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RESIDUATED ALMOST DISTRIBUTIVE LATTICES - I

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ABSTRACT

In this paper, we introduce the concepts of residuation and multiplication in an Almost Distributive Lattice (ADL) L and define a Residuated ADL. We prove important relations between residuation and multiplication in an ADL L.

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1. INTRODUCTION

Swamy, U.M. and Rao, G.C. [3] introduced the concept of an Almost Distributive Lattice as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, p-rings, biregular rings, associate rings, P1-rings etc.) on one hand and distributive lattices on the other. In [4], [5], Ward, M. and Dilworth, R.P., have studied residuated lattices. In this paper, we extend the concepts of a residuation and a multiplication to an ADL ' L ' and we prove that a residuation satisfying one additional condition gives rise to a multiplication, which is unique upto equivalence. We also prove that a multiplication satisfying one additional condition gives rise to a residuation, which is also unique upto equivalence.

In section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL. These are taken from Swamy, U.M. and Rao, G.C. [3] and Rao,G.C. [2].

In section 3, we introduce the concepts of residuation and multiplication in an ADL L and define the concept of residuated almost distributive lattice. We derive important properties of residuation and multiplication in an ADL L.

2. PRELIMINARIES

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1 [2]: A relation L on A satisfying the property reflexive, anti symmetric and transitive is called a partial order relation on A. "$\leq$" is generally used for partial orders.

If "$\leq$" is a partial order on A, then we call $(A, \leq)$ as a partially ordered set (Poset).

Definition 2.2 [2]: A poset $(L, \leq)$ is called a lattice if every subset of L with exactly two elements has supremum and infimum in L. $(L, \leq)$ is a lattice $a, b \in L$ if and only if $\{a, b\}$ has supremum and infimum in L. If $(L, \vee, \wedge)$ be any lattice. Then

(i) A non empty set H of L is called a sub lattice of L, if $a \wedge b,\ a \vee b \in H$, for all $a, b \in H$.
(ii) A sublattice H of L is said to be convex if $a, b \in H$, $c \in L$, $a \leq b$, $a \leq c \leq b \Rightarrow c \in H$.

Definition 2.3 [2]: $(P, \leq)$ is a poset for every $a, b \in P$, either $a \leq$ hold then $(P, \leq)$ is called a chain or simply ordered set. Every chain is a lattice but not vice versa.

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Definition 2.4 [2]: An algebra (L, ∨, ∧) of type (2, 2) is called a lattice if it satisfies the following axioms:

1) a) $x \vee x = x$ b)$x \wedge x = x$
2) a) $x \vee y = y \vee x$ b)$x \wedge y = y \wedge x$
3) a) $(x \vee y) \vee z = x \vee (y \vee z)$ b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
4) a) $(x \vee y) \wedge y = y$ b)$(x \wedge y) \vee y = y$.

In any lattice (L, ∨, ∧) the following are equivalent:

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.5 [2]: A Lattice (L, ∨, ∧) satisfying any of the above four equations is called a distributive lattice.

Definition 2.6 [2]: If (L, ∨, ∧) is a lattice then an element 0∈L is called a zero element or least element of L if $0 \wedge a = 0$ $\forall a \in L$ and an element 1 of L is called 1 element or greatest element if $a \vee 1 = 1$ $\forall a \in L$. If L has 0 and 1 then L is called a bounded lattice.

Definition 2.7 [2]: A bounded lattice L is called complemented if to each a ∈ L there exists b ∈ L such that $a \vee b = 1$ and $a \wedge b = 0$. A lattice L is said to be relatively complemented if for any x, y ∈ L such that $x \leq y$, the bounded lattice $[x, y] = \{z \in L / x \leq z \leq y\}$ is a complemented lattice.

It is well known that a lattice L is distributive if and only if relative compliments of any element in any interval [x, y], $x \leq y$ are unique.

Definition 2.8 [2]: A bounded distributive and complemented lattice is called a Boolean algebra.

Definition 2.9 [2]: A sub lattice I of L is called an ideal of L if i ∈ I, a ∈ L imply $a \wedge i \in I$. An ideal I of L is said to be proper if I = L

An ideal I of L is said to be prime if

1) $a \wedge b \in I$, a, b ∈ L ⇒ either a ∈ I or b ∈ I
2) $I \neq L$

An ideal M of L is called maximal if

(i) $M \neq L$
(ii) If U is an ideal of L such that $M \subseteq U \subseteq L$ ⇒ either M = U or U = L.

