows:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-(\beta+1)\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\left[\frac{(2(\beta+1)\lambda x_i P_i)}{(1-\lambda+2\lambda P_i)}\right]+(a-1)\sum_{i=1}^{n}\left[\frac{(\beta+1)x_i P_i}{(1-P_i)}\right]-$$
$$(a-1)\sum_{i=1}^{n}\left[\frac{\lambda(\beta+1)x_i P_i}{(1+\lambda P_i)}\right]-(b-1)\sum_{i=1}^{n}\left[\frac{(\beta+1)x_i P_i\{1+\lambda P_i-\lambda(1-P_i)\}}{(1-\{[1-P_i][1+\lambda P_i]\})}\right],$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{(\beta+1)}-\alpha\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\left[\frac{(2\alpha\lambda x_i P_i)}{(1-\lambda+2\lambda P_i)}\right]+(a-1)\sum_{i=1}^{n}\left[\frac{\alpha x_i P_i}{(1-P_i)}\right]-$$
$$(a-1)\sum_{i=1}^{n}\left[\frac{\lambda\alpha x_i P_i}{(1+\lambda P_i)}\right]-(b-1)\sum_{i=1}^{n}\left[\frac{\alpha x_i P_i\{1+2\lambda P_i-\lambda\}}{(1-\{[1-P_i][1+\lambda P_i]\})}\right],$$

$$\frac{\partial \ell}{\partial \lambda}=\sum_{i=1}^{n}\left[\frac{(2P_i-1)}{(1-\lambda+2\lambda P_i)}\right]+(a-1)\sum_{i=1}^{n}\left[\frac{P_i}{(1+\lambda P_i)}\right]-(b-1)\sum_{i=1}^{n}\left[\frac{\{P_i\ (1-P_i)\}}{(1-\{[1-P_i][1+\lambda P_i]\})}\right],$$

$$\frac{\partial \ell}{\partial a}=n\,\Psi(a+b)-n\,\Psi(a)+\sum_{i=1}^{n}\log[1-P_i]+\sum_{i=1}^{n}\log[1+\lambda P_i],$$

and

$$\frac{\partial \ell}{\partial b}=n\,\Psi(a+b)-n\,\Psi(b)+\sum_{i=1}^{n}\log[1-\{[1-P_i][1+\lambda P_i]\}],$$

where $\Psi(x)=\frac{d}{dx}[\log\Gamma(x)]$ is the digamma function. Setting these equations to zero and solving the resulting system of non-linear equations to obtain the maximum likelihood estimators of the unknown parameters $\alpha, \beta, \lambda, a$, and b of the BTWE distribution.

8.3 Asymptotic Confidence Bounds

Since the MLE's of the vector θ of the unknown parameters are not obtained in closed forms, then it is not possible to derive the exact distributions of the MLE's $\tilde{\theta}$. The observed information matrix is obtained for approximate confidence intervals and hypothesis tests of the vector θ. Since the expected information matrix of $\alpha, \beta, \lambda, a$, and b involve numerical integration, the observed 5×5 information matrix I is

$$I=\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda} & I_{\alpha a} & I_{\alpha b}\\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\lambda} & I_{\beta a} & I_{\beta b}\\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\lambda} & I_{\lambda a} & I_{\lambda b}\\ I_{a\alpha} & I_{a\beta} & I_{a\lambda} & I_{aa} & I_{ab}\\ I_{b\alpha} & I_{b\beta} & I_{b\lambda} & I_{ba} & I_{bb}\end{pmatrix},$$

Where, $I_{\theta_i\theta_j}=-\left.\frac{\partial^2\ell(\theta)}{\partial\theta_i\partial\theta_j}\right|_{\theta=\tilde{\theta}}$, $i, j=1,...,5$

Which are given in the Appendix . The centered form of the MLE $\sqrt{n}\left(\tilde{\theta}-\theta\right)$ is asymptotically distributed as Normal$\left(0, I^{-1}\left(\tilde{\theta}\right)\right)$, that can be used to create approximate intervals and confidence regions for the parameters. Asymptotic confidence bounds with significance level γ for each parameter θ_i is given by $\tilde{\theta}_i \pm z_{\frac{\gamma}{2}}\sqrt{\theta^{i,i}}$, where $\theta^{i,i}$ is the i-th diagonal element of I^{-1} for $i=1,2,...,5$ and z_{γ} is the $100(1-\gamma)$% percentile of the standard normal distribution.

