

RESIDUATED ALMOST DISTRIBUTIVE LATTICES - II

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ABSTRACT

In this paper, we prove some important properties of residuation ' : ' and multiplication ' . ' in a Residuated Almost Distributive Lattice (RADL)  $L$ . We prove important results in a residuated ADL  $L$ .

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1. INTRODUCTION

Swamy, U.M. and Rao, G.C. [4] introduced the concept of an Almost Distributive Lattice as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, p rings, biregular rings, associate rings,  $P_1$  rings etc.) on one hand and distributive lattices on the other. In [1], Dilworth, R.P., has introduced the concept of a residuation in lattices and in [5], [6] Ward, M. and Dilworth, R.P., have studied residuated lattices. We introduced the concepts of a residuation ' : ' and a multiplication ' . ' in an ADL in our earlier paper [3]. In this paper, we derive some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL  $L$ . We also prove some equivalent conditions and important results in a residuated ADL  $L$ .

In section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL. These are taken from Swamy, U.M. and Rao, G.C. [4] and Rao, G.C. [2]. Also we recall the concepts of residuation and multiplication in an ADL  $L$  and the definition of a residuated almost distributive lattice from our earlier paper [3].

In section 3, we derive some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL  $L$ . We prove important results in a residuated ADL  $L$ .

2. PRELIMINARIES

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

We begin with the definition of an ADL.

**Definition 2.1 ([2]):** An Almost Distributive Lattice (ADL) is an algebra  $(L, \vee, \wedge)$  of type  $(2, 2)$  satisfying

- (1)  $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
- (2)  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- (3)  $(a \vee b) \wedge b = b$
- (4)  $(a \vee b) \wedge a = a$
- (5)  $a \vee (a \wedge b) = a$ , for all  $a, b, c \in L$

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It can be seen directly that every distributive lattice is an ADL.

If there is an element  $0 \in L$  such that  $0 \wedge a = 0$  for all  $a \in L$ , then  $(L, \vee, \wedge, 0)$  is called an ADL with 0.

**Example 2.1 ([2]):** Let  $X$  be a non-empty set. Fix  $x_0 \in X$ . For any  $x, y \in L$ ,

$$\text{define } x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases} \quad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$$

Then  $(X, \vee, \wedge, x_0)$  is an ADL, with  $x_0$  as its zero element. This ADL is called a **discrete ADL**.

For any  $a, b \in L$ , we say that  $a$  is less than or equals to  $b$  and write  $a \leq b$ , if  $a \wedge b = a$ . Then " $\leq$ " is a partial ordering on  $L$ .

**Theorem 2.1 ([2]):** Let  $(L, \vee, \wedge, 0)$  be an ADL with '0'. Then, for any  $a, b \in L$ , we have

- (1)  $a \wedge 0 = 0$  and  $0 \vee a = a$
- (2)  $a \wedge a = a = a \vee a$
- (3)  $(a \wedge b) \vee b = b$ ;  $a \vee (b \wedge a) = a$  and  $a \wedge (a \vee b) = a$
- (4)  $a \wedge b = a \iff a \vee b = b$  and  $a \wedge b = b \iff a \vee b = a$
- (5)  $a \wedge b = b \wedge a$  and  $a \vee b = b \vee a$  whenever  $a \leq b$
- (6)  $a \wedge b \leq b$  and  $a \leq a \vee b$
- (7)  $\wedge$  is associative in  $L$
- (8)  $a \wedge b \wedge c = b \wedge a \wedge c$
- (9)  $(a \vee b) \wedge c = (b \vee a) \wedge c$
- (10)  $a \wedge b = 0 \iff b \wedge a = 0$
- (11)  $a \vee (b \vee a) = a \vee b$ .

It can be observed that an ADL  $L$  satisfies almost all the properties of a distributive lattice except, possibly the right distributivity of  $\vee$  over  $\wedge$ , the com-mutativity of  $\vee$ , the commutativity of  $\wedge$  and the absorption law  $(a \wedge b) \vee a = a$ . Any one of these properties convert  $L$  into a distributive lattice.

**Theorem 2.2 ([2]):** Let  $(L, \vee, \wedge, 0)$  be an ADL with 0. Then the following are equivalent:

- (1)  $(L, \vee, \wedge, 0)$  is a distributive lattice
- (2)  $a \vee b = b \vee a$  for all  $a, b \in L$
- (3)  $a \wedge b = b \wedge a$  for all  $a, b \in L$
- (4)  $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$  for all  $a, b, c \in L$ .