Now, we give the definition of an ADL.

Definition 2.10 [2]: An **Almost Distributive Lattice (ADL)** is an algebra (L, ∨, ∧) of type (2, 2) satisfying

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$, for all a, b, c ∈ L.

It can be seen directly that every distributive lattice is an ADL.

If there is an element 0 ∈ L such that $0 \wedge a = 0$ for all a ∈ L, then (L, ∨, ∧, 0) is called an ADL with 0.

Example 2.1 [2]: Let X be a non-empty set. Fix $x_0 \in X$. For any x, y ∈ L,

define $x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases}$ $\quad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with x_0 as its zero element. This ADL is called a **discrete ADL.**

For any a, b ∈ L, we say that a is less than or equals to b and write $a \leq b$, if $a \wedge b = a$. Then "≤" is a partial ordering on L.

The following hold in any ADL L.

Here onwards by L we mean an ADL (L, ∨, ∧, 0).

Theorem 2.1 [2]: For any a, b ∈ L, we have

(1) $a \wedge 0 = 0$ and $0 \vee a = a$
(2) $a \wedge a = a = a \vee a$
(3) $(a \wedge b) \vee b = b$, $a \vee (b \wedge a) = a$ and $a \wedge (a \vee b) = a$
(4) $a \wedge b = a \Leftrightarrow a \vee b = b$ and $a \wedge b = b \Leftrightarrow a \vee b = a$
(5) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ whenever $a \leq b$
(6) $a \wedge b \leq b$ and $a \leq a \vee b$
(7) $\wedge$ is associative in L
(8) $a \wedge b \wedge c = b \wedge a \wedge c$
(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(10) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(11) $a \vee (b \vee a) = a \vee b$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except, possible the right distributivity of ∨ over ∧, the commutativity of ∨, the commutativity of ∧ and the absorption law $(a \wedge b) \vee a = a$. Any one of these properties convert L into a distributive lattice.

Theorem 2.2 [2]: Let (L, ∨, ∧, 0) be an ADL with 0. Then the following are equivalent:

(1) (L, ∨, ∧, 0) is a distributive lattice
(2) $a \vee b = b \vee a$ for all a, b ∈ L
(3) $a \wedge b = b \wedge a$ for all a, b ∈ L
(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c ∈ L.

Proposition 2.1 [2]: Let (L, ∨, ∧, 0) be an ADL. Then for any a, b, c ∈ L with $a \leq b$, we have

(1) $a \wedge c \leq b \wedge c$
(2) $c \wedge a \leq c \wedge b$
(3) $c \vee a \leq c \vee b$.

Definition 2.11 [2]: An element m ∈ L is called maximal if it is maximal as in the partially ordered set (L,≤). That is, for any a ∈ L, $m \leq a$ implies $m = a$.

Theorem 2.3 [2]: Let L be an ADL and m ∈ L. Then the following are equivalent:

(1) m is maximal with respect to ≤
(2) $m \vee a = m$, for all a ∈ L
(3) $m \wedge a = a$, for all a ∈ L.

Lemma 2.1 [2]: Let L be an ADL with a maximal element m and x, y ∈ L. If $x \wedge y = y$ and $y \wedge x = x$ then x is maximal if and only if y is maximal. Also the following conditions are equivalent:

(i) $x \wedge y = y$ and $y \wedge x = x$
(ii) $x \wedge m = y \wedge m$.

Definition 2.12 [2]: If (L, ∨, ∧, 0, m) is an ADL then the set I(L) of all ideals of L is a complete lattice under set inclusion. In this lattice, for any I, J ∈ I(L), the l.u.b. and g.l.b. of I, J are given by

$$I \vee J = \{(x \vee y) \wedge m / x \in I, y \in J\} \text{ and } I \wedge J = I \cap J.$$

The set P I(L) = {(a] /a ∈ L} of all principal ideals of L forms a sublattice of I(L).
(Since $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$)

Definition 2.13 [2]: An ADL L = (L, ∨, ∧, 0, m) with a maximal element m is said to be a complete ADL, if P I(L) is a complete sub lattice of the lattice I(L).

Theorem 2.4 [2]: Let L = (L, ∨, ∧, 0, m) be an ADL with a maximal element m. Then L is a complete ADL if and only if the lattice ([0, m], ∨, ∧) is a complete lattice.

