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AFFINE TRANSFORMATIONS AND ISOMETRIES IN A COMPLETE RIEMANNIAN MANIFOLD

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ABSTRACT

The groups of affine transformations of an affinely connected manifold were studied by Nomizu and also Hano-Morimato. Further, Myers and Steenrod gave the theory of group of isometries of Riemannian manifold. In the present study we describe certain aspects of a complete Riemannian manifold. We have investigated that in a complete irreducible Riemannian manifolds the group of all affine transformations and the group of all isometries are equal. Further, if X be an infinitesimal affine transformation on a complete Riemannian manifold M, then X is an infinitesimal isometry. Also, we have related the affine transformations on a complete Riemannian manifold to the conditions of isometry.

Key words: *Complete, Riemannian, affine transformation, isometry.*

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1. INTRODUCTION

If M is a differentiable Riemannian manifold with a fundamental metric tensor field G which is positive definite, for any vector field X we denote by $\nabla(X)$ the covariant differentiation in the direction of X with respect to the Riemannian connection.

Now let M_1 and M_2 be two Riemannian manifolds with G_1 and G_2 as their fundamental metric tensor fields and $\nabla_1(X_1)$ and $\nabla_2(X_2)$ are the corresponding covariant differentiations respectively. Let φ be a differentiate homeomorphism of M_1 onto M_2. If φ commutes with the covariant differentiations i. e. for any vector field X on M_1 such that:

$$\varphi\,(\nabla_1 X) = \nabla_2\,(\varphi\, X)\,\varphi,$$

then φ is called an affine transformation. If we have $\varphi\, G_1 = \rho\, G_2$, then φ is said to be an isometric transformation or an isometry. If for some real constant $\rho > 0$ we have $\varphi\, G_1 = \rho\, G_2$, φ is called a homothetic transformation [1].

A manifold with an affine connection (or a Riemannian manifold) *M* is complete if every geodesic curve can be extended for any large value of the canonical parameter. When the completeness is satisfied on *M,* any infinitesimal affine transformation (or a Killing vector field) generates a one-parameter group of affine transformations from *M* onto itself [2].

Let *M* be a manifold with an affine connection. The group A (M) of all affine transformations of *M* onto itself is a Lie group with respect to the compact-open topology ([3], [4], [7]). When M has a Riemannian metric, the group I (M*)* of all isometries of *M* onto itself is a closed subgroup of A (M). I (M) is also a Lie group [5].The mapping from A (M) x M onto M, gives the transformation law which is differentiable, as is known from a theorem of S. Bochner and D. Montgomery [3]. Any one-parameter subgroup in A (M) (i.e. I (M)) induces an infinitesimal affine transformation (i.e. a Killing vector field) on M.

As φ is an affine transformation, the two Riemannian metrics g and g^* determine the same Riemannian connection, say L. Let $\varphi(x)$ be the linear holonomy group of L with reference point x. Since it is irreducible and leaves both g and g^* invariant, there exists a positive constant c_x such that $g^*(X,Y) = c_x^2.\ g(X,Y)$ for all X, Y $\in T_x(M)$, i.e., $g_x^* = c_x^2.\ g_x$. Since both g^* and g are parallel tensor fields with respect to t, c_x is constant.

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Let us assume that φ is non-isometric homothetic transformation of M. Considering the inverse transformation if necessary, we may assume that the constant c associated with φ is less the 1. Take an arbitrary point x of M. If the distance between x and φ(x) is less than δ, then the distance between $\varphi^m(x)$ and $\varphi^{m+1}(x)$ is less than $c^m\delta$. It follows that $\{\varphi^m(x);\ m=1, 2...\}$ is a Cauchy sequence and hence converges to some point x*, since M is complete. We may further conclude at the following two definitions:

Definition (1.1): If M is an irreducible Riemannian manifold, then every affine transformation φ of M homothetic [2].

Definition (1.2): If M is a complete Riemannian manifold which is not locally Euclidian of M is an isometry [2].