9. APPLICATION

In this section, we use a real data set to compare the fits of the BTWE distribution and those of other sub-models, i.e., LTWE $(a=1)$, ETWE $(b=1)$, and TWE $(a=b=1)$ distributions. The data set is obtained from Gupta and Kundu (2009). The data set are the survival times of guinea pigs of sample size (72) injected with different amount of tubercle bacilli:12 15 22 24 24 32 32 33 34 38 38 43 44 48 52 53 54 54 55 56 57 58 58 59 60 60 60 60 61 62 63 65 65 67 68 70 70 72 73 75 76 76 81 83 84 85 87 91 95 96 98 99 109 110 121 127 129 131 143 146 146 175 175 211 233 258 258 263 297 341 341 376. The required numerical evaluations are carried out using the package of Mathcad software.

Table 1 provides the MLE's (with corresponding standard errors in parentheses) of the model parameters. The model selection is carried out using the $\ell(.)$ (log-likelihood function), the AIC (Akaike Information Criteria), the BIC (Bayesian Information Criteria). Table 2 shows the values of $\ell(.)$, AIC, and BIC statistics.

Table-1: MLE's of the model parameters,
the corresponding SEs (given in parentheses), and the statistics $\ell(.)$, AIC, and BIC.

Model	Estimates				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
BTWE	0.069 (6.995E-05)	0.375 (1.224E-03)	0.298 (0.493)	0.893 (0.369)	0.09 (0.021)
LTWE	0.053 (6.046E-05)	0.073 (1.997E-03)	0.191 (1.224)	1	0.127 (0.057)
ETWE	0.011 (4.115E-04)	0.071 (0.041)	0.907 (0.122)	6.859 (1.055)	1
TWE	8.848E-3 (9.86E-04)	0.288 (0.145)	0.929 (0.122)	1	1

The variance covariance matrix of the MLE's under the BTWE distribution is computed as:

$$\begin{pmatrix} 4.893.10^{-9} & -3.405.10^{-6} & -1.969.10^{-5} & 2.661.10^{-6} & 9.781.10^{-7} \\ -3.405.10^{-6} & 1.499.10^{-6} & 2.041.10^{-4} & 4.054.10^{-5} & 3.801.10^{-6} \\ -1.969.10^{-5} & 2.041.10^{-4} & 0.243 & -5.212.10^{-3} & -6.426.10^{-3} \\ 2.66.10^{-6} & 4.054.10^{-5} & -5.212.10^{-3} & 0.136 & 1.989.10^{-3} \\ 9.781.10^{-7} & 3.80.10^{-6} & -6.426.10^{-3} & 1.989.10^{-3} & 4.533.10^{-4} \end{pmatrix}$$

Table-2: Statistics for comparison

Model	Statistic		
	ℓ	AIC	BIC
BTWE	-405.987	821.974	821.261
LTWE	-407.128	822.257	821.686
ETWE	-415.266	838.532	837.961
TWE	-426.817	859.634	859.206

From Table 2,we observe that the BTWE distribution leads to a better fit than the LTWE, ETWE, and TWE distributions. In fact, based on the values of the statistics the BTWE distribution provides the best fit for these data.

10. CONCLUSION

This article introduced the Beta Transmuted Weighted Exponential distribution as a generalized model where some extant models are sub-models. The general mathematical properties of this distribution are studied. The moment generating function and order statistics of the BTWE distribution are derived. The ML estimators and moment estimators of the BTWE parameters are provided. The information matrix and the asymptotic confidence bounds are obtained. Real data set was analyzed and the BTWE has provided a good fit for the data and was more appropriate model compared to the LTWE, ETWE, and TWE sub-models.