3. RESIDUATION AND MULTIPLICATION IN ADL's

In this section, we introduce the concepts of Residuation and Multiplication in an Almost Distributive Lattice (ADL) L and prove important properties of Residuation and Multiplication in an ADL L.

First we begin with the following definition.

Definition 3.1: Let L be an ADL with a maximal element m. A binary operation: on an ADL L is called a **residuation** over L if, for a, b, c ∈ L the following conditions are satisfied.

(R1) $a \wedge b = b$ if and only if a: b is maximal
(R2) $a \wedge b = b \Rightarrow$ (i) $(a: c) \wedge (b: c) = b: c$ and (ii) $(c: b) \wedge (c : a) = c: a$
(R3) $[(a: b): c] \wedge m = [(a: c): b] \wedge m$
(R4) $[(a \wedge b): c] \wedge m = (a: c) \wedge (b: c) \wedge m$
(R5) $[c: (a \vee b)] \wedge m = (c: a) \wedge (c: b) \wedge m$.

Definition 3.2: Let L be an ADL with a maximal element m. Abinary operation. on an ADL L is called a **multiplication** over L if, for $a, b, c \in L$ the following conditions are satisfied.
(M 1) $(a.b) \wedge m = (b.a) \wedge m$
(M 2) $[(a.b).c] \wedge m = [a.(b.c)] \wedge m$
(M 3) $(a.m) \wedge m = a \wedge m$
(M 4) $[a.(b \vee c)] \wedge m = [(a.b) \vee (a.c)] \wedge m$.

In the following, we give the definition of a Residuated ADL.

Definition 3.3: An ADL L with a maximal element m is said to be a **residuated almost distributive lattice (residuated ADL)**, if there exists two binary operations ':' and ' . ' on L satisfying conditions R1 to R5, M1 to M4 and the following condition (A).
(A) $(x: a) \wedge b = b$ if and only if $x \wedge (a.b) = a.b$, for any $x, a, b \in L$.

The following two properties of multiplication and residuation are required for further discussions.

Definition 3.4: Let L be a complete ADL with a maximal element m.
(M 5) For $a, b, x_i \in L$, $\bigvee_{i\in J} \{a \wedge (x_i.b) \wedge m\} = a \wedge [\bigvee_{i\in J}\{(x_i.b) \wedge m\}]$ and
$\bigvee_{i\in J} \{(x_i .b) \wedge m\} = [\bigvee_{i\in J}\{(x_i \wedge m)\}].b$
R6) If A is a non empty subset of L and $a, b \in L$ then,
$\bigwedge_{x\in A} \{(x: a) \wedge m\} = [\bigwedge_{x\in A} \{x \wedge m\}]: a$

We use the following properties frequently later in the results.

Lemma 3.1: Let L be an ADL with a maximal element m and . a binary operation on L satisfying the conditions M 1 − M 4. Then for any $a, b, c, d \in L$,
(i) $a \wedge (a.b) = a.b$ and $b \wedge (a.b) = a.b$
(ii) $a \wedge b = b \Rightarrow (c.a) \wedge (c.b) = c.b$ and $(a.c) \wedge (b.c) = b.c$
(iii) $d \wedge [(a.b).c] = (a.b).c$ if and only if $d \wedge [a.(b.c)] = a.(b.c)$
(iv) $(a.c) \wedge (b.c) \wedge [(a \wedge b).c] = (a \wedge b).c$
(v) $d \wedge (a.c) \wedge (b.c) = (a.c) \wedge (b.c) \Rightarrow d \wedge [(a \wedge b).c] = (a \wedge b).c$
(vi) $d \wedge [(a.c) \vee (b.c)] = (a.c) \vee (b.c) \Leftrightarrow d \wedge [(a \vee b).c] = (a \vee b).c$

Proof: Let $a, b, c, d \in L$
(i) $(a \wedge m) \vee [(a.b) \wedge m] = [(a.m) \wedge m] \vee [(a.b) \wedge m]$ (By M3)
$= [(a.m) \vee (a.b)] \wedge m$
$= [a.(m \vee b)] \wedge m$ (ByM 4)
$= (a.m) \wedge m$ (Since $m \vee b = m$)
$= a \wedge m$ (By M3)

$\Rightarrow a \wedge m \wedge (a.b) \wedge m = (a.b) \wedge m$
$\Rightarrow a \wedge (a.b) = a.b$

Interchanging a and b in the above, we get that $b \wedge (b.a) = b.a$
$\Rightarrow b \wedge (b.a) \wedge m = (b.a) \wedge m$
$\Rightarrow b \wedge (a.b) \wedge m = (a.b) \wedge m$ (By M1)
$\Rightarrow b \wedge (a.b) = a.b$

(ii) Suppose $a \wedge b = b$. Then $a \vee b = a$.