**Proposition 2.1 ([2]):** Let  $(L, \vee, \wedge)$  be an ADL. Then for any  $a, b, c \in L$  with  $a \leq b$ , we have

- (1)  $a \wedge c \leq b \wedge c$
- (2)  $c \wedge a \leq c \wedge b$
- (3)  $c \vee a \leq c \vee b$ .

**Definition 2.2 ([2]):** An element  $m \in L$  is called maximal if it is maximal as in the partially ordered set  $(L, \leq)$ . That is, for any  $a \in L$ ,  $m \leq a$  implies  $m = a$ .

**Theorem 2.3 ([2]):** Let  $L$  be an ADL and  $m \in L$ . Then the following are equivalent:

- (1)  $m$  is maximal with respect to  $\leq$
- (2)  $m \vee a = m$ , for all  $a \in L$
- (3)  $m \wedge a = a$ , for all  $a \in L$ .

**Lemma 2.1 ([2]):** Let  $L$  be an ADL with a maximal element  $m$  and  $x, y \in L$ . If  $x \wedge y = y$  and  $y \wedge x = x$  then  $x$  is maximal if and only if  $y$  is maximal. Also the following conditions are equivalent:

- (i)  $x \wedge y = y$  and  $y \wedge x = x$
- (ii)  $x \wedge m = y \wedge m$ .

**Definition 2.3 ([2]):** If  $(L, \vee, \wedge, 0, m)$  is an ADL with 0 and with a maximal element  $m$ , then the set  $I(L)$  of all ideals of  $L$  is a complete lattice under set inclusion. In this lattice, for any  $I, J \in I(L)$ , the l.u.b. and g.l.b. of  $I, J$  are given by  $I \vee J = \{(x \vee y) \wedge m / x \in I, y \in J\}$  and  $I \wedge J = I \cap J$ . The set  $PI(L) = \{[a] / a \in L\}$  of all principal ideals of  $L$  forms a sub lattice of  $I(L)$ . (Since  $[a] \vee [b] = [a \vee b]$  and  $[a] \cap [b] = [a \wedge b]$ )

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice (ADL) L and the definition of a residuated almost distributive lattice taken from our earlier paper [3].

**Definition 2.4 ([3]):** Let L be an ADL with a maximal element m. A binary operation  $:$  on an ADL L is called a **residuation** over L if, for  $a, b, c \in L$  the following conditions are satisfied.

- (R1)  $a \wedge b = b$  if and only if  $a : b$  is maximal  
 (R2)  $a \wedge b = b \Rightarrow$  (i)  $(a : c) \wedge (b : c) = b : c$  and (ii)  $(c : b) \wedge (c : a) = c : a$  (R3)  $[(a : b) : c] \wedge m = [(a : c) : b] \wedge m$   
 (R4)  $[(a \wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$  (R5)  $[c : (a \vee b)] \wedge m = (c : a) \wedge (c : b) \wedge m$

**Definition 2.5 ([3]):** Let L be an ADL with a maximal element m. A binary operation  $.$  on an ADL L is called a **multiplication** over L if, for  $a, b, c \in L$  the following conditions are satisfied.

- (M1)  $(a.b) \wedge m = (b.a) \wedge m$   
 (M2)  $[(a.b).c] \wedge m = [a.(b.c)] \wedge m$  (M3)  $(a.m) \wedge m = a \wedge m$   
 (M4)  $[a.(b \vee c)] \wedge m = [(a.b) \vee (a.c)] \wedge m$

**Definition 2.6 ([3]):** An ADL L with a maximal element m is said to be a **residuated almost distributive lattice (residuated ADL)**, if there exists two binary operations ' $:$ ' and ' $.$ ' on L satisfying conditions R1 to R5, M1 to M4 and the following condition (A).

- (A)  $(x : a) \wedge b = b$  if and only if  $x \wedge (a.b) = a.b$ , for any  $x, a, b \in L$ .

We use the following properties frequently later in the results.

**Lemma 2.2 ([3]):** Let L be an ADL with a maximal element m and  $.$  a binary operation on L satisfying the conditions M1 - M4. Then for any  $a, b, c, d \in L$ ;

- (i)  $a \wedge (a.b) = a.b$  and  $b \wedge (a.b) = a.b$   
 (ii)  $a \wedge b = b \Rightarrow (c.a) \wedge (c.b) = c.b$  and  $(a.c) \wedge (b.c) = b.c$   
 (iii)  $d \wedge [(a.b).c] = (a.b).c$  if and only if  $d \wedge [a.(b.c)] = a.(b.c)$   
 (iv)  $(a.c) \wedge (b.c) \wedge [(a \wedge b).c] = (a \wedge b).c$   
 (v)  $d \wedge (a.c) \wedge (b.c) = (a.c) \wedge (b.c) \Rightarrow d \wedge [(a \wedge b).c] = (a \wedge b).c$   
 (vi)  $d \wedge [(a.c) \vee (b.c)] = (a.c) \vee (b.c)$ ,  $d \wedge [(a \vee b).c] = (a \vee b).c$

The following result is a direct consequence of M1 of definition 2.6.