REFERENCES

1. Abdelall, Y. Y. (2016). The Transmuted Weighted Exponential Distribution: Properties and Application. International Journal of Mathematical Archive. 7(5). 119-127.
2. Akinsete, A., Famoye, F., and Lee, C. (2008). The Beta Pareto Distribution. Statistics. 42(6). 547-563.
3. Alqallaf, F., Ghitany, M. E., and Agostinelli, C. (2015). Weighted Exponential Distribution: Different methods of Estimations. Applied Math. Inform. Sci. 9. 1167-1173.
4. Azzalini, A. (1985). A class of Distributions which includes the Normal ones. Scandinavian Journal of Statistics. 12. 171-178.
5. Badmus, N. I., and Bamiduro, T. A. (2014). Some Statistical Properties of Exponentiated Weighted Weibull Distribution. SOP Transaction Statistics and Analyziz. 1(2). 1-11.
6. Badmus, N. I., Bamiduro, T. A., Rauf-Animasaun, M. A., and Akingbade, A. A. (2015). Beta Weighted Exponential Distribution: Theory and Application. International Journal of Mathematical Analyziz and Optimization: Theory and Applications. 55-66.

7. Famoye, F., Lee, C., and Olugbenga, O. (2005). The Beta-Weibull Distribution. Journal of Statistical Theory and Applications. 4(2). 121-138.
8. Gradshteyn, I. S., and Ryzhik, I. M. (2000). *Table of Integrals, Series, and Products.* Academic Press, New York.
9. Gupta, R. D., and Kundu, D. (2009). A New Class Some of Weighted Exponential Distributions. Statistics. 43(6). 621-634.
10. Nadarajah, S., and Kotz, S. (2005). On the Moments of the Exponentiated Weibull Distribution. Communication in Statistics, Theory and Methods. 35. 253-256.
11. Nasiru, S. (2015). Another Weighted Weibull Distribution from Azzalini's Family. Eur. Scient. J. 11. 134-144.
12. Oguntunde, P. E. (2015). On the Exponentiated Weighted Exponential Distribution and its Basic Statistical Properties. Applied Sci. Rep. 10. 160-167.
13. Oguntunde, P. E., Owoloko, E. A., and Balogun, O. S. (2016). On A New Weighted Exponential Distribution: Theory and Application. Asian Journal of Applied Sciences. 9(1). 1-12.

APPENDIX

The elements of the observed 5×5 information matrix I are given by $P_i = e^{-\alpha(\beta+1)x_i}$

$$I_{\alpha\alpha} = -\frac{\partial^2 \ell}{\partial\alpha^2} = \frac{n}{\alpha^2} - \sum_{i=1}^{n}\left[\frac{\left(2\lambda(\beta+1)^2 x_i^2 P_i\right)}{(1-\lambda+2\lambda P_i)}\right] + \sum_{i=1}^{n}\left[\frac{\left(2\lambda(\beta+1)x_i P_i\right)^2}{(1-\lambda+2\lambda P_i)^2}\right]$$

$$+(a-1)\sum_{i=1}^{n}\left[\frac{\left((\beta+1)^2 x_i^2 P_i\right)}{(1-P_i)}\left\{1+\frac{P_i}{(1-P_i)}\right\} - \frac{\left(\lambda(\beta+1)^2 x_i^2 P_i\right)}{(1+\lambda P_i)}\left\{1-\frac{\lambda P_i}{(1+\lambda P_i)}\right\}\right]$$

$$-(b-1)\sum_{i=1}^{n}\left[\frac{(\beta+1)^2 x_i^2 P_i}{\left(1-\left\{[1-P_i][1+\lambda P_i]\right\}\right)}\left\{1-\lambda+4\lambda P_i - \frac{(1-\lambda+2\lambda P_i)^2 P_i}{\left(1-\left\{[1-P_i][1+\lambda P_i]\right\}\right)}\right\}\right],$$

$$I_{\alpha\beta} = -\frac{\partial^2 \ell}{\partial\alpha\partial\beta} = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left[\frac{\left(2\lambda x_i P_i\right)}{(1-\lambda+2\lambda P_i)}\right] + \sum_{i=1}^{n}\left[\frac{\left(4\alpha\lambda^2 x_i^2(\beta+1)P_i^2\right)}{(1-\lambda+2\lambda P_i)^2}\right]$$

$$-(a-1)\sum_{i=1}^{n}\left[\frac{x_i P_i}{(1-P_i)}\right] + (a-1)\sum_{i=1}^{n}\left[\begin{array}{l}\dfrac{\alpha(\beta+1)x_i^2 P_i}{(1-P_i)}\left\{1+\dfrac{P_i}{(1-P_i)}\right\} + \dfrac{\lambda x_i P_i}{(1+\lambda P_i)} \\ \times\left\{1-\alpha(\beta+1)x_i + \dfrac{\alpha\lambda x_i(\beta+1)P_i}{(1+\lambda P_i)}\right\}\end{array}\right]$$