Now, $[(c.a) \wedge m] \vee [(c.b) \wedge m] = [(c.a) \vee (c.b)] \wedge m = [c.(a \vee b)] \wedge m$ (By M4)
$= (c.a) \wedge m$

Therefore, $(c.a) \wedge m \wedge (c.b) \wedge m = (c.b) \wedge m$ and hence $(c.a) \wedge (c.b) = c.b$

Similarly, $(a.c) \wedge (b.c) = b.c$

(iii) $d \wedge [(a.b).c] = (a.b).c$

$\Leftrightarrow d \wedge [(a.b).c] \wedge m = [(a.b).c] \wedge m$
$\Leftrightarrow d \wedge [a.(b.c)] \wedge m = [a.(b.c)] \wedge m$ (By M2)
$\Leftrightarrow d \wedge [a.(b.c)] = [a.(b.c)]$

(iv) By (i), above, $a \wedge a \wedge b = a \wedge b \Rightarrow (a.c) \wedge [(a \wedge b).c] = (a \wedge b).c$

and $b \wedge a \wedge b = a \wedge b \Rightarrow (b.c) \wedge [(a \wedge b).c] = (a \wedge b).c$

Thus, $(a.c) \wedge (b.c) \wedge [(a \wedge b).c] = (a \wedge b).c$

(v) follows directly from (iv).

(vi) follows from M4.

The following result is a direct consequence of M1 of definition 3.2.

Lemma 3.2: Let L be an ADL with a maximal element m and . a binary operation on L satisfying the condition M 1. For a, b, x ∈ L, $a \wedge (x.b) = x.b$ if and only if $a \wedge (b.x) = b.x$

Now, we prove the following.

Theorem 3.1: Let L be a complete ADL with a maximal element m. Suppose . is a binary operation on L satisfying conditions M 1 − M 5. For x, b ∈ L,

Let $A = \{a \in L / x \wedge (a.b) = a.b\}$. Define $x : b = \bigvee_{a \in A} (a \wedge m)$. Then we have

(i) $x: b \in A$
(ii) $(x: b) \wedge a = a$ if and only if $x \wedge (a.b) = a.b$ (or, $a \in A$)
(iii) $(x: b) \wedge a = a$ if and only if $(x : a) \wedge b = b$
(iv) $x \wedge \{[(x : b) \wedge a].b\} = [(x : b) \wedge a].b$

Proof: Let a, b, x ∈ L.

Since, $x \wedge (x.b) = x.b$, we get $x \in A$ and hence A is a non empty set.

Also, since [0, m] is a complete lattice, we get that the binary operation: is well defined.

(i) $x \wedge [(x : b).b] = x \wedge [\{ \bigvee_{a \in A} (a \wedge m)\}.b]$
$= x \wedge [\bigvee_{a \in A} \{(a.b) \wedge m\}]$ (By M 5)
$= \bigvee_{a \in A} [x \wedge (a.b) \wedge m]$ (By M 5)
$= \bigvee_{a \in A} [(a.b) \wedge m]$ (Since $a \in A$)
$= [\bigvee_{a \in A} (a \wedge m)].b$ (By M5)
$= (x: b).b$ (Since $x : b = \bigvee_{a \in A} (a \wedge m)$).

Therefore $x: b \in A$

(ii) Suppose that $(x: b) \wedge a = a$.

Then, by Lemma 3.1(ii), $[(x: b).b] \wedge (a.b) = a.b$

Now, $x \wedge (a.b) = x \wedge [(x : b).b] \wedge (a.b)$
$= [(x: b).b] \wedge (a.b)$ (By (i) above)
$= a.b$

On the other hand, if $a \in A$, then $(x: b) \wedge a \wedge m = a \wedge m$ and hence $(x: b) \wedge a = a$

(iii) follows from (ii) and Lemma 3.2

(iv) $(x: b) \wedge (x : b) \wedge a = (x : b) \wedge a$

$\Rightarrow x \wedge \{[(x : b) \wedge a].b\} = [(x : b) \wedge a].b$ (By (ii) above).

