



SOME REFINEMENTS OF ENESTROM-KAKEYA THEOREM

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ABSTRACT

In this paper we extend and generalize some known results concerning the Enestrom-Kakeya theorem.

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INTRODUCTION AND STATEMENT OF RESULTS

The following well-known theorem is due to Enestrom and Kakeya [8].

Theorem A: (Enestrom – Kakeya) Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$0 \leq a_1 \leq a_2 \leq \dots \leq a_n$$

then $p(z)$ has all its zeros in the closed unit disk $|z| \leq 1$

In the literature, there exist several generalizations of this result, (see [1], [3], [4], [7], [8]). Aziz and Zargar [2] relaxed the hypothesis in several ways and proved:

Theorem B: If $P(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n such that for some $k \geq 1$,

$$ka_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| + (k - 1) \leq \frac{ka_n + |a_0| - a_0}{|a_n|}$$

For polynomials, whose coefficients are not necessarily real, Govil and Rehamn [6] proved the following generalization of Theorem A.

Theorem C: If $P(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n with

$$\begin{aligned} \operatorname{Re}(a_j) = \alpha_j \text{ and } \operatorname{Im}(a_j) = \beta_j \text{ such that} \\ \alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0 \geq 0 \end{aligned}$$

where $\alpha_n > 0$, then $p(z)$ has all its zeros in

$$|z| \leq 1 + \left(\frac{2}{\alpha_n}\right) \left(\sum_{j=0}^n |\beta_j|\right)$$

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Theorem D: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j, j = 1, 2, \dots, n$. If for some $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0,$$

then $P(z)$ has all its zeros in

$$|z + k - 1| \leq \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2 \sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

Theorem E: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$. If for some $k \geq 1$

$k\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \dots \geq \beta_0$, then $p(z)$ has all its zeros in

$$|z + k - 1| \leq \frac{k\beta_n - \beta_0 + |\beta_0| + 2 \sum_{j=0}^n |\alpha_j|}{|\beta_n|}$$

In this paper we shall present some interesting generalizations of Theorems D and E, and consequently of Enestrom Kakaye Theorem. We begin with

Theorem 1: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$; if for some real number $\tau, 0 < \tau \leq 1$

$$\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \tau\alpha_0 > 0$$

then $P(z)$ has all its zeros in

$$|z| \leq 1 + \frac{2 \left[(1-\tau)\alpha_0 + \sum_{j=0}^n |\beta_j| \right]}{|\alpha_n|}$$

Remark 1: Taking $\tau = 1$, in Theorem 1, we get Theorem C. If we apply Theorem 1 to the polynomial $\{iP(z)\}$, we easily get the following result.

Theorem 2: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$. If for some real number $\tau, 0 < \tau \leq 1$

$$\beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \tau\beta_0 > 0,$$

then $p(z)$ has all its zeros in

$$|z| \leq 1 + \frac{2 \left[(1-\tau)\beta_0 + \sum_{j=0}^n |\alpha_j| \right]}{|\beta_n|}$$

Remark 2: If we take $\tau = 1$ in Theorem 1 and all the coefficients are real i.e. $\beta_j = 0$ for all j , it reduces to Enestrom-kakeya Theorem.

Theorem 3: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$; if for some real numbers $\tau, 0 < \tau \leq 1$ and $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \tau\alpha_0$$

then P(z) has all its zeros in

$$|z + k - 1| \leq \frac{k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

Remark 3: Taking $\tau = 1$ in theorem 3, we get Theorem D. Applying Theorem 3 to $(-ip(z))$, we get

Theorem 4: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$; if for some real numbers $k \geq 1, 0 < \tau \leq 1$

$$k\beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \tau\beta_0$$

then all the zeros of P (z) lie in

$$|z + k - 1| \leq \frac{k\beta_n + 2|\beta_0| - \tau(|\beta_0| + \beta_0) + 2\sum_{j=0}^n |\alpha_j|}{|\beta_n|}$$

Theorem 5: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$. If for some real numbers $k \geq 1, 0 < \lambda \leq n - 1$,

$$\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_\lambda, \quad k\alpha_\lambda \geq \alpha_{\lambda-1} \geq \dots \geq \alpha_1 \geq \alpha_0 > 0$$

then all the zeros of p (z) lie in

$$|z| \leq 1 + \frac{2(k-1)\alpha_\lambda + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

Taking $\lambda=n$ in Theorem 5, we obtain

Corollary 1: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$. If for some real numbers $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0 > 0$$

then all the zeros of p(z) lie in

$$|z| \leq (2k - 1) + \frac{2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

Remark 4: Taking $k=1$ in corollary 1, we get Theorem D of Govil and McTume If all the coefficients are real i.e., $\beta_j = 0, \forall j$ and $k=1$, we get Enestrom-Kakeya Theorem.

Applying Theorem 5 to $(-iP(z))$, we get the following interesting result.