2. AFFINE TRANSFORMATION AND ISOMETRIES IN A COMPLETE RIEMANNIAN MANIFOLD

In the Euclidian space of three dimensions the distance ds between adjacent points whose rectangular cartesian coordinates are (x, y, z) and (x + dx, y + dy, z + dz) is given by $ds^2 = dx^2 + dy^2 + dz^2$. More generally for any system of oblique curvilinear coordinates (u, v, w) we have, $ds^2 = adu^2 + bdv^2 + cdz^2 + 2fdvdw + 2gdwdu + 2hdudv$, where a, b, c, f, g, h are the function of coordinates. This idea was generalized and extended to space of n-dimensions of Riemannian, who had defined the infinitesimal distance ds between the adjacent points whose coordinates in any system are x^i and $x^i + dx^i$, (i = 1, 2, ..., n), by the relation $ds^2 = g_{ij}\, dx^i dx^j$ (i, j = 1, 2, 3, …, n),where the coefficient g_{ij} are the function of the coordinates x^i. This quadratic differential form is called a Riemannian metric and a space which is characterized by such a metric is called Riemannian manifold. A Riemannian manifold or a Riemannian metric g on M is said to be complete if the Riemannian connection is complete, i. e., if every geodesic of M can be extended for arbitrarily large value of its canonical parameter [6].

For a connect Riemannian manifold the following conditions are mutually equivalent:

(i) M is a complete Riemannian manifold.
(ii) M is a complete metric space with respect to the distance function d.
(iii) Every bounded subset of M is relatively compact.
(iv) For an arbitrary point x of M and for an arbitrary curve C in tangent space T_x (M) starting from x which is developed upon the given curve C.

Definition: If M is a connected complete Riemannian manifold, then any two points x and y of M can be joined by a minimizing geodesic.

Theorem (2.1): *If M is a complete, irreducible Riemannian manifold of dim n >1, then the group of all affine transformations and the group of all isometries are equal.*

Proof: A transformation φ of a Riemannian manifold is said to be homothetic, if there is a positive constant c such that $g(\varphi(X), \varphi(Y)) = c^2 g(X,Y)$ for all $X,Y \in T_x(M)$ Where T_x (M) is a tangent space of M at x, and x∈M. consider the Riemannian metric g^* define by $g^*(X,Y) = g(\varphi\ (X)\ \varphi,(Y))$. By the [2] the Riemannian connection defined by g^* coincides with g. This means that every homothetic transformation of a Riemannian manifold M is an affine transformation of M [def 1.1, def 1.2]. Let U be a neighborhood of x* such that $\overline{U}$ is compact. Let K* be a positive number such that $|g(R(Y_1, Y_2) Y_2, Y_1)| < K^*$ for any unit vector Y_1 and Y_2 at Y∈U, where R denotes the curvature tensor field. Let z ∈ M and q any plane in T_x (M). Let X, Y be an orthonormal basis for q, since φ is an affine transformation and let f: M → M′ be an affine mapping and X, Y, Z the vector fields on M which are f-related to vector fields X′, Y′, Z′ on M′ respectively, then R(X, Y) Z is f-related to R′ (X′, Y′) Z′. Here R and R′ are the curvature tensor fields of M and M′ respectively, which implies

$$R(\varphi^m X, \varphi^m Y)(\varphi^m Y) = \varphi^m(R(X, Y) Y).$$

Hence we have,

$$\begin{aligned} g(R(\varphi^m X, \varphi^m Y)(\varphi^m Y), \varphi^m X) &= g\, \varphi^m((R(X,Y) Y), \varphi^m X) \\ &= c^{2m} g(R(X, Y) Y, X) \\ &= c^{2m} K(q). \end{aligned}$$

Also, the distance between $x^* = \varphi^m(x^*)$ and $\varphi^m(z)$ approaches 0 as m tends to infinity. Thus, there exists an integer m_0 such that $\varphi^m(z) \in U$ for every $m \geq m_0$. Since the lengths of the vectors $\varphi^m X$ and $\varphi^m Y$ are equal to c^m, we have

$$c^{4m} K^* \geq |g(R(\varphi^m X, \varphi^m Y)(\varphi^m Y), \varphi^m X)| \quad \text{for } m \geq m_0$$

This implies

$$c^{2m}K^{*} \geq |K(q)| \text{ for } m \geq m_0$$

Provided m tends to infinity, we have K (q) = 0. In view of def (1.1) and def (1.2) and the above result, we say that if M is a complete, irreducible Riemannian manifold of dim $n \geq 1$, then group of all affine transformations is equal to all group of all isometries.