**Lemma 2.3 ([3]):** Let L be an ADL with a maximal element m and  $.$  a binary operation on L satisfying the condition M1. For  $a, b, x \in L$ ,  $a \wedge (x.b) = x.b$  if and only if  $a \wedge (b.x) = b.x$

### 3. PROPERTIES OF RESIDUATED ADL'S

In this section, we prove some important properties of residuation ' $:$ ' and multiplication ' $.$ ' in a residuated ADL L. We prove some important results in a residuated ADL L.

First we give the following Lemma, whose proof can be obtained from the definition of Residuated ADL.

**Lemma 3.1:** Let L be a residuated ADL. Then

- (1)  $(a : a) : b$  is maximal, for all  $a, b \in L$ ;  
 (2) If an element m of L is maximal then  $m : a$  is maximal, for all  $a \in L$ .

In the following, we prove some important properties of residuation ' $:$ ' and multiplication ' $.$ ' in a residuated ADL L.

**Lemma 3.2:** Let L be a residuated ADL with a maximal element m. For  $a, b, c, d \in L$ , the following hold in L.

- (1)  $(a : b) \wedge a = a$   
 (2)  $[a : (a : b)] \wedge (a \vee b) = a \vee b$   
 (3)  $[(a : b) : c] \wedge [a : (b.c)] = a : (b.c)$   
 (4)  $[a : (b.c)] \wedge [(a : b) : c] = (a : b) : c$   
 (5)  $[(a \wedge b) : b] \wedge (a : b) = a : b$   
 (6)  $(a : b) \wedge [(a \wedge b) : b] = (a \wedge b) : b$   
 (7)  $[a : (a \vee b)] \wedge m = (a : b) \wedge m$   
 (8)  $[c : (a \wedge b)] \wedge [(c : a) \vee (c : b)] = (c : a) \vee (c : b)$   
 (9) If  $a : b = a$  then  $a \wedge (b.d) = b.d \Rightarrow a \wedge d = d$

- (10)  $\{a : [a : (a : b)]\} \wedge (a : b) = a : b$   
 (11)  $[(a \vee b) : c] \wedge [(a : c) \vee (b : c)] = (a : c) \vee (b : c)$   
 (12)  $a \wedge m \geq b \wedge m \Rightarrow (a : c) \wedge m \geq (b : c) \wedge m$   
 (13)  $(a : b) \wedge \{a : [a : (a : b)]\} = a : [a : (a : b)]$   
 (14)  $a \wedge b = b \Rightarrow (a.c) \wedge (b.c) = b.c$   
 (15)  $a \wedge b \wedge (a.b) = a.b$   
 (16)  $[(a.b) : a] \wedge b = b$   
 (17)  $(a.b) \wedge [(a \wedge b).(a \vee b)] = (a \wedge b).(a \vee b)$   
 (18)  $a \vee b$  is maximal  $\Rightarrow (a.b) \wedge a \wedge b = a \wedge b$

**Proof:** Let  $a, b, c, d \in L$ . Then

(1) By R1, we have  $a : a$  is a maximal element. Then  $(a : a) \wedge b = b$ , for all  $b \in L \Rightarrow (a : a) : b$  is maximal  
 $\Rightarrow (a : b) : a$  is maximal  
 $\Rightarrow (a : b) \wedge a = a$

(2)  $[(a : b).(a \vee b)] \wedge m = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow \{[(a : b).a] \vee [(a : b).b]\} \wedge m = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow \{[(a : b).a] \wedge m\} \vee \{[(a : b).b] \wedge m\} = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow \{[a.(a : b)] \wedge m\} \vee \{[b.(a : b)] \wedge m\} = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow \{[a.(a : b)]\} \vee \{[b.(a : b)]\} \wedge m = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow \{[a \wedge [a.(a : b)]] \vee [a \wedge [b.(a : b)]]\} \wedge m = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow \{a \wedge [a.(a : b)] \wedge m\} \vee \{a \wedge [b.(a : b)] \wedge m\} = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow \{a \wedge [(a : b).a] \wedge m\} \vee \{a \wedge [(a : b).b] \wedge m\} = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow \{[a \wedge [(a : b).a]] \vee [a \wedge [(a : b).b]]\} \wedge m = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow [a \wedge \{[(a : b).a] \vee [(a : b).b]\}] \wedge m = [(a : b).(a \vee b)] \wedge m$   
 $\Rightarrow a \wedge [(a : b).(a \vee b)] \wedge m = [(a : b).(a \vee b)] \wedge m \Rightarrow a \wedge [(a : b).(a \vee b)] = [(a : b).(a \vee b)]$   
 $\Rightarrow [a : (a : b)] \wedge (a \vee b) = a \vee b$