Theorem 6 Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$. If for some real numbers $k \geq 1$, $0 < \lambda < n - 1$,

$$\beta_n \geq \beta_{n-1} \geq \dots \geq \beta_\lambda, \quad k\beta_\lambda \geq \beta_{\lambda-1} \geq \dots \geq \beta_1 \geq \beta_0 \geq 0$$

then $p(z)$ has all its zeros in

$$|z| \leq 1 + \frac{2(k-1)\beta_\lambda + 2\sum_{j=0}^n |\alpha_j|}{|\beta_n|}$$

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)p(z) \\ &= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0 \\ &\quad -i\beta_n z^{n+1} + i\beta_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1}) z^j \\ &= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_1 - \alpha_0)z \\ &\quad + (\tau\alpha_0 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1}) z^j \end{aligned}$$

Now

$$\begin{aligned} |F(z)| &= | -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_1 - \tau\alpha_0)z + (\tau\alpha_0 - \alpha_0)z + \alpha_0 \\ &\quad - i\beta_n z^{n+1} + i\beta_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1}) z^j | \\ &\cong |z|^{n+1} |\alpha_n| - |z|^n \left\{ |(\alpha_n - \alpha_{n-1}) + (\alpha_{n-1} - \alpha_{n-2})1/z + \dots + (\alpha_1 - \tau\alpha_0)1/z^{n-1} \right. \\ &\quad \left. + (\tau - 1)\alpha_0 \cdot 1/z^{n-1} + \alpha_0/z^n - i\beta_n z + i\beta_0 1/z^n \right. \\ &\quad \left. + i \sum_{j=1}^n (\beta_j - \beta_{j-1}) 1/z^{n-j} \right\} \end{aligned}$$

For $|z| > 1$, we have

$$\begin{aligned}
 |F(z)| &> |z|^n \left[|\alpha_n||z| - \{|\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| 1/|z| + \dots + |\alpha_1 \right. \\
 &\quad - \tau\alpha_0| 1/|z|^{n-1} + |(\tau - 1)\alpha_0| 1/|z|^{n-1} + |\alpha_0|/|z|^n + |\beta_n||z| \\
 &\quad \left. + |\beta_0| 1/|z|^n + \sum_{j=1}^n (|\beta_j| + |\beta_{j-1}|) 1/|z|^{n-j} \right] \\
 &> |z|^n \left[|\alpha_n||z| - \{(\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \dots + \alpha_1 - \tau\alpha_0 + (1 - \tau)\alpha_0 + \alpha_0 \right. \\
 &\quad \left. + |\beta_n| + |\beta_0| + \sum_{j=1}^n (|\beta_j| + |\beta_{j-1}|) \} \right]
 \end{aligned}$$

(By hypothesis)

$$\begin{aligned}
 &= |z|^n [|\alpha_n||z| - \{\alpha_n + 2(1 - \tau)\alpha_0 + 2\sum_{j=0}^n |\beta_j|\}] > 0 \text{ if} \\
 &\quad |z| > 1 + \frac{2(1 - \tau)\alpha_0 + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}
 \end{aligned}$$

Hence all the zeros of F(z) whose modulus is greater than 1 lie in

$$|z| \leq 1 + \frac{2(1 - \tau)\alpha_0 + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

Since all the zeros of F(z) whose modulus is ≤ 1 already satisfy the inequality, it follows that all the zeros of F(z) and that of P(z) lie in the circle

$$|z| \leq 1 + \frac{2(1 - \tau)\alpha_0 + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

This completes the proof of the Theorem 1.

Proof of Theorem 3: Consider

$$\begin{aligned}
 F(z) &= (1 - z)p(z) = -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + \alpha_1 z + \alpha_0 \\
 &= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_1 - \alpha_0)z + \alpha_0 \\
 &\quad - i\beta_n z^{n+1} + i\beta_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j \\
 &= \alpha_n z^{n+1} - (|\alpha_n| z^n + |\alpha_n - \alpha_{n-1}| z^n + \dots + |\alpha_1 - \tau\alpha_0| z + |\tau\alpha_0 - \alpha_0| z \\
 &\quad + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j
 \end{aligned}$$

