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INTERVAL EQUITABLE EDGE COLORING OF THE GENERALIZED PETERSEN GRAPHS $P(n,2), n > 5$

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ABSTRACT

Coxeter has defined the generalized Petersen graph $P(n,k)$ for the natural numbers n and k, $n > 2k$, with the vertex set $\{u_i, v_i\}$, $1 \leq i \leq n$ and the edge set $\{u_i u_{i+1},\ u_i v_i,\ v_i v_{i+2}\}$, $1 \leq i \leq n$ with subscripts reduced to modulo n. In this paper, we have obtained the interval equitable edge coloring of such generalized Petersen graphs $P(n,2), n > 5$ and found its chromatic number for all $n > 5$ to be 3.

INTRODUCTION

The generalized Petersen graph family was introduced by Coxeter [1] and these graphs were given their name by Mark Watkins [2] in 1969. Mayer [3] has introduced the concept of equitable chromatic number of a graph. Kamalian [4] obtained the interval coloring of complete bipartite graphs and trees. Kamalian [5] also extended the result on cyclically interval edge coloring of simple cycles.

Sudha *et al.* [6] defined the interval equitable edge coloring of graphs by combining the definition of interval coloring and equitable edge coloring. An interval equitable edge coloring is an assignment of colors (positive integers) to the edges of the graph, in such a way that
- (i) no two adjacent edges have the same color,
- (ii) the set of colors defined on the edges incident to any vertex of the graph forms an interval and
- (iii) the number of edges in any two color classes differ by at most one.

Coxeter [1] introduced the generalized Petersen graph in self-dual configurations and regular graphs. Sudha *et al.* [7] discussed the equitable coloring of prisms and the generalized Petersen graphs. Babak Behsaz *et al.* [8] obtained the minimum vertex cover of generalized Petersen graphs.

In this paper, we found the chromatic number of interval equitable edge coloring of the generalized Petersen graph $P(n,2)$ for all $n > 5$ to be 3. The generalized Petersen graph in Watkins notation, $G(n,k)$ is a graph with the vertex set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and the edge set $\{u_i u_{i+1},\ u_i v_i,\ v_i v_{i+k}\}$ for $0 \leq i \leq n-1$, where subscripts are reduced by modulo n and $k < n/2$. Coxeter's has given the notation $\{n\} + \left\{\frac{n}{k}\right\}$ for the same graph.

Definition 1: An edge-coloring of a graph is the coloring of edges of the graph with the minimum number of colors without any two adjacent edges having the same color.

Definition 2: In edge-coloring of a graph, the set of edges with the same color are said to be in the same color class.

In k-edge coloring of a graph, there are k color classes. The color classes are represented by $C[1], C[2]$, ..., if 1,2, ... represent the colors.

Definition 3: An edge-coloring of a graph G with colors 1,2, ... k is called an interval k-edge coloring if all the colors are used in such a way that the colors of the edges incident to any vertex of G are distinct and are consecutive.

The smallest integer k for which the graph G is k-interval edge colored is known as the chromatic number of interval edge coloring and is denoted by $\chi_{ie}(G)$.

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Definition 4: A k-interval edge coloring of a graph is said to be equitable k-interval edge coloring if its edge set E is partitioned into k subsets $E_1, E_2, \dots E_k$ such that each E_i is an independent set and the condition $\left||E_i| - |E_j|\right| \le 1$ holds for all $1 \le i \le k$ and $1 \le j \le k$.

The smallest integer k for which G is interval equitable edge coloring is known as the chromatic number of interval equitable edge coloring and is denoted by $\chi_{iee}(G)$.

Theorem 1: The generalized Petersen graphs $P(n,2), n > 5$ admit interval equitable edge coloring and $\chi_{iee}(P(n,2)) = 3$.

Proof: The inner vertices of the generalized Petersen graph $P(n,2), n > 5$ are denoted by the set $\{v_i\}$, $1 \le i \le n$ and the outer vertices are denoted by the set $\{u_i\}, 1 \le i \le n$ as shown in fig-1.