(3) we have  $[a : (b.c)] \wedge [a : (b.c)] = a : (b.c)$   
 $\Rightarrow a \wedge \{(b.c).[a : (b.c)]\} = (b.c).[a : (b.c)]$   
 $\Rightarrow a \wedge (b.\{c.[a : (b.c)]\}) = b.\{c.[a : (b.c)]\}$   
 $\Rightarrow (a : b) \wedge \{c.[a : (b.c)]\} = c.[a : (b.c)] \Rightarrow [(a : b) : c] \wedge [a : (b.c)] = a : (b.c)$

(4) We have  $[(a : b) : c] \wedge [(a : b) : c] = (a : b) : c$   
 $\Rightarrow (a : b) \wedge \{c.[(a : b) : c]\} = c.[(a : b) : c]$   
 $\Rightarrow a \wedge (b.\{c.[(a : b) : c]\}) = b.\{c.[(a : b) : c]\}$   
 $\Rightarrow a \wedge \{(b.c).[a : (b.c)]\} = (b.c).[a : (b.c)]$   
 $\Rightarrow [a : (b.c)] \wedge [(a : b) : c] = (a : b) : c$

(5)  $[(a \wedge b) : b] \wedge (a : b) = [(a \wedge b) : b] \wedge m \wedge (a : b)$   
 $= [(a : b) \wedge (b : b)] \wedge m \wedge (a : b)$   
 $= [(a : b) \wedge (b : b)] \wedge (a : b)$   
 $= [(b : b) \wedge (a : b)] \wedge (a : b)$   
 $= (a : b) \wedge (a : b)$   
 $= (a : b)$

(6) We have  $(a : b) \wedge (a : b) \wedge (b : b) \wedge m = (a : b) \wedge (b : b) \wedge m$   
 $\Rightarrow (a : b) : [(a : b) \wedge (b : b) \wedge m]$  is maximal  
 $\Rightarrow (a : b) : \{[(a \wedge b) : b] \wedge m\}$  is maximal  
 $\Rightarrow (a : b) \wedge [(a \wedge b) : b] \wedge m = [(a \wedge b) : b] \wedge m \Rightarrow (a : b) \wedge [(a \wedge b) : b] = (a \wedge b) : b$

(7)  $[a : (a \vee b)] \wedge m = [(a : a) \wedge (a : b)] \wedge m = (a : b) \wedge m$

(8) We have  $a \wedge a \wedge b = a \wedge b$   
 $\Rightarrow [c : (a \wedge b)] \wedge (c : a) = c : a$  and  $b \wedge a \wedge b = a \wedge b$   
 $\Rightarrow [c : (a \wedge b)] \wedge (c : b) = c : b$   
 Therefore  $\{[c : (a \wedge b)] \wedge (c : a)\} \vee \{[c : (a \wedge b)] \wedge (c : b)\} = (c : a) \vee (c : b)$   
 Hence  $[c : (a \wedge b)] \wedge [(c : a) \vee (c : b)] = (c : a) \vee (c : b)$

(9) Assume that  $a : b = a$

$$\begin{aligned} \text{Suppose } a \wedge (b.d) &= b.d \Rightarrow (a : b) \wedge d = d \\ \Rightarrow a \wedge d &= d \text{ ( Since } a : b = a \text{ )} \end{aligned}$$

$$\begin{aligned} (10) [a : (a : b)] \wedge [a : (a : b)] &= a : (a : b) \\ \Rightarrow a \wedge \{(a : b).[a : (a : b)]\} &= (a : b).[a : (a : b)] \\ \Rightarrow a \wedge \{(a : b).[a : (a : b)]\} \wedge m &= \{(a : b).[a : (a : b)]\} \wedge m \\ \Rightarrow a \wedge \{[a : (a : b)].(a : b)\} \wedge m &= \{[a : (a : b)].(a : b)\} \wedge m \\ \Rightarrow a \wedge \{[a : (a : b)].(a : b)\} &= \{[a : (a : b)].(a : b)\} \\ \Rightarrow \{a : [a : (a : b)]\} \wedge (a : b) &= a : b \end{aligned}$$

$$\begin{aligned} (11) \text{ We have } (a \vee b) \wedge a &= a \text{ and } (a \vee b) \wedge b = b \\ \Rightarrow [(a \vee b) : c] \wedge (a : c) &= a : c \text{ and } [(a \vee b) : c] \wedge (b : c) = b : c \\ \Rightarrow \{[(a \vee b) : c] \wedge (a : c)\} \vee \{[(a \vee b) : c] \wedge (b : c)\} &= (a : c) \vee (b : c) \\ \Rightarrow [(a \vee b) : c] \wedge [(a : c) \vee (b : c)] &= (a : c) \vee (b : c) \end{aligned}$$