In the following results, we establish the co-existence of a residuation and a multiplication in an ADL in the presence of M5 or R6. First we prove that a multiplication on an ADL L satisfying M5 gives rise to a residuation on L in the following Theorem.

Theorem 3.2: Let L be a complete ADL with a maximal element m and. a muiltiplication on L satisfying condition M 5. Then there exists a residuation: on L such that $x \wedge (a.b) = a.b$ if and only if $(x : b) \wedge a = a$, for any $x, a, b \in L$.

Proof: For $x, b \in L$, let $A = \{a \in L \mid x \wedge (a.b) = a.b\}$ and $x{:}b = \bigvee_{a \in A} (a \wedge m)$ (As in Theorem 3.1)

R1: Suppose $x \wedge b = b$. Then $x \vee b = x$.

Now, for any $a \in L$,
$$\begin{aligned} x \wedge (b.a) \wedge m &= (x \vee b) \wedge (b.a) \wedge m \\ &= [x \wedge (b.a) \wedge m] \vee [b \wedge (b.a) \wedge m] \\ &= [x \wedge (b.a) \wedge m] \vee [(b.a) \wedge m] \\ &= (b.a) \wedge m \end{aligned}$$

So that $x \wedge (a.b) = a.b$ and hence $a \in A$. Then $A = L$

Therefore $x: b = m$ That is $x: b$ is maximal.

On the other hand, $x: b$ is maximal.

Then $x : b = m$ and hence $m \in A$ (By (i) of Theorem 3.1)

$$\begin{aligned} \text{Now, } x \wedge b \wedge m &= x \wedge (b.m) \wedge m \text{ (By M3)} \\ &= x \wedge (m.b) \wedge m \text{ (By M1)} \\ &= (m.b) \wedge m \text{ (Since } m \in A) \\ &= b \wedge m \text{ (By M3)} \end{aligned}$$

Hence $x \wedge b = b$.

R2: Suppose $a, b \in L$ such that $a \wedge b = b$ and $c \in L$

(i) Since $(b: c) \wedge (b: c) = b: c$, we get
$b \wedge [(b : c).c] = (b : c).c$

$$\begin{aligned} \text{Now, } a \wedge [(b: c).c] &= (a \vee b) \wedge [(b: c).c] \\ &= (a \wedge [(b: c).c]) \vee (b \wedge [(b: c).c]) \\ &= (a \wedge [(b: c).c]) \vee [(b: c).c] \\ &= (b: c).c \end{aligned}$$

Hence $(a: c) \wedge (b: c) = b: c$

(ii) Since $(c : a) \wedge (c : a) = c : a$, we get that $[c : (c : a)] \wedge a = a$
$\Rightarrow [c : (c : a)] \wedge a \wedge b = a \wedge b$
$\Rightarrow [c : (c : a)] \wedge b = b$
$\Rightarrow (c : b) \wedge (c : a) = c : a$

Thus (R2) holds in L.

Now, we prove (R3).

Let $a, b, c \in L$. Then
$[(a : c) : b] \wedge [(a : c) : b] = (a : c) : b$

$\Rightarrow (a : c) \wedge \{[(a : c) : b].b\} = [(a : c) : b].b$
$\Rightarrow a \wedge \{([(a : c) : b].b).c\} = \{([(a : c) : b].b).c\}$
$\Rightarrow a \wedge \{[(a : c) : b].(b.c)\} = \{[(a : c) : b].(b.c)\}$
$\Rightarrow [a : (b.c)] \wedge [(a : c) : b] = (a : c) : b$
$\Rightarrow (a : [(a : c) : b]) \wedge (b.c) = b.c$
$\Rightarrow (a : [(a : c) : b]) \wedge (c.b) = c.b$
$\Rightarrow [a : (c.b)] \wedge [(a : c) : b] = (a : c) : b$
$\Rightarrow a \wedge \{[(a : c) : b].(c.b)\} = [(a : c) : b].(c.b)$
$\Rightarrow a \wedge \{([(a : c) : b].c).b\} = ([(a : c) : b].c).b$
$\Rightarrow (a : b) \wedge ([(a : c) : b].c) = ([(a : c) : b].c)$
$\Rightarrow [(a : b) : c] \wedge [(a : c) : b] = (a : c) : b$

Interchanging b and c in the above, we get that
$[(a : c) : b] \wedge [(a : b) : c] = (a : b) : c$

Hence $[(a : b) : c] \wedge m = [(a : c) : b] \wedge m$.

