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U-Γ-SEMIGROUPS AND V- Γ-SEMIGROUPS

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ABSTRACT

In this paper, the terms, Maximal Γ-ideal, primary Γ-semigroup, prime Γ-ideal, simple Γ-semigroup, U- Γ-semigroup and V- Γ-semigroup are introduced. It is proved that Γ-semigroup S is a U- Γ-semigroup if either S has a left (right) identity or S is generated by a Γ-idempotent. Also it is proved that a Γ-semigroup S is a U- Γ-semigroup if and only if every proper Γ-ideal is contained in a proper prime Γ-ideal. Also it is proved that if A is a proper Γ-ideal in the finite dimensional U- Γ-semigroup S, then A is contained in maximal Γ-ideal and also it is proved that if S is a globally idempotent Γ-semigroup with maximal Γ-ideals, then either S is aV- Γ-semigroup or S has a unique maximal Γ-ideal which is prime.

Mathematical subject classification (2010): *20M07; 20M11; 20M12.*

Keywords: *Γ-semigroup, Maximal Γ-ideal, primary Γ-semigroup, commutative Γ-semigroup, left (right) identity, identity, Zero element, Prime Γ-ideal, simple Γ-semigroup, U- Γ-semigroup and V- Γ-semigroup.*

1. INTRODUCTION

Γ- semigroup was introduced by Sen and Saha [8] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of pseudo symmetric ideals and radicals in semigroups. Giri and Wazalwar [4] intiated the study of prime radicals in semigroups. Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [5], [6] initiated the study of prime radicals and primary and semiprimary Γ-ideals in Γ-semigroups. In this paper we introduce the notions of U- Γ-semigroups and V- Γ-semigroups in the class of arbitrary Γ-semigroups. We study prime Γ-ideals and maximal Γ-ideals in a U- Γ-semigroup and we characterize V- Γ-semigroups.

2. PRELIMINARIES

Definition 2.1: Let S and Γ be any two non-empty sets. Then S is said to be a ***Γ-semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \gamma, b) \to a\,\gamma\,b$ satisfying the condition: $(a\alpha b)\beta c = a\,\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

Note 2.2: Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{a\gamma b : a \in A, b \in B$ and $\gamma \in \Gamma\}$ by AΓB.

Definition 2.3: A Γ-semigroup S is said to be ***commutative* Γ-semigroup** provided $a\gamma b = b\gamma a$ for all $a,b \in S$ and $\gamma \in \Gamma$.

Note 2.4: If S is a commutative Γ-semigroup then $a\,\Gamma b = b\,\Gamma a$ for all $a, b \in S$.

Note 2.5: Let S be a Γ-semigroup and $a, b \in S$ and $\alpha \in \Gamma$. Then $a\alpha a\alpha b$ is denoted by $(a\alpha)^2 b$ and consequently $a\,\alpha\,a\,\alpha\,a\,\alpha\ldots$..(*n terms*)$b$ is denoted by $(a\alpha)^n b$.

Definition 2.6: A Γ-semigroup S is said to be ***quasi commutative*** provided for each $a, b \in S$, there exists a natural number n such that $a\gamma b = (b\gamma)^n a \;\; \forall \gamma \in \Gamma$.

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Note 2.7: If a Γ-semigroup S is ***quasi commutative*** then for each $a, b \in$ S, there exists a natural number n such that, $a\Gamma b = (b\,\Gamma)^n a$.

Definition 2.8: An element ***a*** of a Γ- semigroup S is said to be a ***left identity*** of S provided $a\alpha s = s$ for all $s \in$ S and $\alpha \in \Gamma$.

Definition 2.9: An element ***a*** of a Γ- semigroup S is said to be a ***righ tidentity*** of S provided $s\alpha a = s$ for all $s \in$ S and $\alpha \in \Gamma$.

Definition 2.10: An element ***a*** of a Γ- semigroup S is said to be a ***two sided identity*** or an identity provided it is both a left identity and a right identity of S.

Notation 2.11: Let S be a Γ- semigroup. If S has an identity , let S^1 = S and if S does not have an identity, let S^1 be the Γ- semigroup S with identity adjoined, usually denoted by the symbol 1.

Definition 2.12: A non empty subset A of a Γ-semigroup S is said to be a ***left Γ-ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

Note 2.13: A nonempty subset A of a Γ-semigroup S is a ***left Γ- ideal*** of S iff S Γ A ⊆ A.