Theorem (2.2): *If X be an infinitesimal affine transformation on a complete Riemannian manifold M, then X is an infinitesimal isometry.*

Proof: Consider X an infinitesimal affine transformation on a complete Riemannian manifold M. To prove above theorem, we need the theorem (2.1) and the following theorem:

Let $M= M_0 \times M_1 \times \ldots\ldots\ldots \times M_k$ is the de Rham decomposition of a complete, simply connected Riemannian manifold M, then

$$\mathfrak{A}^0(M) \approx \mathfrak{A}^0(M_0) \times \mathfrak{A}^0(M_1) \times \ldots\ldots \times \mathfrak{A}^0(M_k),$$

$$\mathfrak{J}^0(M) \approx \mathfrak{J}^0(M_0) \times \mathfrak{J}^0(M_1) \times \ldots\ldots \times \mathfrak{J}^0(M_k),$$

where $\mathfrak{J}(M)$ is a closed subgroup of the group of all affine transformations $\mathfrak{A}(M)$, while $\mathfrak{J}^0(M)$ and $\mathfrak{A}^0(M)$ are their respective identity components.

Considering that M is connected, let M^* be the universal covering manifold with the naturally induced Riemannian metric $g^*= p^*(g)$, where $p: M^* \rightarrow M$ is natural projection.

Let X^* be the vector field on M^* induced by X and X^* is p-related to X. Then X^* is an infinitesimal affine transformation of M^*. Clearly X^* is an infinitesimal affine transformation of M^* and X^* is an infinitesimal isometry of M^* if and only if X is an infinitesimal isometry of M.

Let $M^*= M_0 \times M_1 \times \ldots\ldots \times M_k$ be the de Rham decomposition of a complete simply connected Riemannian manifold*. By the theorem mentioned above, the lie algebra a (M^*) is isomorphic with a (M_0) + a (M_1) + ….. + a (M_k). Let $(X_0, X_1\ldots X_k)$ be the element of a (M_0) + a (M_1) + ….. + a (M_k) corresponding to $X^* \in$ a (M^*) and $X_1 \ldots X_k$ are all infinitesimal isometries then by theorem (2.1), X is an infinitesimal isometry if and only if X_0 is isometry.

Corollary (2.1): If M is connected, complete Riemannian manifold whose restricted linear holonomy group leaves no any non-zero vector at fix point, then group of all affine transformation is equal to group of all isometries.

Proof: The linear holonomy group of M is naturally isomorphic with the restricted linear holonomy group of M. This means that M_0 reduces to a point and hence $X_0=0$ in the above corollary.

Corollary (2.2): If X is an infinitesimal affine transformation of a complete Riemannian manifold and if the length of X is bounded, then X is an infinitesimal isometry.

Proof: Let M to be connected. If the length of X is bounded on M, the length of X_0 is also bounded on M_0. Let x^1, $x^2 \ldots\ldots\ldots\ldots x^r$ be the Euclidean coordinate system in M_0 and set,

$$X_0=\sum_{\alpha=1}^{r} \xi^{\alpha} \left(\frac{\partial}{\partial x^{\alpha}}\right).$$

Applying the formula

$$(L_{X_0} \circ \nabla_Y - \nabla_Y \circ L_{X_0})Z = \nabla_{[X_0,Y]}Z \text{ to}$$

$$y= \frac{\partial}{\partial x^{\beta}} \text{ and } Z=\frac{\partial}{\partial x^{\gamma}},$$

we see that

$$\frac{\partial^2 \xi^{\alpha}}{\partial x^{\beta} \partial x^{\gamma}} = 0.$$

This means that X_0 is of the form

$$\sum_{\alpha=1}^{r}\left(\sum_{\beta=1}^{r} a_{\beta}^{\alpha} x^{\beta} + b^{\alpha}\right)\left(\frac{\partial}{\partial x^{\alpha}}\right).$$

It is easy to see that length of X_0 is bounded on M_0 if and only if $a^{\alpha}_{\beta}=0$ for $\alpha, \beta = 1, \ldots\ldots, r.$ thus if X_0 is of bounded length, then X_0 is an infinitesimal isometry of M_0 ([2], [4]).

Corollary (2.3): On a compact Riemannian manifold M, we have $\mathfrak{A}^0(M) = \mathfrak{I}^0(M)$.

Proof: On a compact manifold M, every vector field is of bounded length. By Corollary (2.3), every infinitesimal affine transformation X is an infinitesimal isometry.

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