(12) Suppose  $a \wedge m \geq b \wedge m$ . Then  $b \wedge m \wedge a \wedge m = b \wedge m \Rightarrow a \wedge b = b$

$$\text{Now, } (b : c) \wedge m \wedge (a : c) \wedge m = (a : c) \wedge (b : c) \wedge m = [(a \wedge b) : c] \wedge m = (b : c) \wedge m$$

Therefore,  $(a : c) \wedge m \geq (b : c) \wedge m$

$$\begin{aligned} (13) [(a : b) : \{a : [a : (a : b)]\}] \wedge m &= ([a : \{a : [a : (a : b)]\}] : b) \wedge m \\ &\geq [(a \vee [a : (a : b)]) : b] \wedge m \\ &\geq [(a : b) \vee \{[a : (a : b)] : b\}] \wedge m \\ &= \{[a : (a : b)] : b\} \vee (a : b) \wedge m \\ &\geq \{[a : (a : b)] : b\} \wedge m \\ &= [(a : b) : (a : b)] \wedge m = m \end{aligned}$$

But  $[(a : b) : \{a : [a : (a : b)]\}] \wedge m \leq m$  Therefore  $[(a : b) : \{a : [a : (a : b)]\}] \wedge m = m \Rightarrow (a : b) : \{a : [a : (a : b)]\}$  is maximal.

$$\Rightarrow (a : b) \wedge \{a : [a : (a : b)]\} = a : [a : (a : b)]$$

(14) Suppose  $a \wedge b = b$ .

$$\begin{aligned} \text{Then } [(a.c) \wedge m] \vee [(b.c) \wedge m] &= [(c.a) \wedge m] \vee [(c.b) \wedge m] \\ &= [(c.a) \vee (c.b)] \wedge m \\ &= [c.(a \vee b)] \wedge m \\ &= (c.a) \wedge m \\ &= (a.c) \wedge m \end{aligned}$$

$$\begin{aligned} \Rightarrow (a.c) \wedge m \wedge (b.c) \wedge m &= (b.c) \wedge m \\ \Rightarrow (a.c) \wedge (b.c) &= b.c \end{aligned}$$

$$\begin{aligned} (15) [(a \wedge b) : a] \wedge b &= [(a \wedge b) : a] \wedge m \wedge b \\ &= (a : a) \wedge (b : a) \wedge m \wedge b \\ &= (b : a) \wedge b = b \end{aligned}$$

$$\Rightarrow a \wedge b \wedge (a.b) = a.b$$

$$\begin{aligned} (16) (a.b) \wedge (a.b) &= a.b \\ \Rightarrow [(a.b) : a] \wedge b &= b \end{aligned}$$

$$\begin{aligned} (17) b \wedge a \wedge b &= a \wedge b \\ \Rightarrow (b.a) \wedge [(a \wedge b).a] &= (a \wedge b).a \\ \Rightarrow (b.a) \wedge m \wedge [(a \wedge b).a] &= (a \wedge b).a \\ \Rightarrow (a.b) \wedge m \wedge [(a \wedge b).a] &= (a \wedge b).a \\ \Rightarrow (a.b) \wedge [(a \wedge b).a] &= (a \wedge b).a \end{aligned}$$

Similarly,  $(a.b) \wedge [(a \wedge b).b] = (a \wedge b).b$

$$\begin{aligned} &\text{Then } \{(a.b) \wedge [(a \wedge b).a]\} \vee \{(a.b) \wedge [(a \wedge b).b]\} = [(a \wedge b).a] \vee [(a \wedge b).b] \\ &\Rightarrow (a.b) \wedge \{[(a \wedge b).a] \vee [(a \wedge b).b]\} = [(a \wedge b).a] \vee [(a \wedge b).b] \\ &\Rightarrow (a.b) \wedge [(a \wedge b).(a \vee b)] = (a \wedge b).(a \vee b) \end{aligned}$$

(18) Suppose  $a \vee b$  is maximal.

$$\begin{aligned} &\text{By (16), we have } (a.b) \wedge [(a \wedge b).(a \vee b)] = (a \wedge b).(a \vee b) \\ &\Rightarrow (a.b) \wedge [(a \wedge b).(a \vee b)] \wedge m = [(a \wedge b).(a \vee b)] \wedge m \\ &\Rightarrow (a.b) \wedge (a \wedge b) \wedge m = (a \wedge b) \wedge m \\ &\Rightarrow (a.b) \wedge a \wedge b = a \wedge b. \end{aligned}$$

In the definition of a residuated ADL, we have

(R4)  $[(a \wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$  and (R5)  $[a : (b \vee c)] \wedge m = (a : b) \wedge (a : c) \wedge m$ . Regarding  $[(a \vee b) : c] \wedge m$  and  $[a : (b \wedge c)] \wedge m$  we have the following equivalent conditions.