Definition 2.14: A nonempty subset A of a Γ-semigroup S is said to be a ***right Γ-ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

Note 2.15: A nonempty subset A of a Γ-semigroup S is a ***right Γ- ideal*** of S iff AΓS ⊆ A.

Definition 2.16: A nonempty subset A of a Γ-semigroup S is said to be a ***two sided Γ- ideal*** or simply a ***Γ- ideal*** of S if $s \in$ S, $a \in$ A, $\alpha \in \Gamma$ imply $s\alpha a \in$ A, $a\alpha s \in$ A.

Definition 2.17: A Γ-ideal A of a Γ-semigroup S is said to be a ***maximal Γ-ideal*** provided A is a proper Γ-ideal of S and is not properly contained in any proper Γ-ideal of S.

Definition 2.18: A Γ- ideal P of a Γ-semigroup S is said to be a ***prime Γ- ideal*** provided A, B are two Γ-ideals of S and AΓB ⊆ P ⇒ either A ⊆ P or B ⊆ P.

Definition 2.19: A Γ- ideal A of a Γ-semigroup S is said to be a ***semi prime Γ- ideal*** provided $x \in$ S, $x\Gamma S^1\Gamma x \subseteq A$ implies $x \in$ A .

Definition 2.20: If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all prime Γ-ideals of S containing A is called ***prime Γ-radical*** or simply ***Γ-radical*** of A and it is denoted by **√A** or ***rad A***.

Theorem 2.21[5]: If A is a **Γ**-ideal of a **Γ**-semigroup S then √A is a semi prime **Γ**-ideal of S.

Theorem 2.22[5]: A Γ- ideal Q of Γ-semigroup S is a semi prime Γ-ideal of S iff √(Q) = (Q) implies $x\,\Gamma S^1\Gamma y \subseteq$ A.

Definition 2.23: A Γ-ideal A of a Γ- semigroup S is said to be a ***left primary Γ-ideal*** provided

1) If X, Y are two Γ-ideals of S such that X ΓY ⊆ A and Y ⊈ A then X ⊆ √A.
2) √A is a prime Γ-ideal of S.

Definition 2.24: A Γ-ideal A of a Γ- semigroup S is said to be a ***right primary Γ-ideal*** provided

1) If X, Y are two Γ-ideals of S such that X ΓY ⊆ A and X ⊈A then Y ⊆ √A.
2) √A is a prime Γ-ideal of S.

Example 2.25: Let S = {a, b, c} and Γ = {x, y, z}. Define a binary operation. in S as shown in the following table.

.	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

Define a maping S XΓ X S → S by $a\ \alpha\ b = ab$, for all $a, b \in$ S and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup.

Now consider the Γ-ideal <*a*> = S^1 Γ*a* ΓS^1 = {*a*}. Let *p* Γ*q* ⊆ <*a*>, *p* ∉ <*a*> ⇒ *q* ∈ √<*a*> ⇒ $(q\,\Gamma)^{n-1}q$ ⊆ <*a*> for some *n* ∈ N. Since *b* Γ*c* ⊆ <*a*>, *c* ∉ <*a*> ⇒ *b*∈ <*a*>. Therefore <*a*> is left primary. If *b* ∉ <*a*> then $(c\,\Gamma)^{n-1}c$ ∉ <*a*> for any *n* ∈ N ⇒ *c* ∉ √<*a*>. Therefore <*a*> is not right primary.

Definition 2.26: A Γ-ideal A of a Γ- semigroup S is said to be a **primary Γ-ideal** provided A is both left primary Γ-ideal and right primary Γ-ideal.

Definition 2.27: A Γ-ideal A of a Γ- semigroup S is said to be a **principal Γ-ideal** provided A is a Γ-ideal generated by a single element ***a***. It is denoted by J[*a*] = <*a*>.

Definition 2.28: An element ***a*** of a Γ-semigroup S with 1 is said to be ***left invertible*** or ***left unit*** provided there is an element *b* ∈ S such that *b* Γ*a* = 1.

Definition 2.29: An element ***a*** of a Γ-semigroup S with 1 is said to be ***right invertible*** or ***right unit*** provided there is an element *b*∈ S such that *a*Γ*b* = 1.

Definition 2.30: An element ***a*** of a Γ-semigroup S is said to be ***invertible*** or a ***Unit*** in S provided it is both left and right invertible element in S.