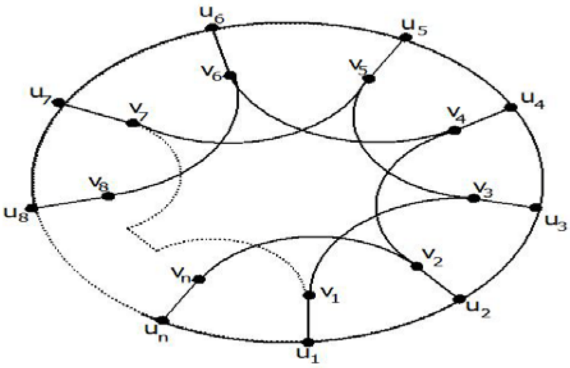

Figure 1.

The function f is defined from the edges of the generalized Petersen graph $P(n,2), n > 5$ to the set of colors $\{1,2,3\}$ as follows:

Case-(i): Let $n \equiv 0(mod\ 4)$.

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 2, & i \equiv 1(mod\ 2) \\ 3, & i \equiv 0(mod\ 2) \end{cases}$$
for $1 \le i \le n-1$

and $f(u_n u_1) = 3$.

The inner edges are colored as

$$f(v_i v_{i+2}) = \begin{cases} 2, & i \equiv 1,2(mod\ 4) \\ 3, & i \equiv 3,0(mod\ 4) \end{cases}$$
for $1 \le i \le n-2$,

$f(v_{n-1} v_1) = 3$,
and $f(v_n v_2) = 3$.

The edges $\{u_i v_i\}$ are colored as
$f(u_i v_i) = 1$ for all i. (1)

The colors $\{1,2,3\}$ are used in coloring $P(n,2)$, $n \equiv 0(mod\ 4)$ satisfying the condition of interval edge coloring. The color classes $C[1]$, $C[2]$, $C[3]$ satisfy the condition $|C[1]| = |C[2]| = |C[3]| = n$. Hence $P(n,2)$ satisfies the interval equitable edge coloring for $n \equiv 0(mod\ 4)$.
Therefore $\chi_{iee}(P(n,2)) = 3$ for $n \equiv 0(mod\ 4)$.

Case-(ii): Let $n \equiv 2(mod\ 4)$.

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 3, & i \equiv 1(mod\ 2) \\ 2, & i \equiv 0(mod\ 2) \end{cases}$$
for $1 < i < n-1$,

$f(u_1u_2) = 1,$
$f(u_{n-1}u_n) = 1,$
and $f(u_nu_1) = 3.$

The inner edges are colored as

$$f(v_iv_{i+2}) = \begin{cases} 3, & i \equiv 1,0 (mod\ 4) \\ 2, & i \equiv 2,3 (mod\ 4) \end{cases}$$
for $1 \le i < n,$

$f(v_{n-1}v_1) = 1,$
and $f(v_nv_2) = 1.$

The edges $\{u_iv_i\}$ are colored as
$f(u_iv_i) = 1,$ for $2 < i < n-1,$
$f(u_1v_1) = 2,$
$f(u_2v_2) = 3,$
$f(u_{n-1}v_{n-1}) = 3,$
and $f(u_nv_n) = 2.$ (2)

The colors $\{1, 2, 3\}$ are used in coloring $P(n, 2),\ n \equiv 2(mod\ 4)$ satisfying the condition of interval edge coloring. The color classes $C[1]$, $C[2]$, $C[3]$ satisfy the condition $|C[1]| = |C[2]| = |C[3]| = n$. Hence $P(n, 2)$ satisfies the interval equitable edge coloring for $n \equiv 2(mod\ 4)$.

Therefore $\chi_{iee}(P(n, 2)) = 3$ for $n \equiv 2(mod\ 4)$.

Case-(iii): Let $n \equiv 1,3(mod\ 4)$.