Thus R3 holds in L.

Now, we prove (R4).

Let $a, b, c \in L$. Then
$(a : c) \wedge (a : c) = a : c$
$\Rightarrow a \wedge [(a : c).c] = (a : c).c$

Similarly, we get $b \wedge [(b : c).c] = (b : c).c$

Thus $a \wedge [(a : c).c] \wedge b \wedge [(b : c).c] = [(a : c).c] \wedge [(b : c).c]$
$\Rightarrow a \wedge b \wedge [(a : c).c] \wedge [(b : c).c] = [(a : c).c] \wedge [(b : c).c]$
$\Rightarrow a \wedge b \wedge \{[(a : c) \wedge (b : c)].c\} = [(a : c) \wedge (b : c)].c$
$\Rightarrow [(a \wedge b) : c] \wedge (a : c) \wedge (b : c) = (a : c) \wedge (b : c)$

Now, we have $a \wedge a \wedge b = a \wedge b$ and $b \wedge a \wedge b = a \wedge b$
$\Rightarrow (a : c) \wedge [(a \wedge b) : c] = (a \wedge b) : c$ and $(b : c) \wedge [(a \wedge b) : c] = (a \wedge b) : c$
$\Rightarrow (a : c) \wedge [(a \wedge b) : c] \wedge (b : c) \wedge [(a \wedge b) : c] = [(a \wedge b) : c] \wedge [(a \wedge b) : c]$
$\Rightarrow (a : c) \wedge (b : c) \wedge [(a \wedge b) : c] = (a \wedge b) : c$

Hence we get $[(a \wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$

Thus (R4) holds in L.

Finally, we prove (R5)

Let $a, b, c \in L$. Then $c \wedge (\{[(c : a) \wedge (c : b)].a\} \vee \{[(c : a) \wedge (c : b)].b\}) = \{[(c : a) \wedge (c : b)].a\} \vee \{[(c : a) \wedge (c : b)].b\}$
$\Rightarrow c \wedge \{[(c : a) \wedge (c : b)].(a \vee b)\}=\{[(c : a) \wedge (c : b)].(a \vee b)\}$
$\Rightarrow [c : (a \vee b)] \wedge (c : a) \wedge (c : b) = (c : a) \wedge (c : b)$

Now, we have $(a \vee b) \wedge a = a$ and $(a \vee b) \wedge b = b$
$\Rightarrow (c : a) \wedge [c : (a \vee b)] = [c : (a \vee b)]$ and $(c : b) \wedge [c : (a \vee b)] = [c : (a \vee b)]$
$\Rightarrow (c : a) \wedge [c : (a \vee b)] \wedge (c : b) \wedge [c : (a \vee b)] = [c : (a \vee b)] \wedge [c : (a \vee b)]$
$\Rightarrow (c : a) \wedge (c : b) \wedge [c : (a \vee b)] = [c : (a \vee b)]$

Hence $[c : (a \vee b)] \wedge m = (c : a) \wedge (c : b) \wedge m$

Thus (R5) holds in L.

Theorem 3.3: Let L be a complete ADL with a maximal element m and : a binary operation on L satisfying conditions R1−R6. For $a, b \in L$, let $B = \{x \in L \mid (x : a) \wedge b = b\}$ and define $a.b = \bigwedge_{x \in B} (x \wedge m)$. Then we have

(i) $[(a.b) : a] \wedge b = b$ and
(ii) $(x : a) \wedge b = b$ if and only if $x \wedge (a.b) = a.b$

Proof: Let $a, b \in L$.

Since $(a : a) \wedge b = b$, we get $a \in B$ and hence B is a non empty set.

Also, since [0, m] is a complete lattice, we get that the binary operation . is well defined.

(i) Let $x \in B$.

Then $(x : a) \wedge b = b$. and $b \wedge m$ $(x : a) \wedge m$.

So that $b \wedge m \leq \bigwedge_{x \in B} [(x : a) \wedge m] = [\bigwedge_{x \in B} (x \wedge m)] : a$ (By R6).

