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ON UNIQUE COMMON FIXED POINT THEOREMS FOR THREE AND FOUR SELF MAPPINGS IN SYMMETRIC G-METRIC SPACE

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ABSTRACT

In this paper, we prove two unique common fixed point theorems for three and four self mappings in symmetric G – metric spaces.

Key words: *Symmetric G-metric space, owc maps, common fixed point theorem.*

2000 Mathematics Subject Classification: *47H10, 54H25.*

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1. INTRODUCTION:

In 1992, Dhage[1] introduced the concept of D – metric space. Recently, Mustafa and Sims [5] shown that most of the results concerning Dhage's D – metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure and called it as G – metric space. For more details on G – metric spaces, one can refer to the papers [5]-[9]. In this paper, we prove two unique common fixed point theorems for three and four self mappings in symmetric G – metric spaces.

Now we give basic definitions and some basic results ([5]-[9]) which are helpful for proving our main result.

In 2006, Mustafa and Sims[6] introduced the concept of G-metric spaces as follows:

Definition: 1.1[6] Let X be a nonempty set, and let $G: X \times X \times X \longrightarrow R^+$ be a function satisfying the following axioms:
(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables) and
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality)

then the function G is called a generalized metric, or, more specifically a G – metric on X and the pair (X, G) is called a G – metric space.

Definition: 1.2[6] A G-metric space (X, G) is symmetric if
(G6) $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$.

Definition: 1.3[6] Let (X,G) be a G-metric space then for $x_0 \in X$, $r > 0$, the G-ball with centre x_0 and radius r is

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}.$$

Proposition: 1.1[6] Let (X,G) be a G-metric space then for any $x_0 \in X$, $r > 0$, we have,

(1) if $G(x_0, y, y) < r$ then $x, y \in B_G(x_0, r)$,
(2) if $y \in B_G(x_0, r)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.

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It follows from (2) of the above proposition that the family of all G-balls, B = {$B_G(x, r)$: x $\in$ X, $r > 0$ } is the base of a topology τ (G) on X, the G-metric topology.

Proposition: 1.2[6] Let (X,G) be a G-metric space then for all $x_0 \in$ X and r > 0, we have,

$$B_G(x_0, \frac{1}{3} r) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r)$$

where $d_G(x,y) = G(x,y,y) + G(x,x,y)$, for all x, y in X.

Consequently, the G-metric topology τ (G) coincides with the metric topology arising from d_G. Thus, while 'isometrically' distinct, every G-metric space is topologically equivalent to a metric space. This allows us to readily transport many results from metric spaces into G-metric spaces settings.

Definition: 1.4[6] Let (X, G) be a G–metric space, and let $\{x_n\}$ a sequence of points in X, a point 'x' in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G–convergent to x.

Thus, that if $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ in a G-metric space (X,G) then for each $\in > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \in$ for all m, n $\geq$ N.

Proposition: 1.3[6] Let (X, G) be a G – metric space. Then the following are equivalent:
(1) $\{x_n\}$ is G-convergent to x,
(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(4) $G(x_m, x_n, x) \to 0$ as m, $n \to \infty$.

Definition: 1.5[6] Let (X, G) be a G – metric space. A sequence $\{x_n\}$ is called G – Cauchy if, for each $\in > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \in$ for all n, m, $l \geq$ N; i.e. if $G(x_n, x_m, x_l) \to 0$ as n, m, $l \to \infty$

Proposition: 1.4[6] If (X, G) is a G – metric space then the following are equivalent:
(1) The sequence $\{x_n\}$ is G – Cauchy,
(2) for each $\in > 0$, there exist a positive integer N such that $G(x_n, x_m, x_m) < \in$ for all n, m $\geq$ N.

Proposition: 1.5 [6] Let (X, G) be a G – metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Definition: 1.6 [6] A G – metric space (X, G) is said to be G–complete if every G-Cauchy sequence in (X,G) is G-convergent in X.