Here there are six cases:

(a) Let $n = 7 + 12j, j = 0, 1, 2, ...$

The outer edges are colored as

$$f(u_iu_{i+1}) = \begin{cases} 2, & i \equiv 1(mod\ 2) \\ 3, & i \equiv 0(mod\ 2) \end{cases}$$
for $1 \le i < n-2,$

$f(u_{n-2}u_{n-1}) = 1,$
$f(u_{n-1}u_n) = 2,$
and $f(u_nu_1) = 1.$

The inner edges are colored as

$$f(v_iv_{i+2}) = \begin{cases} 2, & i \equiv 1, 0(mod\ 4) \\ 3, & i \equiv 2, 3(mod\ 4) \end{cases}$$
for $1 \le i < n-2,$

$f(v_{n-2}v_n) = 1,$
$f(v_{n-1}v_1) = 1,$
And $f(v_nv_2) = 2.$

The edges $\{u_iv_i\}$ are colored as
$f(u_iv_i) = 1,$ for $1 < i < n-2,$
$f(u_1v_1) = 3,$
$f(u_{n-2}v_{n-2}) = 2,$
$f(u_{n-1}v_{n-1}) = 3,$
and $f(u_nv_n) = 3.$ (3)

The colors $\{1, 2, 3\}$ are used in coloring $P(n, 2)$, $n = 7 + 12j, j = 0, 1, 2, ...$ satisfying the condition of interval edge coloring. The color classes $C[1], C[2], C[3]$ satisfy the condition $|C[1]| = |C[2]| = |C[3]| = n$. Hence $P(n, 2)$ satisfies the interval equitable edge coloring for $n = 7 + 12j, j = 0, 1, 2, ...$

Therefore $\chi_{iee}(P(n, 2)) = 3$ for $n = 7 + 12j, j = 0, 1, 2, ...$

(b) Let $n = 9 + 12j, j = 0, 1, 2, \dots$

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 3, & i \equiv 1(mod\ 3) \\ 2, & i \equiv 2(mod\ 3) \\ 1, & i \equiv 0(mod\ 3) \end{cases}, \quad \text{for } 1 \le i \le n-1,$$

and $f(u_n u_1) = 1$.

The inner edges are colored as

$$f(v_i v_{i+2}) = \begin{cases} 1, & i \equiv 1,4(mod\ 6) \\ 3, & i \equiv 2,5(mod\ 6) \\ 2, & i \equiv 0,3(mod\ 6) \end{cases}, \quad \text{for } 1 \le i < n-1,$$

$f(v_{n-1} v_1) = 3$,

and $f(v_n v_2) = 2$.

The edges $\{u_i v_i\}$ are colored as

$$f(u_i v_i) = \begin{cases} 2, & i \equiv 1(mod\ 3) \\ 1, & i \equiv 2(mod\ 3) \\ 3, & i \equiv 0(mod\ 3) \end{cases} \quad \text{for } 1 \le i \le n. \qquad (4)$$

The colors $\{1, 2, 3\}$ are used in coloring $P(n, 2)$, $\mathrm{n} = 9 + 12\mathrm{j}, \mathrm{j} = 0, 1, 2, \dots$ satisfying the condition of interval edge coloring. The color classes $C[1], C[2], C[3]$ satisfy the condition $|C[1]| = |C[2]| = |C[3]| = n$. Hence $P(n, 2)$ satisfies the interval equitable edge coloring for $\mathrm{n} = 9 + 12\mathrm{j}, \mathrm{j} = 0, 1, 2, \dots$.
Therefore $\chi_{iee}(P(n, 2)) = 3$ for $\mathrm{n} = 9 + 12\mathrm{j}, \mathrm{j} = 0, 1, 2, \dots$.

(c) Let $n = 11 + 12j, j = 0, 1, 2, \dots$

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 2, & i \equiv 1(mod\ 2) \\ 3, & i \equiv 0(mod\ 2) \end{cases}, \quad \text{for } 1 \le i < n-2,$$

$f(u_{n-2} u_{n-1}) = 1$,
$f(u_{n-1} u_n) = 3$,
And $f(u_n u_1) = 1$.

The inner edges are colored as

$$f(v_i v_{i+2}) = \begin{cases} 2, & i \equiv 1,2(mod\ 4) \\ 3, & i \equiv 3,0(mod\ 4) \end{cases}, \quad \text{for } 1 \le i < n-2,$$

$f(v_{n-2} v_n) = 1$,
$f(v_{n-1} v_1) = 1$,
and $f(v_n v_2) = 3$.