Definitoin 2.31: A Γ- semigroup S is said to be a ***simple* Γ- *semigroup*** provided S has no proper Γ- ideals.

Definition 2.32: An element ***a*** of a Γ- semigroup S is said to be a **Γ-*idempotent*** provided $a \propto a = a$ for all $\propto \in \Gamma$.

Note 2.33: If an element***a*** of a Γ- semigroup S is a Γ-**idempotent**, then *a* Γ*a* = *a*.

Definition 2.34: A Γ- semigroup S is said to be an **idempotent Γ- semigroup** or a **band** provided every element in S is a Γ-idempotent.

Definition 2.35: A Γ- semigroup S is said to be a **globally idempotent Γ- semigroup** provided S ΓS = S**.**

3). U-Γ-SEMIGROUPS AND V-Γ-SEMIGROUPS

Definition 3.1: A Γ- semigroup S is said to be U-Γ-semigroup, provided for any Γ-ideal A in S, √A = S implies A = S.

Example 3.2: Let S is a Γ-semigroup with S = Γ under the multiplication given in the following table. (S× Γ ×S ⟶ S as *aαb* = *ab*)

.	*a*	*b*	*c*	*d*
a	*a*	*a*	*a*	*a*
b	*a*	*a*	*a*	*b*
c	*a*	*a*	*a*	*a*
d	*a*	*a*	*c*	*d*

Since S = {*a* ,*b* ,*c*, *d* } and S = Γ. Now <*a*>, {*a*, *b*}, {*a*, *c*}, {*a*, *b*, *c*} and {*a*, *b*, *c*, *d*} are the Γ-ideals of S.

If A = <*a*> then √< *a*> = intersection of all prime Γ- ideals containing <*a*> = {*a*, *b*, *c*} ∩ {*a*, *b*, *c*, *d*} = {*a*, *b*, *c*} ≠ S. Similarly √{ *a*, *b*} = {*a*, *b*, *c*} ≠ S. √{ *a*, *b*, *c*} = {*a*, *b*, *c*} ≠ S, √{ *a*, *c*}={*a*, *b*, *c*} ≠ S and if A = { *a*, *b*, *c*, *d*} then √A=√{*a*, *b*, *c*, *d*}={*a*, *b*, *c*, *d*}=S implies A=S. Therefore √A= S is true for only A=S. Therefore S is U-Γ-semigroup.

Theorem 3.3: A Γ-semigroup S is a U-Γ-semigroup if either S has a left (right) identity or S is generated by a Γ- idempotent.

Proof: Suppose S has a left identity e. Let A be any proper Γ-ideal such that √A=S. Since √A ⊆ {*x* ∈ S: $(x\,\Gamma)^{n-1}x$ ⊆ A for some natural number *n*} = S. So there is a natural number *n* such that $(e\,\Gamma)^{n-1}e$ ⊆ A and hence *e* ∈A. Thus S = *e* ΓS ⊆ A, it is a contradiction. Therefore S is a U- Γ-semigroup. Suppose S is generated by a Γ–idempotent *e*. As above we can prove that for any Γ- ideal A in S, if √A = S, then *e* ∈ A and hence A = S. So S is a U- Γ-semigroup.

Theorem 3.4: A Γ-semigroup S is a U- Γ-semigroup if and only if every proper Γ- ideal is contained in a proper prime Γ-ideal.

Proof: Suppose S is a U-Γ-semigroup. Let A be any proper Γ- ideal in S. If A is not contained in any proper prime Γ- ideal, then √A = S. Since S is a U-Γ-semigroup. We have A = S, this is a contradiction. So every proper Γ-ideal is contained in a proper prime Γ-ideal. Conversely if every proper Γ- ideal is contained in a proper prime Γ-ideal, Then √A ≠ S implies A ≠ S then clearly S is a U-Γ-semigroup.

Theorem 3.5: Let S be a U-Γ-semigroup. Then S = S Γ S and hence every maximal Γ- ideal is prime.

Conversely if {P_α} is the collection of all prime Γ- ideals in S and if P is a maximal element in this collection, then P is a maximal Γ- ideal in S.

Proof: Clearly √S Γ S = S. Since S is a U-Γ-semigroup, we have S Γ S = S and hence every maximal Γ- ideal is prime. If P is not a maximal Γ- ideal in S, then there exists a proper Γ- ideal A in S, containing P properly. Since P is a maximal element in the collection of all proper prime Γ- ideals in S, we have A is not contained in any proper prime Γ- ideal. So √A = S. Since S is a U-Γ-semigroup, A = S. This is a contradiction. Therefore P is a maximal Γ- ideal in S.