Proposition: 1.6[6] A G – metric space (X, G) is G – complete if and only if (X, d_G) is a complete metric space.

Proposition: 1.7[6] Let (X, G) be a G – metric space. Then, for any x, y, z, a in X it follows that:

(i) If G(x, y, z) = 0, then x = y = z,
(ii) G(x, y, z) $\leq$ G(x, x, y) + G(x, x, z),
(iii) G(x, y, y) $\leq$ 2G(y, x, x),
(iv) G(x, y, z) $\leq$ G(x, a, z) + G(a, y, z),
(v) G(x, y, z) $\leq$ $^2/_3$ (G(x, y, a) + G(x, a, z) + G(a, y, z)),
(vi) G(x, y, z) $\leq$ (G(x, a, a) + G(y, a, a) + G (z, a, a)).

Definition: 1.7 Let (X, G) be a G-metric space. f and g be self maps on X. A point x in X is called a coincidence point of f and g iff fx = gx. In this case, w = fx = gx is called a point of coincidence of f and g.

Definition: 1.8 A pair of self mappings (f, g) of a G-metric space (X, G) is said to be weakly compatible if they commute at the coincidence points i.e., if fu = gu for some $u \in X$, then fgu = gfu.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition: 1.9 Two self mappings f and g of a G-metric space (X, G) are said to be occasionally weakly compatible (owc) iff there is a point x in X which is coincidence point of f and g at which f and g commute.

2. MAIN RESULTS:

2.1 A unique common fixed point theorem for three mappings

Theorem 2.1: Let (X, G) be symmetric G-metric space. Suppose f, g, and h are three self mappings of (X, G) satisfying the conditions:

(1) for all $x, y \in X$

$$\int_0^{G(fx,gy,gy)} \phi(t)dt \le \int_0^{\alpha G(hx,hy,hy)+\beta[G(fx,hx,hx)+G(gy,hy,hy)]+\gamma[G(hx,gy,gy)+G(hy,fx,fx)]} \phi(t)dt$$

where ϕ : $R^{+} \to R$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that $\int_0^{\in} \phi(t)dt > 0$ for each $\in > 0$, and α, β, γ are non-negative reals such that $\alpha + 2\beta + 2\gamma < 1$

(2) pair of mappings (f, h) or (g, h) is owc.

Then f, g and h have a unique common fixed point.

Proof: Suppose that f and h are owc then there is an element u in X such that fu = hu and fhu = hfu.

First, we prove that fu = gu. Indeed, by inequality (1), we get

$$\int_0^{G(fu,gu,gu)} \phi(t)dt \le \int_0^{\alpha G(hu,hu,hu)+\beta[G(fu,hu,hu)+G(gu,hu,hu)]+\gamma[G(hu,gu,gu)+G(hu,fu,fu)]} \phi(t)dt$$

$$= \int_0^{\beta[G(gu,fu,fu)]+\gamma[G(fu,gu,gu)]} \phi(t)dt$$

$$= \int_0^{\beta[G(gu,gu,fu)]+\gamma[G(fu,gu,gu)]} \phi(t)dt$$

$$= \int_0^{(\beta+\gamma)G(fu,gu,gu)} \phi(t)dt$$

$$< \int_0^{G(fu,gu,gu)} \phi(t)dt$$

which is a contradiction, hence, gu = fu = hu.

Again, suppose that ffu ≠ fu. The use of condition (1), we have

$$\int_0^{G(ffu,gu,gu)} \phi(t)dt \le \int_0^{\alpha G(hfu,hu,hu)+\beta[G(ffu,hfu,hfu)+G(gu,hu,hu)]+\gamma[G(hfu,gu,gu)+G(hu,ffu,ffu)]} \phi(t)dt$$

$$= \int_0^{\alpha G(ffu,gu,gu)+2\gamma[G(ffu,gu,gu)]} \phi(t)dt$$

$$= \int_0^{(\alpha+2\gamma)G(ffu,gu,gu)} \phi(t)dt$$

$$< \int_0^{G(ffu,gu,gu)} \phi(t)dt$$

this contradiction implies that ffu = fu = hfu.