The edges $\{u_i v_i\}$ are colored as
$f(u_i v_i) = 1$, for $1 < i < n-2$,
$f(u_1 v_1) = 3$,
$f(u_{n-2} v_{n-2}) = 2$,
$f(u_{n-1} v_{n-1}) = 2$,
and $f(u_n v_n) = 2$. (5)

The colors $\{1,2,3\}$ are used in coloring $P(n,2)$, $n=11+12j, j=0,1,2,\dots$ satisfying the condition of interval edge coloring. The color classes $C[1]$, $C[2]$, $C[3]$ satisfy the condition $|C[1]|=|C[2]|=|C[3]|=n$. Hence $P(n,2)$ satisfies the interval equitable edge coloring for $n=11+12j, j=0,1,2,\dots$.

Therefore the chromatic number of interval equitable edge coloring of $P(n,2)$ is 3 for $n=11+12j, j=0,1,2,\dots$.

(d) Let $n=13+12j, j=0,1,2,\dots$

The outer edges are colored as

$$f(u_iu_{i+1})=\begin{cases}1, & i\equiv 1,3(mod\ 6)\\ 3, & i\equiv 2,5(mod\ 6)\\ 2, & i\equiv 4,0(mod\ 6)\end{cases}\quad \text{for } 1\le i<n-3,$$

$f(u_{n-3}u_{n-2})=3,$
$f(u_{n-2}u_{n-1})=2,$
$f(u_{n-1}u_n)=3,$
and $f(u_nu_1)=2.$

The inner edges are colored as

$$f(v_iv_{i+2})=\begin{cases}1, & i\equiv 1,2(mod\ 6)\\ 3, & i\equiv 3,0(mod\ 6)\\ 2, & i\equiv 4,5(mod\ 6)\end{cases}\quad \text{for } 1\le i<n,$$

$f(v_{n-3}v_{n-1})=3,$
$f(v_{n-2}v_n)=2,$
$f(v_{n-1}v_1)=2,$
and $f(v_nv_2)=3.$

The edges $\{u_iv_i\}$ are colored as

$$f(u_iv_i)=\begin{cases}3, & i\equiv 1,4(mod\ 6)\\ 2, & i\equiv 2,3(mod\ 6)\\ 1, & i\equiv 5,0(mod\ 6)\end{cases}\quad \text{for } 1\le i<n-3,$$

$f(u_{n-3}v_{n-3})=2,$
$f(u_{n-2}v_{n-2})=1,$
$f(u_{n-1}v_{n-1})=1,$
$f(u_nv_n)=1.$ (6)

The colors $\{1,2,3\}$ are used in coloring $P(n,2)$, $n=13+12j, j=0,1,2,\dots$ satisfying the condition of interval edge coloring. The color classes $C[1]$, $C[2]$, $C[3]$ satisfy the condition $|C[1]|=|C[2]|=|C[3]|=n$. Hence $P(n,2)$ satisfies the interval equitable edge coloring for $n=13+12j, j=0,1,2,\dots$.

Therefore $\chi_{iee}(P(n,2))=3$ for $n=13+12j, j=0,1,2,\dots$.

(e) Let $n=15+12j, j=0,1,2,\dots$

The outer edges are colored as

$$f(u_iu_{i+1})=\begin{cases}3, & i\equiv 1(mod\ 3)\\ 2, & i\equiv 2(mod\ 3)\ ,\\ 1, & i\equiv 0(mod\ 3)\end{cases}\quad \text{for } 1\le i\le n-1,$$

and $f(u_nu_1)=1.$

The inner edges are colored as

$$f(v_i v_{i+2}) = \begin{cases} 1, & i \equiv 1,4 (mod\ 6) \\ 3, & i \equiv 2,5 (mod\ 6) \\ 2, & i \equiv 0,3 (mod\ 6) \end{cases} \quad \text{for } 1 \le i < n-1.$$

$f(v_{n-1} v_1) = 3,$

and $f(v_n v_2) = 2.$

The edges $\{u_i v_i\}$ are colored as

$$f(u_i v_i) = \begin{cases} 2, & i \equiv 1 (mod\ 3) \\ 1, & i \equiv 2 (mod\ 3) \\ 3, & i \equiv 0 (mod\ 3) \end{cases} \quad \text{for } 1 \le i \le n. \tag{7}$$

The colors $\{1, 2, 3\}$ are used in coloring $P(n, 2)$, $n = 15 + 12j, j = 0, 1, 2, \dots$ satisfying the condition of interval edge coloring. The color classes $C[1]$, $C[2]$, $C[3]$ satisfy the condition $|C[1]| = |C[2]| = |C[3]| = n$. Hence $P(n, 2)$ satisfies the interval equitable edge coloring for $n = 15 + 12j, j = 0, 1, 2, \dots$.