$$+(b-1)\sum_{i=1}^{n}\left[\frac{x_i P_i\{1-\lambda+2\lambda P_i\}}{\left(1-\left\{[1-P_i][1+\lambda P_i]\right\}\right)}\left\{\begin{array}{l}1+\alpha(\beta+1)x_i + \dfrac{2\alpha\lambda(\beta+1)x_i}{\{1-\lambda+2\lambda P_i\}} \\ +\dfrac{\alpha(\beta+1)x_i P_i\{1-\lambda+2\lambda P_i\}}{\left(1-\left\{[1-P_i][1+\lambda P_i]\right\}\right)}\end{array}\right\}\right]$$

$$I_{\alpha\lambda} = -\frac{\partial^2 \ell}{\partial\alpha\partial\lambda} = \sum_{i=1}^{n}\left[\frac{\left(2(\beta+1)x_i P_i\right)}{(1-\lambda+2\lambda P_i)}\right] - \sum_{i=1}^{n}\left[\frac{\left(2\lambda(\beta+1)x_i P_i\right)(2P_i-1)}{(1-\lambda+2\lambda P_i)^2}\right]$$

$$+(a-1)\sum_{i=1}^{n}\left[\frac{\left((\beta+1)x_i P_i\right)}{(1+\lambda P_i)}\left\{1-\frac{\lambda P_i}{(1+\lambda P_i)}\right\}\right]$$

$$+(b-1)\sum_{i=1}^{n}\left[\frac{(\beta+1)x_i P_i}{\left(1-\left\{[1-P_i][1+\lambda P_i]\right\}\right)}\left\{(2P_i-1) + \frac{(1-\lambda+2\lambda P_i)(1-P_i)P_i}{\left(1-\left\{[1-P_i][1+\lambda P_i]\right\}\right)}\right\}\right]$$

$$I_{\alpha a}=-\frac{\partial^2\ell}{\partial\alpha\partial a}=-\sum_{i=1}^{n}\left[\frac{\left(\left(\beta+1\right)x_i\,P_i\right)}{\left(1-P_i\right)}\right]+\sum_{i=1}^{n}\left[\frac{\left(\lambda\left(\beta+1\right)x_i\,P_i\right)}{\left(1+\lambda\,P_i\right)}\right],$$

$$I_{\alpha b}=-\frac{\partial^2\ell}{\partial\alpha\partial b}=\left(b-1\right)\sum_{i=1}^{n}\left[\frac{\left(\beta+1\right)x_i\,P_i\left(1-\lambda+2\lambda\,P_i\right)}{\left(1-\left\{\left[1-P_i\right]\left[1+\lambda\,P_i\right]\right\}\right)}\right],$$

$$I_{\beta\beta}=-\frac{\partial^2\ell}{\partial\beta^2}=\frac{n}{\left(\beta+1\right)^2}-\sum_{i=1}^{n}\left[\frac{\left(2\lambda\,\alpha^2\,x_i^{\,2}\,P_i\right)}{\left(1-\lambda+2\lambda\,P_i\right)}\right]+\sum_{i=1}^{n}\left[\frac{\left(2\lambda\,\alpha\,x_i\,P_i\right)^2}{\left(1-\lambda+2\lambda\,P_i\right)^2}\right]$$
$$+\left(a-1\right)\sum_{i=1}^{n}\left[\frac{\left(\alpha^2\,x_i^{\,2}\,P_i\right)}{\left(1-P_i\right)}\left\{1+\frac{P_i}{\left(1-P_i\right)}\right\}-\frac{\left(\lambda\,\alpha^2\,x_i^{\,2}\,P_i\right)}{\left(1+\lambda\,P_i\right)}\left\{1-\frac{\lambda\,P_i}{\left(1+\lambda\,P_i\right)}\right\}\right]$$
$$-\left(b-1\right)\sum_{i=1}^{n}\left[\frac{\alpha^2\,x_i^{\,2}\,P_i}{\left(1-\left\{\left[1-P_i\right]\left[1+\lambda\,P_i\right]\right\}\right)}\left\{1-\lambda+4\lambda\,P_i-\frac{\left(1-\lambda+2\lambda\,P_i\right)^2\,P_i}{\left(1-\left\{\left[1-P_i\right]\left[1+\lambda\,P_i\right]\right\}\right)}\right\}\right],$$