**Theorem 3.1:** Let  $L$  be a residuated ADL with a maximal element  $m$  and  $a, b, c \in L$ . Then the following conditions are equivalent.

$$\begin{aligned} &\text{(R7) } [a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m \\ &\text{(R8) } [(a : b) \vee (b : a)] \wedge m = [(a \wedge b) : (b \wedge a)] \wedge m \\ &\text{(R9) } (a : b) \vee (b : a) \text{ is maximal} \\ &\text{(R10) } [(a \vee b) : c] \wedge m = [(a : c) \vee (b : c)] \wedge m \end{aligned}$$

**Proof:** Let  $a, b, c \in L$ .

**(R7)  $\Rightarrow$  (R8):** Suppose  $[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m$ .

$$\begin{aligned} \text{Then } [(a : b) \vee (b : a)] \wedge m &= [(a : b) \wedge m] \vee [(b : a) \wedge m] \\ &= [(a : b) \wedge (b : b) \wedge m] \vee [(a : a) \wedge (b : a) \wedge m] \\ &= \{[(a \wedge b) : b] \wedge m\} \vee \{[(a \wedge b) : a] \wedge m\} \\ &= [(a \wedge b) : (b \wedge a)] \wedge m \text{ ( By (R7) )} \end{aligned}$$

Hence (R8) holds in  $L$ .

**(R8)  $\Rightarrow$  (R9):** Suppose  $[(a : b) \vee (b : a)] \wedge m = [(a \wedge b) : (b \wedge a)] \wedge m$   $b \wedge a \wedge [(b \wedge a).m] = (b \wedge a).m$

$$\begin{aligned} &\Rightarrow a \wedge b \wedge [(b \wedge a).m] = (b \wedge a).m \\ &\Rightarrow [(a \wedge b) : (b \wedge a)] \wedge m = m \\ &\Rightarrow [(a : b) \vee (b : a)] \wedge m = m \text{ ( By (R8) )} \end{aligned}$$

Hence  $(a : b) \vee (b : a)$  is maximal

Thus (R9) holds in  $L$ .

**(R9)  $\Rightarrow$  (R10):** Suppose  $(a : b) \vee (b : a)$  is maximal.

Consider  $[(a : c) : (a : b)] \wedge m = \{[a : (a : b)] : c\} \wedge m \geq [(a \vee b) : c] \wedge m \rightarrow (1)$

$$\begin{aligned} \text{Now, } \{[(a : c) : [(a \vee b) : c]] : (a : b)\} \wedge m &= \{[(a : c) : (a : b)] : [(a \vee b) : c]\} \wedge m \\ &\geq \{[(a \vee b) : c] : [(a \vee b) : c]\} \wedge m, \text{ Hence it is maximal.} \end{aligned}$$

Therefore,  $\{(a : c) : [(a \vee b) : c]\} : (a : b)$  is maximal.

$$\text{Hence } \{(a : c) : [(a \vee b) : c]\} \wedge (a : b) = (a : b)$$

$$\text{Thus } \{(a : c) : [(a \vee b) : c]\} \wedge m \geq (a : b) \wedge m \rightarrow (2)$$

Similarly, we get  $\{(b : c) : [(a \vee b) : c]\} \wedge m \geq (b : a) \wedge m \rightarrow (3)$

$$\begin{aligned} \text{Now, } \{[(a : c) \vee (b : c)] : [(a \vee b) : c]\} \wedge m &\geq \{[(a : c) : [(a \vee b) : c]] \wedge m\} \vee \{[(b : c) : [(a \vee b) : c]] \wedge m\} \\ &\geq [(a : b) \wedge m] \vee [(b : a) \wedge m] \text{ ( By above (2) and (3) )} \end{aligned}$$

$$\begin{aligned} &= [(a : b) \vee (b : a)] \wedge m, \text{ which is maximal by (R9)} \\ &= \text{Therefore, by (R1), we get} \end{aligned}$$

$$[(a : c) \vee (b : c)] \wedge [(a \vee b) : c] = (a \vee b) : c$$

This combined with property (11) of Lemma 3.2, we get  
 $[(a \vee b) : c] \wedge m = [(a : c) \vee (b : c)] \wedge m$  Thus (R10) holds in L.