Therefore $\chi_{iee}\big(P(n, 2)\big) = 3$ for $n = 15 + 12j, j = 0, 1, 2, \dots$.

(f) Let $n = 17 + 12j, j = 0, 1, 2, \dots$

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 1, & i \equiv 1 (mod\ 2) \\ 2, & i \equiv 0 (mod\ 2) \end{cases}, \quad \text{for } 4 < i < n-4,$$

$f(u_1 u_2) = 2,$
$f(u_2 u_3) = 3,$
$f(u_3 u_4) = 1,$
$f(u_4 u_5) = 3,$
$f(u_{n-4} u_{n-3}) = 3,$
$f(u_{n-3} u_{n-1}) = 1,$
$f(u_{n-2} u_{n-1}) = 3,$
$f(u_{n-1} u_n) = 2,$

and $f(u_n u_1) = 1.$

The inner edges are colored as

$$f(v_i v_{i+2}) = \begin{cases} 1, & i \equiv 1, 0 (mod\ 4) \\ 2, & i \equiv 2, 3 (mod\ 4) \end{cases}, \quad \text{for } 3 < i < n-4,$$

$f(v_1 v_3) = 1,$
$f(v_2 v_4) = 3,$
$f(v_3 v_4) = 3,$
$f(v_{n-4} v_{n-2}) = 3,$
$f(v_{n-3} v_{n-1}) = 3,$
$f(v_{n-2} v_n) = 1,$
$f(v_{n-1} v_1) = 2,$

and $f(v_n v_2) = 2.$

The edges $\{u_i v_i\}$ are colored as

$f(u_i v_i) = 3,$ for $5 < i < n-4,$
$f(u_1 v_1) = 3,$
$f(u_2 v_2) = 1,$
$f(u_3 v_3) = 2,$
$f(u_4 v_4) = 2,$
$f(u_5 v_5) = 2,$

$f(u_{n-4}v_{n-4}) = 1,$
$f(u_{n-3}v_{n-3}) = 2,$
$f(u_{n-2}v_{n-2}) = 2,$
$f(u_{n-1}v_{n-1}) = 1,$
and $f(u_n v_n) = 3.$ (8)

The colors $\{1,2,3\}$ are used in coloring $P(n,2)$, $n = 17 + 12j, j = 0,1,2,\ldots$ satisfying the condition of interval edge coloring. The color classes $C[1]$, $C[2]$, $C[3]$ satisfy the condition $|C[1]| = |C[2]| = |C[3]| = n$. Hence $P(n,2)$ satisfies the interval equitable edge coloring for $n = 17 + 12j, j = 0,1,2,\ldots$.

Therefore $\chi_{iee}(P(n,2)) = 3$ for $n = 17 + 12j, j = 0,1,2,\ldots$.

Illustration 1: Consider the generalized Petersen graph $P(10,2)$. The colors $\{1,2,3\}$ are assigned to the edges of the Petersen graph $P(10,2)$ as per case (ii) of theorem 1 as shown in fig-2.

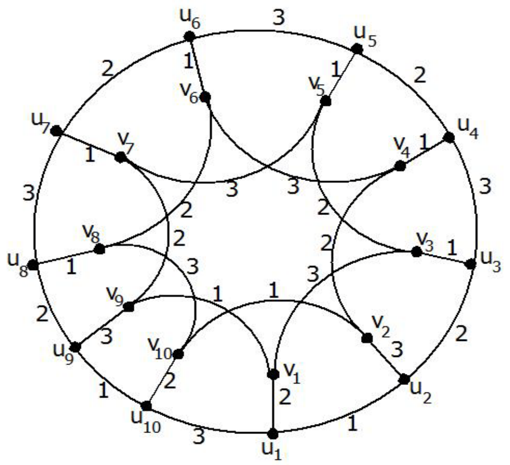

Figure 2.