$$I_{\beta\lambda}=-\frac{\partial^2\ell}{\partial\beta\partial\lambda}=\sum_{i=1}^{n}\left[\frac{\left(2\alpha\,x_i\,P_i\right)}{\left(1-\lambda+2\lambda\,P_i\right)}\right]-\sum_{i=1}^{n}\left[\frac{\left(2\lambda\,\alpha\,x_i\,P_i\right)\left(2P_i-1\right)}{\left(1-\lambda+2\lambda\,P_i\right)^2}\right]$$
$$+\left(a-1\right)\sum_{i=1}^{n}\left[\frac{\left(\alpha\,x_i\,P_i\right)}{\left(1+\lambda\,P_i\right)}\left\{1-\frac{\lambda\,P_i}{\left(1+\lambda\,P_i\right)}\right\}\right]$$
$$+\left(b-1\right)\sum_{i=1}^{n}\left[\frac{\alpha\,x_i\,P_i}{\left(1-\left\{\left[1-P_i\right]\left[1+\lambda\,P_i\right]\right\}\right)}\left\{\left(2P_i-1\right)+\frac{\left(1-\lambda+2\lambda\,P_i\right)\left(1-P_i\right)P_i}{\left(1-\left\{\left[1-P_i\right]\left[1+\lambda\,P_i\right]\right\}\right)}\right\}\right]$$

$$I_{\beta a}=-\frac{\partial^2\ell}{\partial\beta\partial a}=-\sum_{i=1}^{n}\left[\frac{\left(\alpha\,x_i\,P_i\right)}{\left(1-P_i\right)}\right]+\sum_{i=1}^{n}\left[\frac{\left(\lambda\,\alpha\,x_i\,P_i\right)}{\left(1+\lambda\,P_i\right)}\right],$$

$$I_{\beta b}=-\frac{\partial^2\ell}{\partial\beta\partial b}=\left(b-1\right)\sum_{i=1}^{n}\left[\frac{\alpha\,x_i\,P_i\left(1-\lambda+2\lambda\,P_i\right)}{\left(1-\left\{\left[1-P_i\right]\left[1+\lambda\,P_i\right]\right\}\right)}\right],$$

$$I_{\lambda\lambda}=-\frac{\partial^2\ell}{\partial\lambda^2}=\sum_{i=1}^{n}\left[\frac{\left(2P_i-1\right)^2}{\left(1-\lambda+2\lambda\,P_i\right)^2}\right]+\left(a-1\right)\sum_{i=1}^{n}\left[\frac{P_i^{\,2}}{\left(1+\lambda\,P_i\right)^2}\right]$$
$$+\left(b-1\right)\sum_{i=1}^{n}\left[\frac{\left(P_i\left(1-P_i\right)\right)}{\left(1-\left\{\left[1-P_i\right]\left[1+\lambda\,P_i\right]\right\}\right)}\right]^2,$$

$$I_{\lambda a}=-\frac{\partial^2\ell}{\partial\lambda\partial a}=-\sum_{i=1}^{n}\left[\frac{P_i}{\left(1+\lambda\,P_i\right)}\right],$$

$$I_{\lambda b}=-\frac{\partial^2\ell}{\partial\lambda\partial b}=\sum_{i=1}^{n}\left[\frac{\left(P_i\left(1-P_i\right)\right)}{\left(1-\left\{\left[1-P_i\right]\left[1+\lambda\,P_i\right]\right\}\right)}\right],$$

$$I_{aa}=-\frac{\partial^2\ell}{\partial a^2}=-n\frac{\partial\Psi\left(a+b\right)}{\partial a}+n\,\Psi'\left(a\right),$$

$$I_{ab} = -\frac{\partial^2 \ell}{\partial a \partial b} = -n\frac{\partial \Psi(a+b)}{\partial b},$$

$$I_{bb} = -\frac{\partial^2 \ell}{\partial b^2} = -n\frac{\partial \Psi(a+b)}{\partial b} + n\,\Psi'(b).$$