Now, suppose that gfu ≠ fu. By inequality (1), we have

$$\int_0^{G(fu,gfu,gfu)}\phi(t)dt \le \int_0^{\alpha G(hu,hfu,hfu)+\beta[G(fu,hu,hu)+G(gfu,hfu,hfu)]+\gamma[G(hu,gfu,gfu)+G(hfu,fu,fu)]}\phi(t)dt$$
$$=\int_0^{\beta G(gfu,fu,fu)+\gamma[G(fu,gfu,gfu)]}\phi(t)dt$$
$$=\int_0^{(\beta+\gamma)G(fu,gfu,gfu)}\phi(t)dt$$
$$<\int_0^{G(fu,gfu,gfu)}\phi(t)dt$$

This above contradiction implies that gfu = fu. Put fu = gu = hu = t, so, t is a common fixed point of mappings f, g and h.

Now, let t and z be two distinct common fixed points of f, g and h. That is ft = gt =ht = t and fz = gz = hz = z. As t ≠ z, then from condition (1), we have

$$\int_0^{G(t,z,z)}\phi(t)dt=\int_0^{G(ft,gz,gz)}\phi(t)dt \le \int_0^{\alpha G(ht,hz,hz)+\beta[G(ft,ht,ht)+G(gz,hz,hz)]+\gamma[G(ht,gz,gz)+G(hz,ft,ft)]}\phi(t)dt$$
$$=\int_0^{\alpha G(t,z,z)+2\gamma G(t,z,z)]}\phi(t)dt$$
$$=\int_0^{(\alpha+2\gamma)G(t,z,z)}\phi(t)dt$$
$$<\int_0^{G(t,z,z)}\phi(t)dt$$

Contradiction, hence z = t. Thus the common fixed point is unique.

If we put $\phi(t)=1$ in the above theorem, we get the following result:

Corollary 2.1: Let (X,G) be symmetric G-metric space. Suppose f, g, and h are three self mappings of (X,G) satisfying the conditions:

(1) for all $x, y \in X$
$G(fx,gy,gy)\le \alpha G(hx,hy,hy)+\beta[G(fx,hx,hx)+G(gy,hy,hy)]+\gamma[G(hx,gy,gy)+G(hy,fx,fx)]$ and α,β,γ are non-negative reals such that $\alpha+2\beta+2\gamma<1$

(2) pair of mappings (f, h) or (g, h) is owc.

Then f, g and h have a unique common fixed point.

2.2 A unique common fixed point theorem for four mappings

Now, we give our second main result:

Theorem 2.2: Let (X, G) be symmetric G-metric space. Suppose f, g , h and k are four self mappings of (X,G) satisfying the following conditions: (1)

$$\int_0^{G(fx,gy,gy)}\phi(t)dt \le \int_0^{\alpha G(hx,ky,ky)+\beta[G(fx,hx,hx)+G(gy,ky,ky)]+\gamma[G(hx,gy,gy)+G(ky,fx,fx)]}\phi(t)dt$$ for all $x, y \in X$, where ϕ : $R^+ \to R$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that $\int_0^{\epsilon}\phi(t)dt>0$ for each $\epsilon>0$, and α,β,γ are non-negative reals such that $\alpha+2\beta+2\gamma<1$

(2) pair of mappings (f, h) and (g, k) are owc.

Then f, g, h and k have a unique common fixed point.

Proof: Since pairs of mappings (f, h) and (g , k) are owc, then , there exists two points u and v in X such that fu = hu and fhu = hfu, gv = kv and gkv = kgv.