Now for $|z| > 1$

$$\begin{aligned}
 |F(z)| &= \left| -\alpha_n z^{n+1} - k\alpha_n z^n + \alpha_n z^n + (k\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_1 - \tau\alpha_0)z + (\tau\alpha_0 \right. \\
 &\quad \left. - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j \right| \\
 &\geq |z|^n |\alpha_n z + k\alpha_n - \alpha_n| - |z|^n \\
 &\quad \left| (k\alpha_n - \alpha_{n-1}) + (\alpha_{n-1} - \alpha_{n-2})1/z + \dots + (\alpha_1 - \tau\alpha_0)1/z^{n-1} + (\tau\alpha_0 - \alpha_0)1/z^{n-1} \right. \\
 &\quad \left. + \alpha_0/z^n - i\beta_n z + \frac{i\beta_0}{z^n} + i \sum_{j=1}^n (\beta_j - \beta_{j-1})1/z^{n-j} \right| \\
 &> |z|^n [|\alpha_n||z + (k-1)| \\
 &\quad - |z|^n \left(|k\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| \frac{1}{|z|} + \dots + |\alpha_1 - \tau\alpha_0| \frac{1}{|z|^{n-1}} \right. \\
 &\quad \left. + |1 - \tau||\alpha_0| \frac{1}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + |\beta_n||z| + \frac{|\beta_0|}{|z|^n} + \sum_{j=1}^n |\beta_j| + |\beta_{j-1}| \frac{1}{|z|^{n-j}} \right)] \\
 &> |z|^n \left[|\alpha_n| |z + (k-1)| \right. \\
 &\quad \left. - |z|^n \left\{ (k\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \dots + (\alpha_1 - \tau\alpha_0) + (1 - \tau)|\alpha_0| \right. \right. \\
 &\quad \left. \left. + |\alpha_0| + |\beta_n| + |\beta_0| + \sum_{j=1}^n |\beta_j| + |\beta_{j-1}| \right\} \right] \\
 &= |z|^n \left[|\alpha_n| |z + k - 1| - \left(k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2 \sum_{j=0}^n |\beta_j| \right) \right] \\
 &\quad > 0, \text{ if} \\
 &\quad |z + k - 1| > \frac{k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2 \sum_{j=0}^n |\beta_j|}{|\alpha_n|}
 \end{aligned}$$

Hence all the zeros of $F(z)$ whose modulus is greater than 1 lie in

$$|z + k - 1| \leq \frac{k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2 \sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

But all the zeros of F (z) whose modulus is ≤ 1 already satisfy the inequality. This shows that all the zeros of F (z) and hence of P (z) lie in the disk

$$|z + k - 1| \leq \frac{k\alpha_n + 2|\alpha_c| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

That proves the result.

Proof of Theorem 5: Consider the polynomial

$$\begin{aligned} F(z) &= (1 - z)\varphi(z) \\ &= (1 - z)(\alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_{\lambda+1} z^{\lambda+1} + \alpha_\lambda z^\lambda + \alpha_1 z + \alpha_0) \\ &= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \dots + \\ &\quad (\alpha_1 - \alpha_0)z + \alpha_0 \\ &= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \dots \\ &\quad + (\alpha_1 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^n (\beta_j - \beta_{j-1}) \end{aligned}$$

For $|z| > 1$, we have

$$\begin{aligned} |F(z)| &= \left| -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} - k\alpha_\lambda z^\lambda + \alpha_\lambda z^\lambda + (k\alpha_n - \alpha_{n-1})z^n \right. \\ &\quad \left. + \dots + (\alpha_1 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^n (\beta_j - \beta_{j-1})z^j \right| \\ &\geq |z|^n \left[|\alpha_n||z| - \left\{ (\alpha_n - \alpha_{n-1}) + \dots + (\alpha_{\lambda+1} - \alpha_\lambda) \frac{1}{z^{n-\lambda-1}} - (k-1)\alpha_\lambda \frac{1}{z^{n-\lambda}} + (k\alpha_\lambda - \alpha_{\lambda-1}) \frac{1}{z^{n-\lambda}} \right. \right. \\ &\quad \left. \left. + \dots + (\alpha_1 - \alpha_0) \frac{1}{z^{n-1}} + \alpha_0 \frac{1}{z^n} - i\beta_n z + i\beta_0 \frac{1}{z^n} + i\sum_{j=1}^n (\beta_j - \beta_{j-1}) \frac{1}{z^{n-j}} \right\} \right] \\ &> |z|^n \left[|\alpha_n||z| - \left\{ (\alpha_n - \alpha_{n-1}) + \dots + (\alpha_{\lambda+1} - \alpha_\lambda) + (k-1)|\alpha_\lambda| + (k\alpha_\lambda - \alpha_{\lambda-1}) \right. \right. \\ &\quad \left. \left. + \dots + (\alpha_1 - \alpha_0) + |\alpha_0| - |\beta_n| + |\beta_0| + \sum_{j=1}^n |\beta_j| + |\beta_{j-1}| \right\} \right] \\ &= |z|^n \left[|\alpha_n||z| - \left\{ (\alpha_n - \alpha_{n-1}) + \dots + (\alpha_{\lambda+1} - \alpha_\lambda) + (k-1)\alpha_\lambda + (k\alpha_\lambda - \alpha_{\lambda-1}) \right. \right. \\ &\quad \left. \left. + \dots + (\alpha_1 - \alpha_0) + \alpha_0 + 2\sum_{j=0}^n |\beta_j| \right\} \right] \\ &= |z|^n \left[|\alpha_n||z| - \left\{ -\alpha_n + 2(k-1)\alpha_\lambda + 2\sum_{j=0}^n |\beta_j| \right\} \right] > 0, \text{ if} \\ &\quad |z| > 1 + \frac{2(k-1)\alpha_\lambda + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|} \end{aligned}$$

Hence all the zeros of F(z) whose modulus is greater than 1 lie in

$$|z| \leq 1 + \frac{2(k-1)a_\lambda + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

But all those zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Thus, all the zeros of F(z) and therefore P(z) lie in the circle defined

$$|z| \leq 1 + \frac{2(k-1)a_\lambda + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}$$

This completes the proof of Theorem 5.

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