$= (a.b) : a$

$\Rightarrow b \wedge m \wedge [(a.b) : a] \wedge m = b \wedge m$.

Hence $[(a.b) : a] \wedge b = b$.

Now, we prove (ii).

Suppose $(x : a) \wedge b = b$.

Then $x \in B$ and hence $a.b \leq x \wedge m$.

So that $x \wedge (a.b) = a.b$

On the other hand, $x \wedge (a.b) = a.b$
$\Rightarrow (x : a) \wedge [(a.b) : a] = (a.b) : a$
$\Rightarrow (x : a) \wedge [(a.b) : a] \wedge b = [(a.b) : a] \wedge b$
$\Rightarrow (x : a) \wedge b = b$ (By (i) above).

We have proved, in Theorem 3.2, that a multiplication on an ADL L satisfying M5 gives rise to a residuation on L.

Now we prove, in the following Theorem, that a residuation on an ADL L satisfying R6 gives rise to a multiplication on L.

Theorem 3.4: Let L be a complete ADL with a maximal element m. If : is a residuation on L satisfying R6, then there exists a multiplication . on L such that $(x : a) \wedge b = b$ if and only if $x \wedge (a.b) = a.b$, for any $x, a, b \in L$.

Proof: Suppose : is a binary operation on L satisfying conditions R1 − R6 and let $a, b \in L$.

Define $a.b = \bigwedge_{x \in B} (x \wedge m)$, where $B = \{x \in L \mid (x : a) \wedge b = b\}$

We prove (M1).

Let $a, b \in L$. Then
$(a.b) \wedge (a.b) = a.b$
$\Rightarrow [(a.b) : a] \wedge b = b$
$\Rightarrow [(a.b) : a] : b$ is maximal
$\Rightarrow [(a.b) : b] : a$ is maximal
$\Rightarrow [(a.b) : b] \wedge a = a$
$\Rightarrow (a.b) \wedge (b.a) = b.a$

Interchanging a and b in the above, we get that $(b.a) \wedge (a.b) = a.b$

Hence $(a.b) \wedge m = (b.a) \wedge m$

Now, we prove the following property.

(iii) $(x : a) \wedge b = b$ if and only if $(x : b) \wedge a = a$
$(x : a) \wedge b = b \Leftrightarrow x \wedge (a.b) = a.b \Leftrightarrow x \wedge (b.a) = b.a \Leftrightarrow (x : b) \wedge a = a$

Now, we prove (M2).

Let $a, b, c \in L$.

Since $[(a.b).c] \wedge [(a.b).c] = [(a.b).c]$, we get

$$[(a.b).c] \wedge [c.(a.b)] = [c.(a.b)] \rightarrow (1)$$

Also, $[c.(a.b)] \wedge [c.(a.b)] = c.(a.b)$
$\Rightarrow \{[c.(a.b)] : c\} \wedge (a.b) = a.b$
$\Rightarrow \{([c.(a.b)] : c) : a\} \wedge b = b$
$\Rightarrow \{([c.(a.b)] : a) : c\} \wedge b = b$
$\Rightarrow \{[c.(a.b)] : a\} \wedge (c.b) = c.b$
$\Rightarrow \{[c.(a.b)] : a\} \wedge (b.c) = b.c$
$\Rightarrow [c.(a.b)] \wedge [a.(b.c)] = a.(b.c) \rightarrow (2)$

Now, $[(a.b).c] \vee [a.(b.c)] = [(a.b).c] \vee \{[c.(a.b)] \wedge [a.(b.c)]\}$ (By (2))
$= \{[(a.b).c] \vee [c.(a.b)]\} \wedge \{[(a.b).c] \vee [a.(b.c)]\}$
$= [(a.b).c] \wedge \{[(a.b).c] \vee [a.(b.c)]\}$ (By (1))
$= (a.b).c$

Thus $[a.(b.c)] \wedge m \leq [(a.b).c] \wedge m \rightarrow (3)$

Now, $[(a.b).c] \wedge m = [c.(a.b)] \wedge m$
$\leq [(c.a).b] \wedge m$ (By (3))
$= [b.(c.a)] \wedge m$
$\leq [(b.c).a] \wedge m$ (By (3))
$= [a.(b.c)] \wedge m$

$\Rightarrow [(a.b).c] \wedge m \leq [a.(b.c)] \wedge m \rightarrow (4)$

From (3) and (4), we get $[(a.b).c] \wedge m = [a.(b.c)] \wedge m$

Now, we prove (M3).