First, we prove that fu = gv. Indeed, by inequality (1), we get

$$\int_0^{G(fu,gv,gv)} \phi(t)dt \le \int_0^{\alpha G(hu,kv,kv)+\beta[G(fu,hu,hu)+G(gv,kv,kv)]+\gamma[G(hu,gv,gv)+G(kv,fu,fu)]} \phi(t)dt$$

$$= \int_0^{\alpha[G(hu,kv,kv)]+\gamma[G(fu,gv,gv)]} \phi(t)dt$$

$$= \int_0^{(\alpha+\gamma)G(fu,gv,gv)} \phi(t)dt$$

$$< \int_0^{G(fu,gv,gv)} \phi(t)dt$$

which is a contradiction, hence, gv = fu = hu = kv.

Again, suppose that ffu = fhu = hfu $\neq$ fu. The use of condition (1), we have

$$\int_0^{G(ffu,gv,gv)} \phi(t)dt \le \int_0^{\alpha G(hfu,kv,kv)+\beta[G(ffu,hfu,hfu)+G(gv,kv,kv)]+\gamma[G(hfu,gv,gv)+G(kv,ffu,ffu)]} \phi(t)dt$$

$$= \int_0^{\alpha G(ffu,fu,fu)+2\gamma[G(ffu,gv,gv)]} \phi(t)dt$$

$$= \int_0^{(\alpha+2\gamma)G(ffu,gv,gv)} \phi(t)dt$$

$$< \int_0^{G(ffu,gv,gv)} \phi(t)dt$$

this contradiction implies that ffu = fu = hfu = fhu.

Similarly gfu = kfu = fu. Put fu = t , therefore t is a common fixed point of mappings f, g, h and k.

Now, let t and z be two distinct common fixed points of f, g, h and k. That is ft = gt =ht = kt = t and fz = gz = hz = kz = z. As t $\neq$ z, then from condition (1), we have

$$\int_0^{G(t,z,z)} \phi(t)dt = \int_0^{G(ft,gz,gz)} \phi(t)dt \le \int_0^{\alpha G(ht,hz,hz)+\beta[G(ft,ht,ht)+G(gz,hz,hz)]+\gamma[G(ht,gz,gz)+G(hz,ft,ft)]} \phi(t)dt$$

$$= \int_0^{\alpha G(t,z,z)+2\gamma G(t,z,z)]} \phi(t)dt$$

$$= \int_0^{(\alpha+2\gamma)G(t,z,z)} \phi(t)dt$$

$$< \int_0^{G(t,z,z)} \phi(t)dt$$

a contradiction, hence z = t. Thus the common fixed point is unique.

If we put $\phi(t) = 1$ in the above theorem, we get the following result:

Corollary: 2.2 Let (X,G) be symmetric G-metric space. Suppose f, g , h and k are four self mappings of (X,G) satisfying the following conditions:

(1) $G(fx, gy, gy) \le \alpha G(hx, ky, ky) + \beta[G(fx, hx, hx) + G(gy, ky, ky)] + \gamma[G(hx, gy, gy) + G(ky, fx, fx)]$
for all $x, y \in X$, and α, β, γ are non-negative reals such that $\alpha + 2\beta + 2\gamma < 1$.

(2) pair of mappings (f, h) and (g, k) are owc.

Then f, g, h and k have a unique common fixed point.

Example 2.1: Let X = $[0,\infty)$ with the symmetric G-metric G(x, y, z) = $(x-y)^2 + (y-z)^2 + (z-x)^2$. Define

$$f(x) = g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x \in [1,\infty) \end{cases}, h(x) = \begin{cases} 3 & x \in [0,1) \\ \dfrac{1}{x} & x \in [1,\infty) \end{cases},$$

$$h(x) = \begin{cases} 9 & x \in [0,1) \\ \dfrac{1}{\sqrt{x}} & x \in [1,\infty) \end{cases}$$

Clearly (f,h) and (g,k) are occasionally weakly compatible. By taking $\phi(x) = 3x^2, \alpha = \frac{1}{4}, \beta = \frac{1}{5}, \gamma = \frac{1}{6}$, all the hypothesis of theorem 2.2 are satisfied and x = 1 is the unique common fixed point of mappings f, g, h and k.

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