Here $|C[1]| = |C[2]| = |C[3]| = 10$, these satisfying the condition $\big||C[i]| - |C[j]|\big| \le 1$ for all $1 \le i \le 3$ and $1 \le j \le 3$.

Hence this type of coloring of the generalized Petersen graph $P(10,2)$ satisfies the definition of the interval equitable edge coloring. Here $\chi_{iee}(P(10,2)) = 3$.

Illustration 2: Consider the generalized Petersen graph $P(13,2)$. The colors $\{1,2,3\}$ are assigned to the edges of the Petersen graph $P(13,2)$ as per case (iii) (d) of theorem 1 as shown in fig-3.

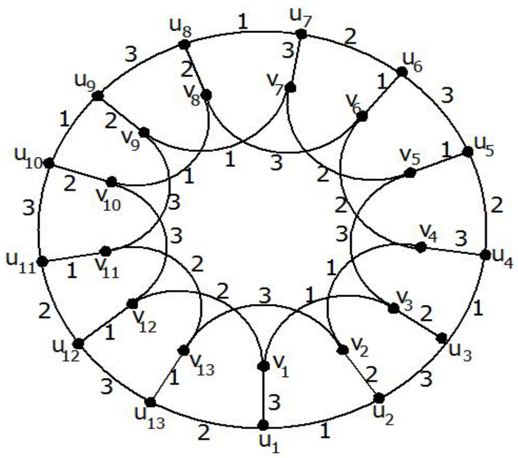

Figure 3.

Here $|C[1]| = |C[2]| = |C[3]| = 13$, these satisfying the condition $\big||C[i]| - |C[j]|\big| \le 1$ for all $1 \le i \le 3$ and $1 \le j \le 3$.

Hence this type of coloring of the generalized Petersen graph $P(13,2)$ satisfies the definition of the interval equitable edge coloring. Here $\chi_{iee}(P(13,2)) = 3$.

Illustration 3: Consider the generalized Petersen graph $P(17, 2)$. The colors $\{1, 2, 3\}$ are assigned to the edges of the Petersen graph $P(17, 2)$ as per case (iii) (f) of theorem 1 as shown in fig-4.

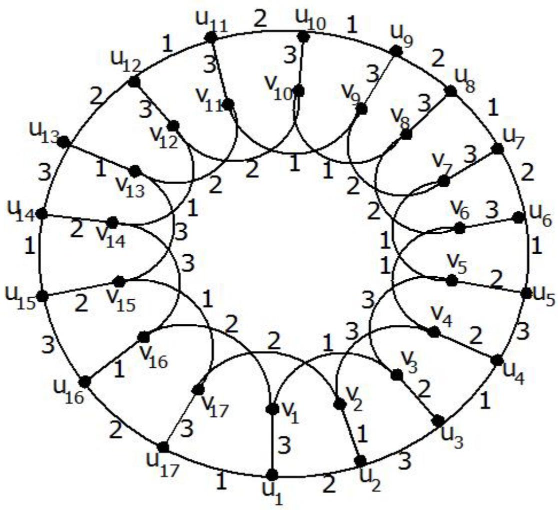

Figure 4.

Here $|C[1]| = |C[2]| = |C[3]| = 17$, these satisfying the condition $||C[i]| - |C[j]|| \leq 1$ for all $1 \leq i \leq 3$ and $1 \leq j \leq 3$.

Hence this type of coloring of the generalized Petersen graph $P(17, 2)$ satisfies the definition of the interval equitable edge coloring. Here $\chi_{iee}(P(17, 2)) = 3$.

CONCLUSION

We discussed above the interval equitable edge coloring of the generalized Petersen graph $P(n, 2)$ for all $n > 5$ by considering the interval edge coloring along with equitable coloring condition. We have given the interval equitable edge coloring of the generalized Petersen graph $P(n, 3)$ and $P(n, 4)$ in the forth coming articles.

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