Since a: a is a maximal element.

Then $(a : a) \wedge m = m$

$\Rightarrow a \wedge (a.m) = a.m$

Now, for any $p \in L$, $[(a.m) : a] \wedge p = [(a.m) : a] \wedge m \wedge p = m \wedge p = p$.

Therefore $(a.m) : a$ is maximal.

Thus $(a.m) \wedge a = a$.

Hence $a \wedge m = (a.m) \wedge m$

Finally, we prove (M4).

Let $a, b, c \in L$.

Now, $[(a.b) \vee (a.c)] \wedge (a.b) = a.b$

$\Rightarrow ([(a.b) \vee (a.c)] : a) \wedge b = b \rightarrow (1)$

Similarly, $\{[(a.b) \vee (a.c)] : a\} \wedge c = c \rightarrow (2)$

From (1) and (2), we get that

$$[([(a.b) \vee (a.c)] : a) \wedge b] \vee [\{[(a.b) \vee (a.c)] : a\} \wedge c] = b \vee c$$

Thus, $\{[(a.b) \vee (a.c)] : a\} \wedge (b \vee c) = b \vee c$

$\Rightarrow [(a.b) \vee (a.c)] \wedge [a.(b \vee c)] = [a.(b \vee c)]$

Now, $\{[a.(b \vee c)] : a\} \wedge b = \{[a.(b \vee c)] : a\} \wedge (b \vee c) \wedge b$
$= (b \vee c) \wedge b = b$

Thus $[a.(b \vee c)] \wedge (a.b) = a.b \rightarrow (3)$

Similarly, $[a.(b \vee c)] \wedge (a.c) = a.c \rightarrow (4)$

Therefore, $[a.(b \vee c)] \wedge [(a.b) \vee (a.c)] = (a.b) \vee (a.c)$ (By (3) and (4))

Hence $[a.(b \vee c)] \wedge m = [(a.b) \vee (a.c)] \wedge m$.

Definition 3.5: Let $(L, \vee, \wedge)$ be an ADL. For $a, b \in L$, we say that a is equivalent to b or $a \sim b$, if $a \wedge b = b$ and $b \wedge a = a$. Then ' $\sim$ ' is an equivalence relation on L.

In the following Theorem, we prove that the correspondence between a residuation and a multiplication on an ADL L given in Theorem 3.3 and Theorem 3.4 is unique upto equivalence.

Theorem 3.5: Let L be a complete ADL with a maximal element m and $x, a, b \in L$.

(i) Suppose ' : ' is a residuation on L satisfying R6. If ' o ' is a binary operation on L such that $a \wedge (x.b) = x.b$ if and only if $(aob) \wedge x = x$ then $aob \sim a : b$

(ii) Suppose ' . ' is a multiplication on L satisfying M5. If ' * ' is a binary operation on L such that $(x : b) \wedge a = a$ if and only if $x \wedge (a * b) = a * b$ then $a * b \sim a.b$

Proof: Let $x, a, b \in L$.

We prove (i) and (ii).

(i) We have $(a : b) \wedge (a : b) = a : b$
$\Rightarrow a \wedge [b.(a : b)] = b.(a : b)$
$\Rightarrow a \wedge [(a : b).b] = (a : b).b$
$\Rightarrow (aob) \wedge (a : b) = a : b$

Now, we have $(aob) \wedge (aob) = aob$
$\Rightarrow a \wedge [(aob).b] = (aob).b$
$\Rightarrow a \wedge [b.(aob)] = b.(aob)$
$\Rightarrow (a : b) \wedge (aob) = aob$

Hence $aob \sim a : b$

(ii) We have $(a * b) \wedge (a * b) = a * b$
$\Rightarrow [(a * b) : b] \wedge a = a$
$\Rightarrow (a * b) \wedge (b.a) = b.a$
$\Rightarrow (a * b) \wedge (a.b) = a.b$

Now, we have $(a.b) \wedge (a.b) = a.b$
$\Rightarrow (a.b) \wedge (b.a) = b.a$
$\Rightarrow [(a.b): b] \wedge a = a$
$\Rightarrow (a.b) \wedge (a * b) = a * b$

Hence $a * b \sim a.b$.

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