Definition 3.6: A Γ-semigroup S is said to have dimension n or n – dimensional if there exist a strictly ascending chain $P_0 \subset P_1 \subset P_2 \subset \ldots\ldots \subset P_n$ of prime (proper) Γ- ideals in S, but no such a chain of n+2 proper prime Γ- ideals exists in S.

Theorem 3.7: If A is a proper Γ- ideal in the finite dimensional U-Γ-semigroup S, then A is contained in a maximal Γ-ideal.

Proof: By theorem 3.4, A is contained in a proper prime Γ- ideal P_0, If P_0 is not a maximal Γ- ideal, then by theorem 3.5, there exists a proper prime Γ- ideal P such that $P_0 \subset P_1$. If P_1 is maximal we are through. Otherwise P_1 is properly contained in a proper prime Γ- ideal P_2 in S. The process of choosing P_i's must cease in a finite number of steps because of the finite dimensionality of S. Hence A is contained in a maximal Γ- ideal.

Note 3.8: In a commutative ring, it is proved that every finite dimentional v -ring is a union of maximal Γ- ideals. But in Γ- semigroups this is not true, as the Γ- semigroup S in example 3.2 is a finite dimensional U-Γ-semigroup with the unique maximal Γ- ideal {a, b, c}.

Definition 3.9: A Γ- semigroup S is said to be V- Γ- semigroup provided for any element *a* ∈ S, √<*a*> = S implies <*a*> = S.

Note 3.10: Every U-Γ-semigroup is a V- Γ-semigroup. But a V- Γ-semigroup is not necessarily a U- Γ-semigroup.

Example 3.11: Let S be the Γ-semigroup of all natural numbers greater than 1, under usual multiplication. The Γ-ideal A = {3, 4…..} is not contained in any proper prime Γ-ideal and hence by theorem 3.4, S is not a U-Γ-semigroup. Clearly every principal Γ-ideal is contained in a proper prime Γ-ideal. So S is a V- Γ-semigroup.

Theorem 3.12: If S is a globally idempotent Γ-semigroup with maximal Γ-ideals, then either S is a V-Γ-semigroup or S has a unique maximal Γ-ideal which is prime.

Proof: Let T = {*a* ∈ S: √<*a*> ≠ S} If T = ϕ, then for every *a* ∈ S, √< *a*> = S and so S has no proper prime Γ-ideals. But maximal Γ-ideals are prime. Hence this case is inadmissible. Clearly T is a Γ-ideal in S. If T ≠ S then T is the unique maximal Γ-ideal. Since S = S ΓS, M is a prime Γ-ideal and so√M = M. Now if *a* ∈ M\T then S =√< *a*> ⊆√M = M. Thus M ⊆ T and so M = T. Then only other possibility is T = S, in which case S is a V- Γ-semigroup.

Note 3.13: It is clear that a Γ-semigroup S is globally idempotent if and only if maximal Γ-ideals in S is prime. So if a Γ-semigroup S contains unique maximal Γ-ideal which is prime, then S is globally idempotent. But from the example 3.11, we remark that there are V- Γ-semigroups containing maximal Γ-ideals which are not globally idempotent.

Theorem 3.14: A Γ-semigroup S is a V- Γ-semigroup if and only if S has atmost one proper prime Γ-ideal and if {P_α} is the family of all proper prime Γ-ideals then <*x*> = S for *x* ∈ S\∪ P_α or S is a simple Γ-semigroup.

Proof: Let S be a V- Γ-semigroup which is not a simple Γ-semigroup. If S has no proper prime Γ-ideals, then √<*a*> = S for *a* ∈ S. This implies <*a*> = S and hence S is a simple Γ-semigroup. So assume S has proper prime Γ-ideals. Then for any *a* ∈ S\∪ $P_{\alpha,}$ √<*a*> = S, since *a* does not belong to any proper prime Γ-ideals. Then <*a*> = S. Conversly let '*a*' be any element of S such that <*a*> ≠ S. If *a* ∈ S\∪ $P_{\alpha,}$ then <*a*> = S. So *a* ∈ ∪ P_α and hence √<*a*> ≠ S. Therefore S is a V- Γ-semigroup.

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