**(R10)  $\Rightarrow$  (R7):** Suppose  $[(a \vee b) : c] \wedge m = [(a : c) \vee (b : c)] \wedge m$

Since  $[a \vee (b \wedge c)] \wedge m \geq (b \wedge c) \wedge m$ , we get  
 $\{[a \vee (b \wedge c)] : b\} \wedge m \geq [(b \wedge c) : b] \wedge m$  and  $\{[a \vee (b \wedge c)] : c\} \wedge m \geq [(b \wedge c) : c] \wedge m$

Now,  $\{[(a : b) \vee (a : c)] : [a : (b \wedge c)]\} \wedge m \geq \{[(a : b) : [a : (b \wedge c)]] \wedge m\} \vee \{[(a : c) : [a : (b \wedge c)]] \wedge m\}$   
 $= \{[(a : [a : (b \wedge c)]) : b] \wedge m\} \vee \{[(a : [a : (b \wedge c)]) : c] \wedge m\}$   
 $\geq \{[(a \vee (b \wedge c)) : b] \wedge m\} \vee \{[(a \vee (b \wedge c)) : c] \wedge m\}$   
 $\geq \{[(b \wedge c) : b] \wedge m\} \vee \{[(b \wedge c) : c] \wedge m\}$  ( By (3) and (4) )  
 $= [(b : b) \wedge (c : b) \wedge m] \vee [(b : c) \wedge (c : c) \wedge m]$   
 $= [(c : b) \wedge m] \vee [(b : c) \wedge m]$   
 $= \{[c : (b \vee c)] \wedge m\} \vee \{[b : (b \vee c)] \wedge m\}$   
 $= \{[c : (b \vee c)] \vee [b : (b \vee c)]\} \wedge m$   
 $= [(c \vee b) : (b \vee c)] \wedge m$  ( By R10 )

Thus  $[(a : b) \vee (a : c)] : [a : (b \wedge c)]$  is maximal.  
 $\Rightarrow [(a : b) \vee (a : c)] \wedge [a : (b \wedge c)] = a : (b \wedge c)$

Now, by property (8) of Lemma 3.2, we get  
 $[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m$ .

Thus (R7) holds in L.

**Definition 3.1:** Let L be an ADL and  $a \in L$ . An element  $a^1 \in L$  is said to be a complement of a in L if  $a \wedge a^1 = 0$  and  $a \vee a^1$  is maximal. In this case we say that a is a complemented element of L.

**Theorem 3.2:** Let L be a residuated ADL with a maximal element m and  $a, b \in L$ . If  $b^1$  is a complement of b in L then  
 $(a : b) \wedge m = (a \vee b^1) \wedge m$

**Proof:** Suppose  $b^1$  is a complement of b in L.

Then  $\{(a : b) : (a \vee b^1)\} \wedge m = f[(a : b) : a] \wedge [(a : b) : b^1] \wedge m$   
 $= [(a : a) : b] \wedge [(a : b) : b^1] \wedge m$   
 $= [(a : b) : b^1] \wedge m$   
 $\geq [a : (b.b^1)] \wedge m$   
 $= (a : 0) \wedge m$  ( Since  $b.b^1 = b \wedge b^1 \wedge (b.b^1) = 0$  )  
 $= m$  ( Since  $(a : 0) \wedge m = (a : a) \wedge (a : 0) \wedge m = [a : (a \vee 0)] \wedge m = (a : a) \wedge m = m$  )

Therefore  $[(a : b) : a \vee b^1] \wedge m = m$  and

Hence  $(a : b) \wedge (a \vee b^1) = a \vee b^1$

Now,  $[(a \vee b^1) : (a : b)] \wedge m \geq \{[a : (a : b)] \vee [b^1 : (a : b)]\} \wedge m$   
 $\geq [(a \vee b) \wedge m] \vee \{[b^1 : (a : b)] \wedge m\} \geq (b \wedge m) \vee (b^1 \wedge m)$   
 $= (b \vee b^1) \wedge m$   
 $= m$

$\Rightarrow (a \vee b^1) : (a : b)$  is maximal  
 $\Rightarrow (a \vee b^1) \wedge (a : b) = a : b$

Hence  $(a : b) \wedge m = (a \vee b^1) \wedge m$

**Theorem 3.3:** Let L be a residuated ADL with a maximal element m and  $a, b \in L$ . If b is a complemented element of L, then  $(a.b) \wedge m = a \wedge b \wedge m$

**Proof:** Let L be a residuated ADL with a maximal element m and  $a, b \in L$ . Suppose b is a complement element of L and  $b^1$  be a complement of b in L. Then  $b \wedge b^1 = 0$  and  $b \vee b^1$  is maximal.

By property (15) of Lemma 3.2, we get  $a \wedge b \wedge (a.b) = a.b$

Now,  $[a.(b \vee b^1)] \wedge (b \vee b^1) = a \wedge (b \vee b^1)$

$$\begin{aligned} \Rightarrow [a.(b \vee b^1)] \wedge (b \vee b^1) \wedge m &= a \wedge (b \vee b^1) \wedge m \\ \Rightarrow [a.(b \vee b^1)] \wedge m &= a \wedge m \\ \Rightarrow a \wedge m &= [(a.b) \vee (a.b^1)] \wedge m \\ &= [(a.b) \vee (b^1.a)] \wedge m \\ &\leq [(a.b) \vee b^1] \wedge m \quad (\text{Since } (b^1.a) \wedge m \leq b^1 \wedge m) \\ \Rightarrow a \wedge b \wedge m &= b \wedge a \wedge m \\ &\leq b \wedge [(a.b) \vee b^1] \wedge m \\ &= \{[b \wedge (a.b)] \vee (b \wedge b^1)\} \wedge m \\ &= \{[b \wedge (b.a) \wedge m] \vee (0 \wedge m)\} \\ &= [(b.a) \vee 0] \wedge m \\ &= (b.a) \wedge m \\ &= (a.b) \wedge m \end{aligned}$$

Therefore  $a \wedge b \wedge m \leq (a.b) \wedge m$

Hence  $(a.b) \wedge m = a \wedge b \wedge m$

In Theorem 3.1, we have proved that (R8), (R9) and (R10) are equivalent to (R7). Now we prove (R7) in a residuated ADL L under certain condition. We conclude this paper with the following Theorem.

**Theorem 3.4:** Let L be a residuated ADL with a maximal element m and  $a, b, c \in L$ . If  $b \vee c$  is a maximal element of L then  $[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m$

**Proof:** Let L be a residuated ADL with a maximal element m and  $a, b, c \in L$  such that  $b \vee c$  is maximal.

$$\begin{aligned} \text{Now, } b \wedge b \wedge c &= b \wedge c \text{ and } c \wedge b \wedge c = b \wedge c \\ \Rightarrow [a : (b \wedge c)] \wedge (a : b) &= a : b \text{ and } [a : (b \wedge c)] \wedge (a : c) = a : c \\ \Rightarrow [a : (b \wedge c)] \wedge [(a : b) \vee (a : c)] &= (a : b) \vee (a : c) \end{aligned}$$

First, we prove the following property (i).

$$\begin{aligned} \text{(i) } [(a : b) \vee (a : c)] \wedge m &= \{[(a : b) \vee (a : c)] : (b \vee c)\} \wedge m \\ \text{(i) } \{[(a : b) \vee (a : c)] : [(a : b) \vee (a : c)] : (b \vee c)\} \wedge m &\geq \{[(a : b) \vee (a : c)] \vee (b \vee c)\} \wedge m \\ &= (b \vee c) \wedge m \\ &= m \\ \Rightarrow [(a : b) \vee (a : c)] : \{[(a : b) \vee (a : c)] : (b \vee c)\} &\text{ is maximal} \\ \Rightarrow [(a : b) \vee (a : c)] \wedge m &\geq \{[(a : b) \vee (a : c)] : (b \vee c)\} \wedge m \end{aligned}$$

By property (1) of Lemma 3.2, we get

$$\{[(a : b) \vee (a : c)] : (b \vee c)\} \wedge m \geq [(a : b) \vee (a : c)] \wedge m$$

$$\begin{aligned} \text{Therefore, } [(a : b) \vee (a : c)] \wedge m &= \{[(a : b) \vee (a : c)] : (b \vee c)\} \wedge m \\ &= \{[(a : b) \vee (a : c)] : b\} \wedge \{[(a : b) \vee (a : c)] : c\} \wedge m \\ &\geq \{[(a : c) : b] \wedge m\} \wedge \{[(a : b) : c] \wedge m\} \\ &= [(a : b) : c] \wedge m \wedge [(a : b) : c] \wedge m \\ &= (a : b) : c \wedge m \geq [a : (b.c)] \wedge m \geq [a : (b \wedge c)] \wedge m. \end{aligned}$$

Therefore,  $[(a : b) \vee (a : c)] \wedge m \geq [a : (b \wedge c)] \wedge m$

Hence  $[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m$ .

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